CHAPTER I

THEORY OF VECTORS

I. OPERATIONS ON VECTORS

§ 1. Preliminary definitions. Magnitudes which can be characterized by means of one real number are called *scalars*. Examples of scalars are: mass, work, kinetic energy, etc.

A *vector* is a line segment in which the initial point is distinguished from the terminal point. Points are classified as *zero vectors*.

Magnitudes such as velocity, acceleration and force can be represented by means of vectors. A vector will be denoted by bold face type, for example $\mathbf{a}$; a vector whose origin is A and terminus is B will be denoted by $\overline{AB}$ (Fig. 1). In a drawing an arrow serves to mark the terminus of a vector. The origin of a vector is also called *a point of application.*

By the *length* or *absolute value* of the vector $\overline{AB}$ is meant the length of the line segment AB and it is denoted by $|\overline{AB}|$.

Two vectors having the same direction (i. e. parallel vectors) can have the same or opposite *senses* (Fig. 2).

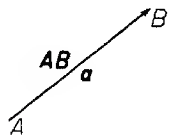

Fig. 1.

Fig. 2.

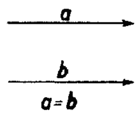

Fig. 3.

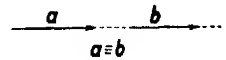

Fig. 4.

The vectors $\mathbf{a}$ and $\mathbf{b}$ having equal lengths, directions and senses are said to be *equal* (Fig. 3) and we write

$$\mathbf{a} = \mathbf{b}.$$

Two vectors having equal lengths and directions but opposite senses are called *opposite* vectors. The vector opposite to $\mathbf{a}$ is denoted by $-\mathbf{a}$ (Fig. 11).

The straight line on which a vector lies is called the *position* of the vector.

Equal vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ having the same position (i. e. lying on the same line) are termed *equipollent* (Fig. 4):

$$\boldsymbol{a} \equiv \boldsymbol{b}.$$

Two zero vectors are considered to be equal and equipollent.

Equal vectors will often be denoted by one and the same letter (whenever there is no likelihood of committing an error).

The *projection* of the vector $\boldsymbol{a}$ on a line (or plane) is the vector whose initial and terminal points are the projections of the corresponding points of $\boldsymbol{a}$.

Suppose that there is given in space a coordinate system $O(x, y, z)$ which is either rectangular or oblique. Rotate the x-axis about O in the xy-plane through an angle $< \pi$ so that the positive side of the x-axis falls on the positive side of the y-axis. If to an observer situated on the same side of the xy-plane as the positive side of the z-axis, the rotation is clockwise, then the coordinate system $O(x, y, z)$ is said to be *left-handed* and in the contrary case *right-handed.*

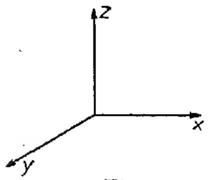

Fig. 5.

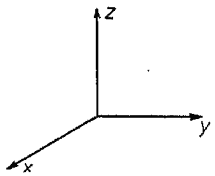

Fig. 6.

In this book we shall consistently use a left-handed rectangular system (i. e. as in Fig. 5, and not as in Fig. 6).

We say that *the system of vectors* ($\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$), not parallel to the same plane, has a *left (or right) sense,* if upon passing the x, y and z axes through an arbitrary point O parallel to and in the same direction as the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, we obtain a left-handed (or right-handed) system.

§ 2. Components of a vector. Let $\boldsymbol{a}$ represent an arbitrary vector and $\boldsymbol{a}'$ its projection on a given x-axis.

The component of the vector $\boldsymbol{a}$ *with respect to the x-axis,* which we shall denote by a_x, is a number defined in the following way: $a_x = |\boldsymbol{a}'|$ if $\boldsymbol{a}'$ has the same direction as the x-axis, but $a_x = -|\boldsymbol{a}'|$ in the contrary case.

We obviously have

$$a_x = |\boldsymbol{a}| \cos \alpha, \tag{1}$$

where α denotes the angle between the vector $\boldsymbol{a}$ and the x-axis (Fig. 8).

Suppose that a rectangular coordinate system (x, y, z) is given. Denoting the components of $\boldsymbol{a}$ with respect to the coordinate axes by a_x, a_y, a_z, and the angles which the vector makes with the axes by α, β, γ (Fig. 7), we obtain by (1):

$$a_x = |\boldsymbol{a}| \cos \alpha, \; a_y = |\boldsymbol{a}| \cos \beta, \; a_z = |\boldsymbol{a}| \cos \gamma. \tag{I}$$

Therefore: *equal vectors have equal components* with respect to the coordinate axes.

By the identity $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ well-known from analytic geometry and by (I)

$$|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}, \tag{II}$$

$$\cos \alpha = a_x / |\boldsymbol{a}|, \; \cos \beta = a_y / |\boldsymbol{a}|, \; \cos \gamma = a_z / |\boldsymbol{a}|. \tag{III}$$

From equations (II) and (III) it follows that the components of a vector define its length, direction and sense.

Hence, two *vectors* $\boldsymbol{a}$ and $\boldsymbol{b}$ *having correspondingly equal components* with respect to a rectangular coordinate system (i. e. for which $a_x = b_x$, $a_y = b_y$, $a_z = b_z$) *are equal.*

If the vector $\boldsymbol{a}$ lies in the xy-plane (Fig. 8), then

$$a_x = |\boldsymbol{a}| \cos \alpha, \; a_y = |\boldsymbol{a}| \sin \alpha, \tag{IV}$$

$$|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2}, \; \cos \alpha = a_x / |\boldsymbol{a}|, \; \sin \alpha = a_y / |\boldsymbol{a}|. \tag{V}$$

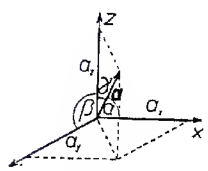

Fig. 7.

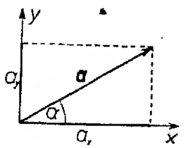

Fig. 8.

Often (when an error is precluded) the projections of the vector $\boldsymbol{a}$ on the coordinate axes are also termed the components a_x, a_y, a.

It is easy to show that if the points A and A' have coordinates x, y, z and x', y', z' respectively, then the vector $\boldsymbol{a} = \overline{AA'}$ has components: $a_x = x' - x$, $a_y = y' - y$, and $a_z = z' - z$.

§ 3. Sum and difference of vectors. Every vector which can be obtained in the following manner is said to be the *sum* of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

From an arbitrary point O we draw a vector equal to $\boldsymbol{a}$ and from the terminal point of this vector a second vector equal to $\boldsymbol{b}$; the vector whose initial point is O and whose terminal point is the terminal point of the

second vector we call *the sum of the vectors* $\boldsymbol{a}$ and $\boldsymbol{b}$ (Fig. 9) and we denote it by

$$\boldsymbol{a} + \boldsymbol{b}.$$

For opposite vectors (Fig. 11) we therefore obtain in particular

$$\boldsymbol{a} + (-\boldsymbol{a}) = 0.$$

The sum of several vectors, for example $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$, is obtained by forming the sum $\boldsymbol{b} + \boldsymbol{c}$ and then adding the result to the vector $\boldsymbol{a}$ (Fig. 10).

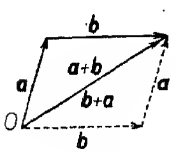

Fig. 9.

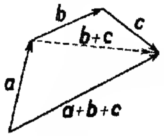

Fig. 10.

Vectors obey the commutative and associative laws of addition. Hence:

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a},$$

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}).$$

From these laws it follows that the sum of any number of vectors remains unaltered if the order of the terms is changed, or if several are enclosed by a parenthesis. For example:

$$\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d} + \boldsymbol{e} = \boldsymbol{a} + \boldsymbol{c} + \boldsymbol{e} + \boldsymbol{b} + \boldsymbol{d} =$$
$$= (\boldsymbol{a} + \boldsymbol{c}) + \boldsymbol{e} + (\boldsymbol{b} + \boldsymbol{d}).$$

The *difference* $\boldsymbol{a} - \boldsymbol{b}$ is defined as the sum $\boldsymbol{a} + (-\boldsymbol{b})$. Therefore from the definition

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b}).$$

Figures 12 and 13 show how to determine the difference.

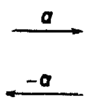

Fig. 11.

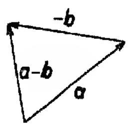

Fig. 12.

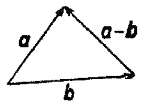

Fig. 13.

Since

$$(\boldsymbol{a} - \boldsymbol{b}) + \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b}) + \boldsymbol{b} = \boldsymbol{a},$$

it follows that *the difference added to the subtrahend gives as a result the minuend.*

§ 4. Product of a vector by a number. *The product of a vector* $\boldsymbol{a}$ *by a number* m is defined as the vector which has the same direction as $\boldsymbol{a}$, a length $|m|$ times that of $\boldsymbol{a}$, and a sense agreeing with or opposite to that

of $\boldsymbol{a}$, depending on whether $m > 0$ or $m < 0$. The product of $\boldsymbol{a}$ by m is denoted by

$$m\boldsymbol{a}.$$

If $m = 0$ or $\boldsymbol{a} = 0$, then $m\boldsymbol{a} = 0$.

We evidently have (Fig. 14 for $m = 2$):

$$(-m)\,\boldsymbol{a} = -m\boldsymbol{a}.$$

Hence it follows that

$$(-1)\,\boldsymbol{a} = -\boldsymbol{a}.$$

For a product it is easy to demonstrate the distributive law for multiplication with respect to addition and the associative law:

$$m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}, \quad (m + p)\,\boldsymbol{a} = m\boldsymbol{a} + p\boldsymbol{a}, \quad m(p\boldsymbol{a}) = (mp)\,\boldsymbol{a},$$

where m and p denote numbers (Figs 15 and 16).

From the above laws follow the usual algebraic rules of addition and multiplication.

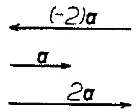

Fig. 14.

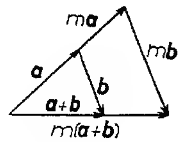

Fig. 15.

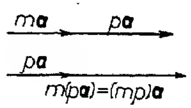

Fig. 16.

Division of a vector by a number (different from zero) is defined as multiplication by the reciprocal of that number. Therefore:

$$\frac{\boldsymbol{a}}{m} = \frac{1}{m}\boldsymbol{a}.$$

§ 5. Components of a sum and product. It is easy to show that the *projection* (on a line or plane) of a *sum* of vectors is equal to the *sum of the projections* of these vectors (Fig. 17). Hence:

$$\text{Proj}\,(\boldsymbol{a} + \boldsymbol{b}) = \text{Proj}\,\boldsymbol{a} + \text{Proj}\,\boldsymbol{b}.$$

Similarly, *the projection of a product* of a vector by a number is equal to *the product of the projection* of the vector by this number (Fig. 18). Therefore:

$$\text{Proj}\,(m\boldsymbol{a}) = m\,\text{Proj}\,\boldsymbol{a}.$$

If the vector $\boldsymbol{a}$ has components a_x, a_y, a_z, and the vector $\boldsymbol{b}$ components b_x, b_y, b_z, then the vector $\boldsymbol{s} = \boldsymbol{a} + \boldsymbol{b}$ has components $s_x = a_x + b_x$, $s_y = a_y + b_y$, $s_z = a_z + b_z$.

This follows from the theorem on the projection of a sum of vectors.

Similarly, from the theorem on the projection of the product of a vector by a number, it follows that the vector $\boldsymbol{c} = m\boldsymbol{a}$ has components

$$c_x = ma_x, \quad c_y = ma_y, \quad c_z = ma_z.$$

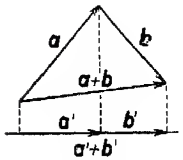

Fig. 17.

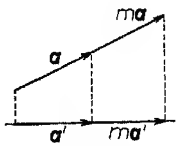

Fig. 18.

For example, if $\boldsymbol{d} = 5\boldsymbol{a} - 3\boldsymbol{b} - 2\boldsymbol{c}$, then:

$$d_x = 5a_x - 3b_x - 2c_x,$$
$$d_y = 5a_y - 3b_y - 2c_y,$$
$$d_z = 5a_z - 3b_z - 2c_z.$$

§ 6. Resolution of a vector. The sum of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ having a common origin, but not lying on the same line, represents the diagonal of a parallelogram with these vectors as sides. Similarly, the sum of the three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ having a common origin, but not lying in the same plane, represents the diagonal of a parallelepiped having these vectors as edges.

The resolution of a given vector into the sum of two or three vectors having given directions is based on the preceding theorems.

Let us suppose that a vector $\boldsymbol{s}$ and two non-parallel lines l and m lying in a certain plane parallel to $\boldsymbol{s}$ are given. If we want to represent the vector $\boldsymbol{s}$ as a sum of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ parallel to l and m, then let us form a parallelogram whose sides are parallel to l and m, and whose diagonal is $\boldsymbol{s}$. For this purpose we draw lines from the initial and terminal points of the vector $\boldsymbol{s}$ parallel to l and m. The sides of the parallelogram obtained will determine the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (Fig. 19).

It is easy to see that such a resolution is possible in only one way.

Similarly, if a vector $\boldsymbol{s}$ and three lines l, m, n not parallel to the same plane are given and we want to represent $\boldsymbol{s}$ as the sum of the three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ parallel to l, m, n, then we form a parallelepiped with edges parallel to l, m, n whose diagonal is $\boldsymbol{s}$. We therefore draw lines l', m', n' from the initial point O of the vector $\boldsymbol{s}$ parallel to l, m, n; then from the terminal point of $\boldsymbol{s}$ we draw a line parallel to n to the point of intersection G of this line with the plane formed by l', m'; finally from the point G we draw parallels to l and m. The points of intersection of these lines with l' and m' are the end points of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ whose initial point is O. Vector $\boldsymbol{c}$ is equal to the vector joining point G with the end of vector $\boldsymbol{s}$ (Fig. 20).

Only one such resolution is possible, since there exists only one parallelepiped having edges parallel to l, m, n, and a diagonal $\boldsymbol{s}$.

A particular case of such a resolution is the representation of a vector by means of unit vectors. We denote the projections of the vector $\boldsymbol{a}$ on the axes of the system (x, y, z) by $\boldsymbol{a}', \boldsymbol{a}'', \boldsymbol{a}'''$. We obviously have (Fig. 21):

$$\boldsymbol{a} = \boldsymbol{a}' + \boldsymbol{a}'' + \boldsymbol{a}'''.$$

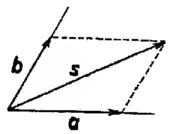

Fig. 19.

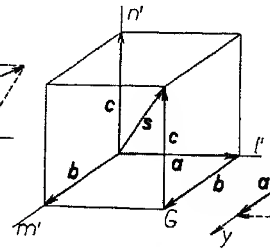

Fig. 20.

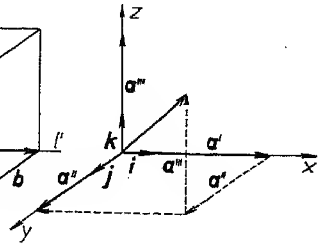

Fig. 21.

On the coordinate axes let us select vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of unit length agreeing in direction with the corresponding axes. From the definition of a_x, a_y, a_z (§ 2, p. 2) it follows that

$$\boldsymbol{a}' = a_x\boldsymbol{i}, \quad \boldsymbol{a}'' = a_y\boldsymbol{j}, \quad \boldsymbol{a}''' = a_z\boldsymbol{k}.$$

Therefore

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}. \tag{I}$$

The vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are called *unit* vectors. Formula (I) expresses the vector $\boldsymbol{a}$ in terms of components and unit vectors.

§ 7. Scalar product. The *scalar product* of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ forming an angle φ (Fig. 22) is defined as the number $|\boldsymbol{a}||\boldsymbol{b}| \cos \varphi$.

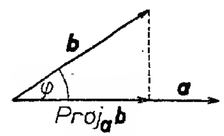

Fig. 22.

We denote the scalar product by $\boldsymbol{a} \cdot \boldsymbol{b}$ or $\boldsymbol{ab}$. Therefore

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \varphi. \tag{I}$$

The scalar product is zero not only when $\boldsymbol{a} = 0$ or $\boldsymbol{b} = 0$, but also when $\boldsymbol{a} \perp \boldsymbol{b}$, because then $\varphi = \pi/2$ and hence $\cos \varphi = 0$. However, if $\boldsymbol{a} \neq 0$ and $\boldsymbol{b} \neq 0$, then the scalar product can be positive or negative depending on whether φ is acute or obtuse.

The *scalar product is commutative* because we have

$$\boldsymbol{b} \cdot \boldsymbol{a} = |\boldsymbol{b}||\boldsymbol{a}| \cos \varphi = |\boldsymbol{a}||\boldsymbol{b}| \cos \varphi = \boldsymbol{a} \cdot \boldsymbol{b}.$$

The expression $|\boldsymbol{b}| \cos \varphi$ represents the projection of the vector $\boldsymbol{b}$ on the axis determined by the vector $\boldsymbol{a}$ and agreeing with it in direction. This

projection is called the *projection of* $\boldsymbol{b}$ *on the direction of* $\boldsymbol{a}$ and is denoted by $\mathrm{Proj}_a \boldsymbol{b}$. Therefore

$$\mathrm{Proj}_a \boldsymbol{b} = |\boldsymbol{b}| \cos \varphi, \quad \mathrm{Proj}_b \boldsymbol{a} = |\boldsymbol{a}| \cos \varphi.$$

Hence by (I)

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \,\mathrm{Proj}_a \boldsymbol{b} = |\boldsymbol{b}| \,\mathrm{Proj}_b \boldsymbol{a}. \tag{1}$$

Therefore: *the scalar product is equal to the product of the length of one vector by the projection of the other on the direction of the first.*

Distributive law. From the definition of a scalar product we have

$$(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = |\boldsymbol{c}| \,\mathrm{Proj}_c(\boldsymbol{a} + \boldsymbol{b}).$$

Since the $\mathrm{Proj}_c(\boldsymbol{a} + \boldsymbol{b}) = \mathrm{Proj}_c \boldsymbol{a} + \mathrm{Proj}_c \boldsymbol{b}$, it follows that

$$(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = |\boldsymbol{c}| \,\mathrm{Proj}_c \boldsymbol{a} + |\boldsymbol{c}| \,\mathrm{Proj}_c \boldsymbol{b}.$$

But $|\boldsymbol{c}| \,\mathrm{Proj}_c \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{c}$ and $|\boldsymbol{c}| \,\mathrm{Proj}_c \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{c}$, therefore

$$(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}. \tag{II}$$

Proceeding similarly, we obtain

$$(\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{b} \cdot \boldsymbol{c}. \tag{III}$$

Hence the distributive law with respect to multiplication holds for sums and differences. From these result the usual laws of multiplying sums by sums.

For example:

$$\begin{aligned}(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{c} + \boldsymbol{d}) &= (\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} + (\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{d} = \\ &= \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{a} \cdot \boldsymbol{d} + \boldsymbol{b} \cdot \boldsymbol{d}.\end{aligned}$$

Associative law. Let m denote any number. Then $(m\boldsymbol{a}) \cdot \boldsymbol{b} = |\boldsymbol{b}| \cdot \mathrm{Proj}_b(m\boldsymbol{a}) = m\boldsymbol{b}\,\mathrm{Proj}_b \boldsymbol{a}$, whence $(m\boldsymbol{a}) \cdot \boldsymbol{b} = m(\boldsymbol{a} \cdot \boldsymbol{b})$.

Now let m and n denote numbers. By the preceding formula $(m\boldsymbol{a}) \cdot (n\boldsymbol{b}) = m\boldsymbol{a} \cdot (n\boldsymbol{b}) = mn(\boldsymbol{a} \cdot \boldsymbol{b})$, hence

$$(m\boldsymbol{a}) \cdot (n\boldsymbol{b}) = (mn)(\boldsymbol{a} \cdot \boldsymbol{b}). \tag{IV}$$

From these follow the usual laws of multiplying a polynomial by a polynomial.

For example:

$$\begin{aligned}(2\boldsymbol{a} - 3\boldsymbol{b}) \cdot 5\boldsymbol{c} &= 10\boldsymbol{a} \cdot \boldsymbol{c} - 15\boldsymbol{b} \cdot \boldsymbol{c}, \\ (4\boldsymbol{a} - 2\boldsymbol{b}) \cdot (3\boldsymbol{c} + \boldsymbol{d}) &= 12\boldsymbol{a} \cdot \boldsymbol{c} - 6\boldsymbol{b} \cdot \boldsymbol{c} + 4\boldsymbol{a} \cdot \boldsymbol{d} - 2\boldsymbol{b} \cdot \boldsymbol{d}.\end{aligned}$$

Square of a vector. The scalar product $\boldsymbol{a} \cdot \boldsymbol{a}$ is denoted by $\boldsymbol{a}^2$. Since $\boldsymbol{a}^2 = |\boldsymbol{a}| \cdot |\boldsymbol{a}| \cos 0$, then $\boldsymbol{a}^2 = |\boldsymbol{a}|^2$ and therefore $\boldsymbol{a} = \sqrt{\boldsymbol{a}^2}$.

Hence we obtain:

$$\begin{aligned}(\boldsymbol{a} + \boldsymbol{b})^2 &= (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a}^2 + 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b}^2, \\ (\boldsymbol{a} - \boldsymbol{b})^2 &= (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{a}^2 - 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b}^2, \\ (\boldsymbol{a} + \boldsymbol{b}) &\cdot (\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{a}^2 - \boldsymbol{b}^2.\end{aligned} \tag{V}$$

The first two formulae can be written in the following form:

$$\begin{aligned}|\boldsymbol{a} + \boldsymbol{b}|^2 &= |\boldsymbol{a}|^2 + 2|\boldsymbol{a}||\boldsymbol{b}| \cos \varphi + |\boldsymbol{b}|^2, \\ |\boldsymbol{a} - \boldsymbol{b}|^2 &= |\boldsymbol{a}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}| \cos \varphi + |\boldsymbol{b}|^2.\end{aligned} \tag{VI}$$

These formulae express the so-called *theorem of Carnot* known from trigonometry.

Analytic representation of a scalar product. Let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ denote unit vectors (p. 7). From the definition of a scalar product we obtain:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1, \quad \boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0. \tag{2}$$

Representing $\boldsymbol{a}$ and $\boldsymbol{b}$ in the form $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$, $\boldsymbol{b} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}$ (p. 7), we can write $\boldsymbol{a} \cdot \boldsymbol{b}$ in the form

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}) \cdot (b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}).$$

Performing the indicated multiplication and using formulae (2) we get

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_xb_x + a_yb_y + a_zb_z. \tag{VII}$$

The above formula enables one to find the scalar product of two vectors when their components are known.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ and therefore

$$a_xb_x + a_yb_y + a_zb_z = 0. \tag{VIII}$$

Conversely, if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other provided they are different from zero. Therefore formula (VIII) represents the *condition of perpendicularity* of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (different from zero).

§ 8. Vector product. The *vector product* of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as the vector $\boldsymbol{c}$ which satisfies the following conditions:

(1) Length. If φ denotes the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \varphi. \tag{I}$$

(2) Direction. The vector $\boldsymbol{c}$ is perpendicular to the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Hence if the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ radiate from the same point, then the vector $\boldsymbol{c}$ is perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$ (Fig. 23).

(3) Sense. The sense of the system of vectors $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ agrees with that of the chosen coordinate system, i. e. the system is left-handed.

We denote the vector product by

$$\boldsymbol{a} \times \boldsymbol{b}.$$

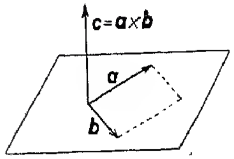

Fig. 23.

From (I) it follows that $|\boldsymbol{c}|$ is zero, if and only if

$$\boldsymbol{a} = 0 \text{ or } \boldsymbol{b} = 0 \text{ or } \varphi = 0 \text{ or } \varphi = \pi.$$

Therefore: *the vector product is zero, if and only if one of its factors is zero, or if the factors are parallel to each other.*

Conditions (1) and (2) are obviously dropped if the vector product is zero. In particular we have

$$\boldsymbol{a} \times \boldsymbol{a} = 0. \tag{II}$$

Remark. The absolute value of the vector product is $|\boldsymbol{a}||\boldsymbol{b}| \sin \varphi$ (formula (I)). This expression represents the area of a parallelogram constructed on vectors correspondingly equal to the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and radiating from one point (Fig. 23).

Change of order of factors. If the order of the factors is altered, then we get the product

$$\boldsymbol{b} \times \boldsymbol{a}.$$

The product $\boldsymbol{a} \times \boldsymbol{b}$ has (by the definition of a vector product) the same length and direction as $\boldsymbol{b} \times \boldsymbol{a}$ but an opposite sense. Hence

$$\boldsymbol{b} \times \boldsymbol{a} = -(\boldsymbol{a} \times \boldsymbol{b}). \tag{III}$$

Therefore: *a change in the order of the factors changes the sign before the vector product.*

Associative law. On the basis of the definition of a vector product it is easy to demonstrate the following relations (where m and n denote numbers):

$$m(\boldsymbol{a} \times \boldsymbol{b}) = (m\boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (m\boldsymbol{b}), \tag{IV}$$

$$(m\boldsymbol{a}) \times (n\boldsymbol{b}) = (mn)(\boldsymbol{a} \times \boldsymbol{b}). \tag{V}$$

For example:

$$3\boldsymbol{a} \times \boldsymbol{b} = 3(\boldsymbol{a} \times \boldsymbol{b}), \quad 2\boldsymbol{a} \times 3\boldsymbol{b} = 6(\boldsymbol{a} \times \boldsymbol{b}).$$

Distributive law with respect to a sum. The following formulae hold for vector products:

$$\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{c} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b}, \quad (\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}. \tag{VI}$$

We shall now derive the first formula. We can obviously suppose that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ have a common origin O.

For the time being let us assume that $|\boldsymbol{c}| = 1$. Pass a plane Π through O perpendicular to $\boldsymbol{c}$. Let

$$\boldsymbol{s} = \boldsymbol{a} + \boldsymbol{b} \tag{1}$$

and denote the projections of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{s}$ on the plane Π by $\boldsymbol{a}'$, $\boldsymbol{b}'$, $\boldsymbol{s}'$ (Fig. 24). We obviously have

$$\boldsymbol{s}' = \boldsymbol{a}' + \boldsymbol{b}'. \tag{2}$$

Let φ denote the angle between $\boldsymbol{c}$ and $\boldsymbol{a}$. Therefore $|\boldsymbol{a}'| = |\boldsymbol{a}| \sin \varphi = |\boldsymbol{a}||\boldsymbol{c}| \sin \varphi$, since we assumed that $|\boldsymbol{c}| = 1$. Hence

$$|\boldsymbol{a}'| = |\boldsymbol{c} \times \boldsymbol{a}| \text{ and similarly } |\boldsymbol{b}'| = |\boldsymbol{c} \times \boldsymbol{b}|, \quad |\boldsymbol{s}'| = |\boldsymbol{c} \times \boldsymbol{s}|. \tag{3}$$

Now rotate $\boldsymbol{a}'$, $\boldsymbol{b}'$, $\boldsymbol{s}'$ through 90° in the plane Π about O from left to right with respect to a person whose feet are at the origin and whose head is at the terminus of $\boldsymbol{c}$. We thus obtain $\boldsymbol{a}''$, $\boldsymbol{b}''$, $\boldsymbol{s}''$. By (2)

$$\boldsymbol{s}'' = \boldsymbol{a}'' + \boldsymbol{b}'', \tag{4}$$

$$|\boldsymbol{a}''| = |\boldsymbol{a}'|, \quad |\boldsymbol{b}''| = |\boldsymbol{b}'|, \quad |\boldsymbol{s}''| = |\boldsymbol{s}'|. \tag{5}$$

The vector $\boldsymbol{a}''$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{c}$; the sense of the system of vectors $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{a}'')$ is left-handed. Moreover, since $|\boldsymbol{a}''| = |\boldsymbol{c} \times \boldsymbol{a}|$ by (3) and (5), it follows $\boldsymbol{a}'' = \boldsymbol{c} \times \boldsymbol{a}$ and similarly $\boldsymbol{b}'' = \boldsymbol{c} \times \boldsymbol{b}$, $\boldsymbol{s}'' = \boldsymbol{c} \times \boldsymbol{s}$. Therefore in virtue of (4) and (1) we obtain

$$\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{c} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b}.$$

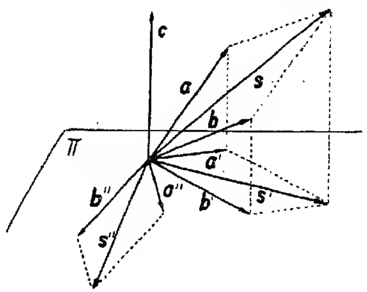

Fig. 24.

We obtained the above relation by assuming that $|\boldsymbol{c}| = 1$. We shall now prove it for the general case. Let $\boldsymbol{h}$ be a unit vector agreeing in direction with $\boldsymbol{c}$. Then

$$|\boldsymbol{h}| = 1 \text{ and } \boldsymbol{c} = |\boldsymbol{c}|\boldsymbol{h}, \tag{6}$$

whence according to the associative law

$$\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b}) = |\boldsymbol{c}|\boldsymbol{h} \times (\boldsymbol{a} + \boldsymbol{b}) = |\boldsymbol{c}|\{\boldsymbol{h} \times (\boldsymbol{a} + \boldsymbol{b})\}. \tag{7}$$

But from the formula proved on the assumption that $|\boldsymbol{c}| = 1$, and from the associative law we have in succession:

$$|\boldsymbol{c}|\{\boldsymbol{h} \times \boldsymbol{a} + \boldsymbol{h} \times \boldsymbol{b}\} = |\boldsymbol{c}|(\boldsymbol{h} \times \boldsymbol{a}) + |\boldsymbol{c}|(\boldsymbol{h} \times \boldsymbol{b}) =$$
$$= (|\boldsymbol{c}|\boldsymbol{h}) \times \boldsymbol{a} + (|\boldsymbol{c}|\boldsymbol{h}) \times \boldsymbol{b}.$$

whence by (6) and (7) we obtain in all generality:

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}.$$

The second of the relations (VI) we can obtain from the first by applying formula (III) as follows

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = -\{\mathbf{c} \times (\mathbf{a} + \mathbf{b})\} = -\{\mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}\} = \\ = -(\mathbf{c} \times \mathbf{a}) - (\mathbf{c} \times \mathbf{b}) = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}.$$

From (VI) follows easily

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{d}. \tag{VII}$$

For example:

$$(2\mathbf{a} - 3\mathbf{b}) \times (5\mathbf{c} + 2\mathbf{d}) = 10\mathbf{a} \times \mathbf{c} + 4\mathbf{a} \times \mathbf{d} - 15\mathbf{b} \times \mathbf{c} - 6\mathbf{b} \times \mathbf{d},$$
$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = \mathbf{a} \times \mathbf{a} - \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b} = -2\mathbf{a} \times \mathbf{b},$$
$$(3\mathbf{a} + 2\mathbf{b}) \times (5\mathbf{a} - 2\mathbf{b}) = -16\mathbf{a} \times \mathbf{b}.$$

Components of a vector product. Denoting unit vectors by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ we have:

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0, \tag{8}$$

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}, \\ \mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}. \tag{9}$$

Setting

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}, \qquad \mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k},$$

we obtain

$$\mathbf{a} \times \mathbf{b} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \times (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}).$$

Performing the multiplication and using (8) and (9) we get:

$$\mathbf{a} \times \mathbf{b} = (a_yb_z - a_zb_y)\,\mathbf{i} + (a_zb_x - a_xb_z)\,\mathbf{j} + (a_xb_y - a_yb_x)\,\mathbf{k}.$$

For $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ we therefore have

$$c_x = a_yb_z - a_zb_y, \quad c_y = a_zb_x - a_xb_z, \quad c_z = a_xb_y - a_yb_x. \tag{VIII}$$

§ 9. Product of several vectors. 1° Let us first consider the product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Setting $\mathbf{r} = \mathbf{b} \times \mathbf{c}$, we obtain

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{r} = a_xr_x + a_yr_y + a_zr_z.$$

Since $r_x = b_yc_z - b_zc_y$ etc.,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_x(b_yc_z - b_zc_y) + a_y(b_zc_x - b_xc_z) + a_z(b_xc_y - c_xb_y).$$

The above formula can be written in the form

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{I}$$

From well-known properties of determinants it follows easily

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \tag{II}$$

Suppose that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ have their initial points at the origin of the coordinate system. From analytic geometry it is known that the volume V of a parallelepiped having edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$, is 1/6 of the determinant (I). Hence $V = \frac{1}{6}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

Therefore: *the necessary and sufficient condition that the vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (*having a common origin*) *be in the same plane is that* $V = 0$, *or that* $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

However, if we do not assume that $\mathbf{a}, \mathbf{b}, \mathbf{c}$, have a common origin, then — as is quite evident — the condition $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ *is the necessary and sufficient condition that the vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be parallel to the same plane.*

2° Let us now consider the product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Let us denote this product by $\mathbf{u}$ and set $\mathbf{r} = \mathbf{b} \times \mathbf{c}$. Then

$$u_x = a_yr_z - a_zr_y = a_y(b_xc_y - b_yc_x) - a_z(b_zc_x - b_xc_z).$$

Adding and subtracting $a_xb_xc_x$ we obtain

$$u_x = b_x(a_xc_x + a_yc_y + a_zc_z) - c_x(a_xb_x + a_yb_y + a_zb_z);$$

hence $u_x = b_x(\mathbf{a} \cdot \mathbf{c}) - c_x(\mathbf{a} \cdot \mathbf{b})$ and similarly $u_y = b_y(\mathbf{a} \cdot \mathbf{c}) - c_y(\mathbf{a} \cdot \mathbf{b})$, $u_z = b_z(\mathbf{a} \cdot \mathbf{c}) - c_z(\mathbf{a} \cdot \mathbf{b})$. Therefore

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} \cdot (\mathbf{a} \cdot \mathbf{b}). \tag{III}$$

3° From (I), (II) and (III) follow the relations:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\ (\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\ (\mathbf{b} \cdot \mathbf{c}), \tag{IV}$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{b}[\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})] - \mathbf{a}[\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})]. \tag{V}$$

§ 10. Vector functions. If to each number t in the interval (t', t'') there corresponds a vector $\mathbf{w}$, then we say that a *vector function* is defined in the interval (t', t'') and we write

$$\mathbf{w} = \mathbf{F}(t). \tag{1}$$

The components w_x, w_y, w_z are also functions (i. e. scalar functions) of the variable t. Therefore:

$$w_x = f(t), \quad w_y = \varphi(t), \quad w_z = \psi(t). \tag{2}$$

The three preceding functions define the vector function (1) precisely.

Limit. The vector function (1) is said to have *the limit* $\mathbf{w}_0$ *as* t *tends to* t_0, and we write

$$\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{w}_0,$$

when

$$\lim_{t\to t_0} f(t) = w_{0,x}, \quad \lim_{t\to t_0} \varphi(t) = w_{0,y}, \quad \text{and} \quad \lim_{t\to t_0} \psi(t) = w_{0,z}.$$

Continuity. The vector function (1) is *continuous at* t_0, if $\lim\limits_{t\to t_0} \mathbf{F}(t) = \mathbf{w}_0$, where $\mathbf{w}_0 = \mathbf{F}(t_0)$.

The following relations obviously hold:

$$\lim_{t\to t_0} f(t) = f(t_0), \quad \lim_{t\to t_0} \varphi(t) = \varphi(t_0), \quad \lim_{t\to t_0} \psi(t) = \psi(t_0).$$

The functions f, φ, ψ are therefore continuous at $t = t_0$. Conversely, if f, φ, ψ are continuous at t_0, then $\mathbf{w} = \mathbf{F}(t)$ is also continuous at t_0.

Derivative. Let Δt denote the increment of the variable t, and $\Delta \mathbf{w}$ the corresponding increment of the vector $\mathbf{w}$. Then $\mathbf{w} + \Delta\mathbf{w} = \mathbf{F}(t + \Delta t)$, and $\Delta\mathbf{w} = \mathbf{F}(t + \Delta t) - \mathbf{F}(t)$, whence

$$\frac{\Delta \mathbf{w}}{\Delta t} = \frac{\mathbf{F}(t+\Delta t) - \mathbf{F}(t)}{\Delta t}.$$

The limit $\lim\limits_{\Delta t\to 0} \dfrac{\Delta\mathbf{w}}{\Delta t}$ is called the *derivative* of the function $\mathbf{F}(t)$ at the point t.

We denote the derivative by $\dfrac{\mathrm{d}\mathbf{w}}{\mathrm{d}t}$, $\mathbf{w}'$ or $\mathbf{F}'(t)$.

Since $\Delta\mathbf{w}$ has components

$$\Delta w_x = f(t+\Delta t) - f(t), \quad \Delta w_y = \varphi(t+\Delta t) - \varphi(t),$$
$$\Delta w_z = \psi(t+\Delta t) - \psi(t),$$

it follows that

$$w'_x = f'(t), \quad w'_y = \varphi'(t), \quad w'_z = \psi'(t).$$

Higher ordered derivatives are defined in the usual manner: the second derivative as the derivative of the first derivative, the third derivative as the derivative of the second derivative etc. We denote higher ordered derivatives by

$$\frac{\mathrm{d}^2\mathbf{w}}{\mathrm{d}t^2}, \ \frac{\mathrm{d}^3\mathbf{w}}{\mathrm{d}t^3}, \ \ldots \ \text{or} \ \mathbf{w}'', \mathbf{w}''', \ldots \ \text{etc.}$$

It is easy to show that

$$w''_x = f''(t), \quad w''_y = \varphi''(t), \quad w''_z = \psi''(t) \ \text{etc.}$$

If the functions $\mathbf{w} = \mathbf{F}(t)$ and $\mathbf{v} = \mathbf{\Phi}(t)$ possess derivatives, then the following relations obtain:

$$\frac{\mathrm{d}(\mathbf{w}\pm\mathbf{v})}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} \pm \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \tag{I}$$

$$\frac{\mathrm{d}(m\mathbf{w})}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} \quad \text{(where } m \text{ is a number)}, \tag{II}$$

$$\frac{\mathrm{d}(\mathbf{w}\cdot\mathbf{v})}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t}\cdot\mathbf{v} + \mathbf{w}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \tag{III}$$

$$\frac{\mathrm{d}(\mathbf{w}\times\mathbf{v})}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t}\times\mathbf{v} + \mathbf{w}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}. \tag{IV}$$

We shall demonstrate for instance relation (III). We have $\Delta(\mathbf{w}\cdot\mathbf{v}) = (\mathbf{w} + \Delta\mathbf{w})\cdot(\mathbf{v} + \Delta\mathbf{v}) - \mathbf{w}\cdot\mathbf{v}$; hence

$$\frac{\Delta(\mathbf{w}\cdot\mathbf{v})}{\Delta t} = \frac{\Delta\mathbf{w}}{\Delta t}\cdot\mathbf{v} + \mathbf{w}\cdot\frac{\Delta\mathbf{v}}{\Delta t} + \Delta\mathbf{w}\cdot\frac{\Delta\mathbf{v}}{\Delta t},$$

whence, upon passing to the limit we obtain (III).

Vector functions of many variables. We can also consider vector functions of many variables. For instance, the vector function

$$\mathbf{w} = \mathbf{F}(\xi, \eta, \zeta)$$

is a function of the three variables ξ, η, ζ. The projections of $\mathbf{w}$ are then defined by certain functions

$$w_x = f(\xi,\eta,\zeta), \quad w_y = \varphi(\xi,\eta,\zeta), \quad w_z = \psi(\xi,\eta,\zeta).$$

The limit, continuity and partial derivatives of vector functions of several variables can easily be given by analogy with the case for one variable.

§ 11. Moment of a vector.

Moment of a vector with respect to a point. Let us suppose that a vector $\overline{AB}$ and a point O are given. *The moment of the vector* $\overline{AB}$ *with respect to the point* O is defined as the vector $\mathbf{M}$ satisfying the following conditions:

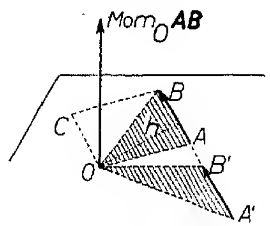

Fig. 25.

(1) $|\mathbf{M}|$ is equal to twice the area of the triangle OAB or

$$|\mathbf{M}| = |\overline{AB}|\cdot h,$$

where h denotes the distance of the point O from $\overline{AB}$.

(2) The direction of the vector $\boldsymbol{M}$ is perpendicular to the plane passing through O and $\overline{AB}$.

(3) The system of vectors $(\overline{AB}, \overline{OA}, \boldsymbol{M})$ has a sense agreeing with that of the coordinate system, i. e. a left sense.

We shall denote the moment of a vector $\overline{AB}$ with respect to the point O by the symbol

$$\mathrm{Mom}_o\overline{AB}.$$

The moment is zero only in the case when $\overline{AB} = 0$ or when the prolongation of the vector $\overline{AB}$ passes through O. If the moment is zero then conditions (2) and (3) are dropped.

For equipollent vectors we can establish the following

Theorem 1. *Equipollent vectors have equal moments with respect to the same point.*

Proof. By hypothesis $\overline{AB} \equiv \overline{A'B'}$. Therefore $\overline{AB}$ and $\overline{A'B'}$ are equal and lie on the same line. It is easy to verify that the moments of both vectors with respect to O have the same direction and sense. They also have the same length because triangles OAB and $OA'B'$ have equal areas (equal bases and a common altitude). Hence $\mathrm{Mom}_o\overline{AB} = \mathrm{Mom}_o\overline{A'B'}$, q. e. d.

Moment as a vector product. Let us consider the vector product $\overline{AB} \times \overline{OA}$. Let us note that the preceding product has the same direction and sense as the $\mathrm{Mom}_o\overline{AB}$. We also have $|\overline{AB} \times \overline{OA}| = |\mathrm{Mom}_o\overline{AB}|$ because the absolute value of the vector is equal to the area of the parallelogram $OABC$ (Fig. 25) and hence to twice the area of the triangle ABC. Therefore

$$\mathrm{Mom}_o\overline{AB} = \overline{AB} \times \overline{OA}.$$

Had we taken the equipollent vector $\overline{A'B'}$ instead of the vector $\overline{AB}$, then we would have

$$\mathrm{Mom}_o\overline{A'B'} = \overline{A'B'} \times \overline{OA'}.$$

By the preceding theorem

$$\mathrm{Mom}_o\overline{AB} = \overline{A'B'} \times \overline{OA'} = \overline{AB} \times \overline{OA}.$$

Therefore: *if A' is an arbitrary point of the line on which the vector $\overline{AB}$ lies, then*

$$\mathrm{Mom}_o\overline{AB} = \overline{AB} \times \overline{OA'}.$$

Theorem 2. *If two equal vectors have equal moments with respect to a point, then they are equipollent.*

Proof. By hypothesis $\overline{AB} = \overline{A'B'}$ and $\mathrm{Mom}_o\overline{AB} = \mathrm{Mom}_o\overline{A'B'}$. Therefore $\overline{AB} \times \overline{OA} = \overline{A'B'} \times \overline{OA'}$, hence $\overline{AB} \times \overline{OA} = \overline{AB} \times \overline{OA'}$ and therefore $\overline{AB} \times (\overline{OA} - \overline{OA'}) = 0$. Moreover, since $\overline{OA} - \overline{OA'} = \overline{A'A}$, it follows

$$\overline{AB} \times \overline{A'A} = 0.$$

But $\overline{AB} \times \overline{A'A} = \mathrm{Mom}_{A'}\overline{AB}$; hence

$$\mathrm{Mom}_{A'}\overline{AB} = 0.$$

It follows from this that the point A' lies on the prolongation of the vector $\overline{AB}$. Since, in addition, $\overline{AB}$ is parallel to $\overline{A'B'}$, then vectors $\overline{AB}$ and $\overline{A'B'}$ lie on the same line.

Moment of a sum of vectors having a common origin. Let us assume that $\overline{AB}$ and $\overline{AC}$ (both having initial points at A) are given. Let $\overline{AD}$ be their sum. We have

$$\mathrm{Mom}_o\overline{AD} = \overline{AD} \times \overline{OA} = (\overline{AB} + \overline{AC}) \times \overline{OA};$$

consequently

$$\mathrm{Mom}_o\overline{AD} = \overline{AB} \times \overline{OA} + \overline{AC} \times \overline{OA},$$

and therefore

$$\mathrm{Mom}_o\overline{AD} = \mathrm{Mom}_o\overline{AB} + \mathrm{Mom}_o\overline{AC}.$$

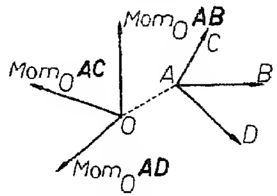

Fig. 26.

We obtain a similar formula for the sum of several vectors. Hence: *the sum of the moments of several vectors having a common origin is equal to the moment of their sum having the same origin.*

Components of a moment. The position of a vector $\boldsymbol{a}$ is defined if ist projections and the coordinates x, y, z of an arbitrary point A of the line l on which the vector $\boldsymbol{a}$ lies are given.

Let x_0, y_0, z_0 be the coordinates of the point O. We have

$$\mathrm{Mom}_o\boldsymbol{a} = \boldsymbol{a} \times \overline{OA}.$$

The projections of the vector $\overline{OA}$ are $x - x_0$, $y - y_0$, $z - z_0$. Hence, denoting the moment with respect to O by $\boldsymbol{M}$, we obtain:

$$M_x = a_y(z - z_0) - a_z(y - y_0), \quad M_y = a_z(x - x_0) - a_x(z - z_0), \quad M_z = a_x(y - y_0) - a_y(x - x_0). \tag{I}$$

If, in particular, the origin of the coordinate system is O, then $x_0 = 0$, $y_0 = 0$, $z_0 = 0$, and therefore

$$M_x = a_y z - a_z y, \quad M_y = a_z x - a_x z, \quad M_z = a_x y - a_y x. \qquad \text{(II)}$$

Suppose that $\boldsymbol{a} \equiv \boldsymbol{a}'$. Then, denoting the moments of the vectors with respect to an arbitrary point by $\boldsymbol{M}$ and $\boldsymbol{M}'$, we have $\boldsymbol{a} = \boldsymbol{a}'$, $\boldsymbol{M} = \boldsymbol{M}'$ or

$$a_x = a'_x, \quad a_y = a'_y, \quad a_z = a'_z, \quad M_x = M'_x, \quad M_y = M'_y, \quad M_z = M'_z.$$

Conversely, if the above relations hold, then $\boldsymbol{a} = \boldsymbol{a}'$, $\boldsymbol{M} = \boldsymbol{M}'$ and therefore by theorem 2, p. 16, the vectors $\boldsymbol{a}$ and $\boldsymbol{a}'$ are equipollent.

Therefore: *the projections of the vector* $\boldsymbol{a}$ *and the projections of the moment* $\boldsymbol{M}$ *with respect to an arbitrary point determine the length, direction, sense, and position of* $\boldsymbol{a}$.

Moment of a vector with respect to a line. Let the vector $\boldsymbol{a}$ and the line l be given. Through an arbitrary point O on the line l pass a plane Π perpendicular to l. Form the projection $\boldsymbol{a}'$ of the vector $\boldsymbol{a}$ on the plane Π.

The moment of the vector $\boldsymbol{a}'$ with respect to O is called as *the moment of the vector* $\boldsymbol{a}$ *with respect to* l and it is denoted by the symbol

$$\mathrm{Mom}_l \boldsymbol{a}.$$

Obviously $\mathrm{Mom}_l \boldsymbol{a}$ does not depend on the choice of point O.

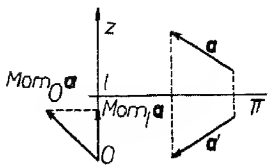

Fig. 27.

$\mathrm{Mom}_l \boldsymbol{a}$ is zero only in the following cases:

1° when $\boldsymbol{a} = 0$,
2° when $\boldsymbol{a} \parallel l$, because then $\boldsymbol{a}' = 0$,
3° when $\boldsymbol{a}$ produced cuts l, because then $\boldsymbol{a}'$ produced passes through O.

If d denotes the distance of $\boldsymbol{a}$ from l and φ the angle between $\boldsymbol{a}$ and l, then it is easy to show that

$$|\mathrm{Mom}_l \boldsymbol{a}| = d|\boldsymbol{a}| \sin\varphi. \qquad \text{(III)}$$

Let us choose the line l as the z-axis and the plane Π as the xy-plane. Let $\boldsymbol{M} = \mathrm{Mom}_O \boldsymbol{a}$ and $\boldsymbol{L} = \mathrm{Mom}_l \boldsymbol{a}$. Since $\boldsymbol{a}'$ has the projections $a'_x = a_x$, $a'_y = a_y$, $a'_z = 0$, then $L_x = 0$, $L_y = 0$ and $L_z = a_x y - a_y x$, where x, y, z are the coordinates of the initial point of $\boldsymbol{a}$. We then see that $M_z = L_z$.

Therefore: $\mathrm{Mom}_l \boldsymbol{a}$ *is the projection on the line* l *of the moment of the vector* $\boldsymbol{a}$ *with respect to an arbitrary point of this line.*

II. SYSTEMS OF VECTORS

§ 12. Total moment of a system of vectors. Let

$$\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$$

be a given system of vectors. Let us denote *the sum of the system* (i. e. the sum of the vectors of the system) by $\boldsymbol{s}$. Thus

$$\boldsymbol{s} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \ldots + \boldsymbol{a}_n.$$

Choose an arbitrary point O.

The *total moment* or briefly *the moment of the system with respect to* O is defined as the sum of the moments of the separate vectors with respect to O. We shall denote it by

$$\boldsymbol{M}_O.$$

We therefore have

$$\boldsymbol{M}_O = \mathrm{Mom}_O \boldsymbol{a}_1 + \mathrm{Mom}_O \boldsymbol{a}_2 + \ldots + \mathrm{Mom}_O \boldsymbol{a}_n.$$

The total moment we sometimes also denote by

$$\mathrm{Mom}_O(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n).$$

Let us select another point O'.

We have

$$\boldsymbol{M}_{O'} = \mathrm{Mom}_{O'} \boldsymbol{a}_1 + \mathrm{Mom}_{O'} \boldsymbol{a}_2 + \ldots + \mathrm{Mom}_{O'} \boldsymbol{a}_n.$$

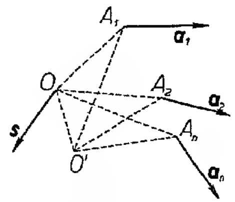

Fig. 28

Since $\mathrm{Mom}_O \boldsymbol{a}_1 = \boldsymbol{a}_1 \times \overline{O'A}_1$, where A_1 is the initial point of $\boldsymbol{a}_1$ etc., it follows that

$$\boldsymbol{M}_{O'} = \boldsymbol{a}_1 \times \overline{O'A} + \boldsymbol{a}_2 \times \overline{O'A}_2 + \ldots$$

But $\overline{O'A}_1 = \overline{O'O} + \overline{OA}_1$ etc. hence

$$\boldsymbol{M}_{O'} = \boldsymbol{a}_1 \times (\overline{O'O} + \overline{OA}_1) + \boldsymbol{a}_2 \times (\overline{O'O} + \overline{OA}_2) + \ldots$$

After performing the multiplication we obtain:

$$\boldsymbol{M}_{O'} = (\boldsymbol{a}_1 \times \overline{O'O} + \boldsymbol{a}_2 \times \overline{O'O} + \ldots) + (\boldsymbol{a}_1 \times \overline{OA}_1 + \boldsymbol{a}_2 \times \overline{OA}_2 + \ldots). \quad (1)$$

But $\boldsymbol{a}_1 \times \overline{O'O} + \boldsymbol{a}_2 \times \overline{O'O} + \ldots = (\boldsymbol{a}_1 + \boldsymbol{a}_2 + \ldots) \times \overline{O'O} = \boldsymbol{s} \times \overline{O'O}$.

The sum enclosed in the second parenthesis of (1) represents the moment of the system with respect to O. Therefore

$$\boldsymbol{M}_{O'} = \boldsymbol{s} \times \overline{O'O} + \boldsymbol{M}_O. \qquad \text{(I)}$$

The product $\boldsymbol{s} \times \overline{O'O}$ is the moment with respect to O' of the sum of the system of vectors with initial point at O.

Hence: *if we change the point with respect to which we find the total moment of the system, then this moment changes by the moment of the sum of the system whose initial point is at the old point taken with respect to the new point.*

The following corollaries are consequences of the preceding theorem:

1. *If the sum of the system is zero, then the total moment is constant* (i. e. it does not depend on the point with respect to which it is determined).

Because if $\boldsymbol{s} = 0$, then $\boldsymbol{s} \times \overline{O'O} = 0$, and hence $\boldsymbol{M}_{o'} = \boldsymbol{M}_o$.

2. *If the total moments with respect to three non-collinear points are equal, then the sum of the system of vectors is zero.*

For let us assume that the total moments with respect to the non-collinear points O, O', O'' are equal. Then $\boldsymbol{M}_o = \boldsymbol{M}_{o'} = \boldsymbol{M}_{o''}$, whence $\boldsymbol{s} \times \overline{O'O} = 0$ and $\boldsymbol{s} \times \overline{O''O} = 0$. Hence, if $\boldsymbol{s} \neq 0$, then $\boldsymbol{s} \parallel OO'$ and $\boldsymbol{s} \parallel O''O$, which is impossible when O, O', O'', are non-collinear.

3. *If the point with respect to which the total moment is determined is moved along a line parallel to the sum of the system, then the moment does not undergo a change.*

For if $\boldsymbol{s} \parallel \overline{O'O}$, then $\boldsymbol{s} \times \overline{O'O} = 0$ and hence $\boldsymbol{M}_{o'} = \boldsymbol{M}_o$.

4. *The scalar product of the total moment by the sum of the system is constant* (i. e. it is independent of the point with respect to which it is determined).

For let us multiply both sides of (I) scalarly by $\boldsymbol{s}$. We obtain $\boldsymbol{s} \cdot \boldsymbol{M}_{o'} = \boldsymbol{s} \cdot (\boldsymbol{s} \times \overline{O'O}) + \boldsymbol{s} \cdot \boldsymbol{M}_o$, but $\boldsymbol{s} \times \overline{O'O} \perp \boldsymbol{s}$; therefore $\boldsymbol{s} \cdot (\boldsymbol{s} \times \overline{O'O}) = 0$, whence

$$\boldsymbol{s} \cdot \boldsymbol{M}_{o'} = \boldsymbol{s} \cdot \boldsymbol{M}_o .$$

The scalar product of the total moment by the sum is called the *parameter* of the system.

5. *The projection of the moment on the direction of the sum is constant in magnitude* (under the assumption that the sum is different from zero).

For by corollary 4 and the definition of a scalar product we have $|\boldsymbol{s}| \operatorname{Proj}_{\boldsymbol{s}} \boldsymbol{M}_{o'} = |\boldsymbol{s}| \operatorname{Proj}_{\boldsymbol{s}} \boldsymbol{M}_o$, whence

$$\operatorname{Proj}_{\boldsymbol{s}} \boldsymbol{M}_{o'} = \operatorname{Proj}_{\boldsymbol{s}} \boldsymbol{M}_o .$$

§ 13. Parameter. We shall presently determine the parameter (i. e. the scalar product of the total moment by the sum) for certain systems appearing frequently in mechanics.

A *central* system is one in which the prolongations of the separate vectors all pass through a fixed point O called the *centre* (Fig. 29).

The moment of the system with respect to the centre is zero, because the moment of each vector is zero. Therefore the parameter is zero.

Hence: *the parameter of a central system is zero.*

A *plane* system is one in which every vector lies in the same plane Π (Fig. 30).

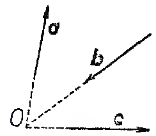

Fig. 29.

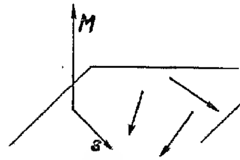

Fig. 30.

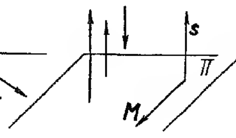

Fig. 31.

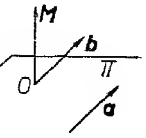

Fig. 32.

The total moment of the system with respect to an arbitrary point O in the plane Π is perpendicular to Π, because the moments of the individual vectors with respect to O are perpendicular to Π. Since the sum lies in the plane Π, then the sum is perpendicular to the total moment. It follows that the parameter is zero.

Therefore: *the parameter of a plane system is zero.*

A *parallel* system is one in which all the vectors are parallel (Fig. 31).

If the sum $\boldsymbol{s}$ is zero, then the parameter is obviously zero. Let us assume then, that $\boldsymbol{s} \neq 0$. Let O be an arbitrary point. The moments of the separate vectors with respect to O lie in the plane perpendicular to the vectors of the system and passing through O. Therefore the total moment also lies in the plane Π. Since $\boldsymbol{s} \perp \Pi$, then $\boldsymbol{s}$ is perpendicular to the total moment and consequently the parameter is zero.

Hence: *the parameter of a plane system is zero.*

Let us now assume that vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are *skew* (i. e. do not lie in the same plane). Let O be the initial point of the vector $\boldsymbol{b}$ (Fig. 32).

The moment $\boldsymbol{M}$ of the system $(\boldsymbol{a}, \boldsymbol{b})$ with respect to O is obviously equal to $\operatorname{Mom}_o \boldsymbol{a}$. The parameter $K = \boldsymbol{M} \cdot \boldsymbol{s} = \boldsymbol{M} \cdot (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{M} \cdot \boldsymbol{a} + \boldsymbol{M} \cdot \boldsymbol{b}$. But $\boldsymbol{M} = \operatorname{Mom}_o \boldsymbol{a}$ is perpendicular to the plane Π which passes through O and the vector $\boldsymbol{a}$. Since $\boldsymbol{a}$ lies in Π and $\boldsymbol{b}$ does not, the moment $\boldsymbol{M}$ is perpendicular to $\boldsymbol{a}$, but not to $\boldsymbol{b}$, and consequently from the last equality $K = \boldsymbol{M} \cdot \boldsymbol{b} \neq 0$.

Therefore: *the parameter of a system consisting of two skew vectors is different from zero.*

§ 14. Equipollent systems. Two systems of vectors $(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots)$ and $(\boldsymbol{a}_1', \boldsymbol{a}_2', \ldots)$ are said to be *equipollent* if they have equal sums and equal total moments with respect to every point.

If we have a system ($\boldsymbol{a}$) consisting of only one vector $\boldsymbol{a}$ and a system ($\boldsymbol{a}'$) consisting of only one vector $\boldsymbol{a}'$, then — as follows from theorem 2, p. 16 — the necessary and sufficient condition that systems ($\boldsymbol{a}$) and ($\boldsymbol{a}'$) be equipollent is that $\boldsymbol{a} \equiv \boldsymbol{a}'$. Therefore, in this case, the notion of equipollence of systems coincides with the notion of equipollence of vectors.

In the general case we have the following theorems:

1. *If two systems have equal sums and equal total moments with respect to a certain point, then these systems are equipollent.*

This follows from formula (I), p. 19. For if the moments with respect to the point O are equal and the sums are equal, then the moments with respect to every point O' will be equal, since in replacing point O by O' they undergo equal changes in both systems.

2. *If two systems have equal moments with respect to three non-collinear points, then these systems are equipollent.*

Because if we denote the points with respect to which the total moments of both systems are equal by O_1, O_2, O_3 and the sums of these systems by $\boldsymbol{s}$ and $\boldsymbol{s}'$, then from fromula (I), p. 19, we shall obtain $\boldsymbol{s} \times \overline{O_1O_2} = \boldsymbol{s}' \times \overline{O_1O_2}$ and $\boldsymbol{s} \times \overline{O_1O_3} = \boldsymbol{s}' \times \overline{O_1O_3}$, whence

$$(\boldsymbol{s} - \boldsymbol{s}') \times \overline{O_1O_2} = 0 \quad \text{and} \quad (\boldsymbol{s} - \boldsymbol{s}') \times \overline{O_1O_3} = 0.$$

Were $\boldsymbol{s} - \boldsymbol{s}' \neq 0$, then we should have $\boldsymbol{s} - \boldsymbol{s}' \parallel \overline{O_1O_2}$ and $\boldsymbol{s} - \boldsymbol{s}' \parallel \overline{O_1O_3}$, which is impossible because O_1, O_2, O_3 are non-collinear. Hence $\boldsymbol{s} - \boldsymbol{s}' = 0$, or $\boldsymbol{s} = \boldsymbol{s}'$, whence, by the preceding theorem, the equipollence of the systems follows.

That *equipollent systems have equal parameters* is an immediate consequence of the definition of a parameter.

The converse of this statement is obviously false.

Systems equipollent to zero. If the sum of a system is zero, then — as we know — the total moment is constant. If the sum of the system is zero and the total moment is zero, then the system is said to be a *system equipollent to zero.*

A system equipollent to zero is equipollent to a zero vector.

In order to ascertain whether a system is equipollent to zero it is sufficient to see whether its sum and moment with respect to some arbitrary point are equal to zero.

It follows easily from theorem 2, p. 20, that *a system is equipollent to zero if the total moment with respect to three non-collinear points is zero.*

System of three vectors equipollent to zero. *If a system consisting of three vectors is equipollent to zero, then the prolongations of these vectors pass through one point (or the vectors are parallel).*

Let us suppose that the system of vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is equipollent to zero. The total moment with respect to A (the initial point of $\boldsymbol{a}$) is therefore zero, whence $\mathrm{Mom}_A\boldsymbol{b} + \mathrm{Mom}_A\boldsymbol{c} = 0$, and hence $\mathrm{Mom}_A\boldsymbol{b} = -\mathrm{Mom}_A\boldsymbol{c}$. From this it follows that the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ lie in the plane Π passing through A. Since $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = 0$, then $\boldsymbol{a} = -\boldsymbol{b} - \boldsymbol{c}$, and therefore $\boldsymbol{a}$ also lies in the plane Π. Let O denote the point of intersection of $\boldsymbol{a}$ and $\boldsymbol{b}$. Since the total moment of the system with respect to O is reduced to the moment of the vector $\boldsymbol{c}$ with respect to O, $\mathrm{Mom}_0\boldsymbol{c} = 0$ and hence $\boldsymbol{c}$ also passes through O. Finally, if $\boldsymbol{a} \parallel \boldsymbol{b}$, then $\boldsymbol{a} \parallel \boldsymbol{c}$ also, because $\boldsymbol{c} = -\boldsymbol{a} - \boldsymbol{b}$ (Fig. 29).

§ 15. Vector couple. A *vector couple* is a system consisting of two parallel vectors $\boldsymbol{a}$ and $-\boldsymbol{a}$ oppositely directed and of equal length.

Since the sum of a vector couple is zero, the moment of the couple is constant. Computing it with respect to the initial point of $\boldsymbol{a}$, we see that the moment of the vector $\boldsymbol{a}$ is zero, but the moment of the vector $-\boldsymbol{a}$ is perpendicular to the plane of the couple and equal in magnitude to the area of the parallelogram constructed on the vectors forming the couple.

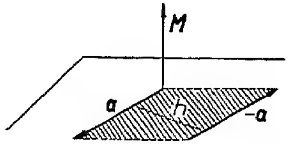

Fig. 33.

Therefore: *the moment of a couple is perpendicular to the plane of the couple and in magnitude equal to the area of the parallelogram constructed on the vectors of the couple.*

If the vectors of the couple lie on the same straight line, then obviously the moment is zero.

If h denotes the distance between vectors $\boldsymbol{a}$ and $-\boldsymbol{a}$ and $\boldsymbol{M}$ the moment of the couple, then

$$|\boldsymbol{M}| = |\boldsymbol{a}| \cdot h. \tag{1}$$

Corresponding to a given vector $\boldsymbol{M}$ there can always be found a couple whose moment is equal to $\boldsymbol{M}$. On the plane perpendicular to $\boldsymbol{M}$ it is sufficient to select a parallelogram whose area is equal to $|\boldsymbol{M}|$. The opposite sides, suitably directed, form the sought for couple. Clearly, the problem can be solved in an infinite number of ways.

Two couples whose moments are equal form an equipollent system. Hence, if a couple is arbitrarily translated or rotated in the plane of the couple, then an equipollent couple is obtained.

§ 16. Reduction of a system of vectors. Let a system S consisting of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ be given. We shall consider the problem of determining the simplest system equipollent to S.

Let O be an arbitrary point. Denote the sum of the system S by $\boldsymbol{s}$ and the total moment with respect to O by $\boldsymbol{M}$. Let us consider the system R consisting of the couple $(\boldsymbol{a}, -\boldsymbol{a})$ whose moment equals $\boldsymbol{M}$ and the vector $\boldsymbol{s}$ with initial point at O. Systems R and S are obviously equipollent because they have equal sums $\boldsymbol{s}$ and equal moments $\boldsymbol{M}$ with respect to O.

Therefore: *every system of vectors is equipollent to a system consisting of a sum with initial point at an arbitrary point O and a couple whose moment is equal to the moment of the system with respect to O.*

The latter is the so-called *reduction theorem*. The point O is called the *centre of reduction*.

The couple $(\boldsymbol{a}, -\boldsymbol{a})$ can be chosen so that the point O is the initial point of $-\boldsymbol{a}$. Let us replace the vectors $\boldsymbol{s}$ and $-\boldsymbol{a}$ by their sum $\boldsymbol{b}$ whose initial point is at O (Fig. 34). The system consisting of $\boldsymbol{a}$ and $\boldsymbol{b}$ is obviously equipollent to system S.

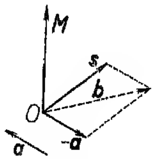

Fig. 34.

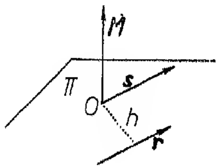

Fig. 35.

Hence: *every system of vectors is equipollent to a system of two vectors one of which has its origin at an arbitrary point.*

Every system of vectors is therefore equipollent to a certain system composed of a vector and a couple or two vectors. Let us now consider conditions under which a given system is equipollent to only one vector or one couple.

Let us examine in succession cases in which the parameter is different from zero and equal to zero.

1° Parameter different from zero. A system consisting of one vector or one couple is a plane system and hence its parameter $K = 0$. Therefore, if the parameter of the system S is different from zero, then the system S cannot be equipollent to one vector or one couple because equipollent systems have equal parameters.

Let us assume now that the system S whose parameter $K \neq 0$ is equipollent to the system R consisting of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The parameter R of the system is therefore also different from zero. It follows from this that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ cannot lie in one plane and are therefore skew (*vide* § 13, p. 21).

Hence: *if the parameter of a system is different from zero, then the system is equipollent to a system of two skew vectors.*

2° Parameter equal to zero, sum different from zero. Let us suppose that the parameter K of the system S is zero but the sum $\boldsymbol{s} \neq 0$. Select an arbitrary point O and denote the moment of the system S with respect to O by $\boldsymbol{M}$. Since $K = \boldsymbol{M} \cdot \boldsymbol{s} = 0$, then $\boldsymbol{M} \perp \boldsymbol{s}$. Pass a plane Π through O and perpendicular to $\boldsymbol{M}$ (Fig. 35). On Π we can choose a vector $\boldsymbol{r}$ equal to the vector $\boldsymbol{s}$ and such that $\mathrm{Mom}_o \boldsymbol{r} = \boldsymbol{M}$. The distance h from $\boldsymbol{r}$ to O is obtained from $|\boldsymbol{M}| = h|\boldsymbol{r}|$. It is easy to see that system S is equipollent to the vector $\boldsymbol{r}$.

Therefore: *if the parameter of a system is equal to zero but the sum is different from zero, then the system is equipollent to one vector.*

The vector $\boldsymbol{r}$, to which the entire system S is equipollent, is called the *resultant vector* or briefly the *resultant* of the system S.

The sum is not to be confused with the resultant. The sum has only a definite length, direction and sense; the resultant has in addition a definite position, i. e. the line on which it lies.

3° Parameter and sum equal to zero. Finally, let us assume that the parameter as well as the sum of the system are equal to zero. From the reduction theorem it follows that the system is equipollent to a couple. Since the sum is zero, the total moment $\boldsymbol{M}$ is constant.

If $\boldsymbol{M} \neq 0$, then the couple is the simplest system equipollent to the given one. If $\boldsymbol{M} = 0$, and as by the hypothesis the sum is equal to zero then the system is equipollent to zero, i. e. to a zero vector.

Therefore: *a system whose parameter and sum are equal to zero is equipollent to a couple of vectors or to a zero vector, depending on whether the total moment is different from zero or equal to zero.*

The above results are compiled in the following table:

Parameter	Sum	Moment	Simplest equipollent system
$K \neq 0$	—	—	vector and couple or two skew vectors
$K = 0$	$\boldsymbol{s} \neq 0$	—	resultant vector
	$\boldsymbol{s} = 0$	$\boldsymbol{M} \neq 0$	couple
	$\boldsymbol{s} = 0$	$\boldsymbol{M} = 0$	zero vector

The following theorems are easy consequences of the preceding:

1. *If the moment of a system with respect to a certain point O is zero, then the system has a resultant with its initial point at O.*

2. *A central system has a resultant whose initial point is at the centre.*

These theorems follow from the reduction theorem (p. 24) if we take point O (or the centre of the system) respectively, as the centre of reduction.

3. *A plane system either has a resultant or is equipollent to a couple.*

4. *A parallel system either has a resultant or is equipollent to a couple.*

Theorems 3 and 4 are obtained at once from the table because in both cases K is zero.

§ 17. Central axis. Wrench. Let S be a given system having a sum different from zero. Let us determine the geometric locus of points with respect to which the total moment is parallel to $\mathbf{s}$ (or $= 0$).

For this purpose choose an arbitrary point O. Let $\mathbf{M}_o = \overline{OA}$ be the total moment of the system with respect to point O and $\overline{OB}$ the projection of $\mathbf{M}_o$ on $\mathbf{s}$.

Let us now determine the point O' with respect to which the moment of the sum $\mathbf{s}$ with initial point at O is equal to $\overline{AB}$. Such a point is found at a distance d from O on a line perpendicular at O to $\overline{AB}$ and $\mathbf{s}$, where d satisfies the condition:

$$d \cdot |\mathbf{s}| = |\overline{AB}|.$$

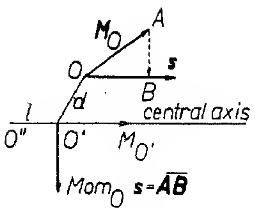

Fig. 36.

Therefore $\mathrm{Mom}_{o'}\mathbf{s} = \overline{AB}$ or

$$\mathbf{s} \times \overline{O'O} = \overline{AB},$$

and hence by (I), p. 19

$$\mathbf{M}_{o'} = \mathbf{s} \times \overline{O'O} + \mathbf{M}_o,$$

whence

$$\mathbf{M}_{o'} = \overline{AB} + \overline{OA} = \overline{OB}.$$

Therefore $\mathbf{M}_{o'}$ is parallel to $\mathbf{s}$ (or $= 0$, when $\mathbf{M}_o \perp \mathbf{s}$).

Let us pass a line l through O' parallel to the sum $\mathbf{s}$. The relation $\mathbf{s} \parallel \overline{O'O''}$ holds for an arbitrary point O'' of line l; hence $\mathbf{s} \times \overline{O'O''} = 0$, whence $\mathbf{M}_{o'} = \mathbf{M}_o$ (p. 20, corollary 3).

Therefore: *the total moment with respect to an arbitrary point of l is parallel to $\mathbf{s}$ (or $= 0$).*

Points not on the line l do not possess the above mentioned property, because if the moment $\mathbf{M}_{o_1}$ for some point O_1 is parallel to $\mathbf{s}$ or equal to

zero, then by theorem 5, p. 20, the $\mathrm{Proj}_s\mathbf{M}_{o_1} = \mathrm{Proj}_s\mathbf{M}_{o'}$. Hence $\mathbf{M}_{o_1} = \mathbf{M}_{o'}$. By formula (I), p. 19, it follows that $\mathbf{s} \times \overline{O'O_1} = 0$, or that $\mathbf{s} \parallel \overline{O'O_1}$. Therefore point O^1 lies on l.

We have thus proved that the sought for geometric locus is a line parallel to $\mathbf{s}$. This line is called the *central axis* of the system.

Therefore the central axis of the szstem is a straight line with the property, that the total moment with respect to an arbitrary point of this line is parallel to the sum or equal to zero.

Hence: *a system whose sum is different from zero possesses one (and only one) central axis.*

A system consisting of a vector and a couple whose moment is parallel to the vector is called a *wrench.*

In particular, a vector or a couple is called a wrench.

Selecting a point on the central axis, we see by the reduction theorem, p. 24, that the system is reduced to a wrench. If the sum of the system is zero, then the system is reduced to a couple and hence also to a wrench.

Therefore: *every system is equipollent to a certain wrench.*

§ 18. Centre of parallel vectors. Let a system of parallel vectors $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$, whose sum is different from zero, be given. Denote a unit vector parallel to the vectors of the system by $\mathbf{w}$. The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ can be represented in the form

$$\mathbf{a}_1 = a_1\mathbf{w}, \quad \mathbf{a}_2 = a_2\mathbf{w}, \quad \ldots, \quad \mathbf{a}_n = a_n\mathbf{w},$$

where by $a_1, a_2, \ldots, a_n$ we denote the lengths of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Therefore $\mathbf{s} = (a_1 + a_2 + \ldots + a_n)\,\mathbf{w}$. Since $\mathbf{s} \neq 0$, then $a_1 + a_2 + \ldots + a_n \neq 0$.

Select an arbitrary point O' and denote the vectors $\overline{O'A_1}, \overline{O'A_2}, \ldots, \overline{O'A_n}$ by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, where $A_1, A_2, \ldots, A_n$ are the initial points of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Hence

$$\mathbf{M}_{o'} = a_1\mathbf{w} \times \mathbf{r}_1 + a_2\mathbf{w} \times \mathbf{r}_2 + \ldots + a_n\mathbf{w} \times \mathbf{r}_n,$$

or

$$\mathbf{M}_{o'} = \mathbf{w} \times \Sigma a_i\mathbf{r}_i. \qquad (1)$$

Choose a point O such that

$$\mathbf{r} = \overline{O'O} = \frac{\Sigma a_i\mathbf{r}_i}{\Sigma a_i}. \qquad (2)$$

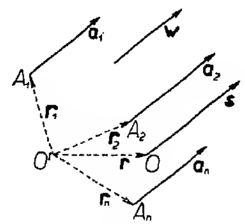

Fig. 37.

From (I), p. 19,

$$\mathbf{M}_o = \mathbf{s} \times \overline{OO'} + \mathbf{M}_{o'}. \qquad (3)$$

Since

$$s \times \overline{OO'} = (\Sigma a_i \mathbf{w}) \times (-\mathbf{r}) = -\mathbf{w} \times \mathbf{r}\, \Sigma a_i,$$

therefore according to (2), $\mathbf{s} \times \overline{OO'} = -\mathbf{w} \times \Sigma a_i \mathbf{r}_i$. Hence by (1) and (3) it follows that $\mathbf{M}_o = 0$.

The resultant of the system therefore passes through O (theorem 1, p. 26).

Let us note that according to (2) the position of the point O does not depend on the direction $\mathbf{w}$ of the vectors $\mathbf{a}_i$. Therefore, if the vectors $\mathbf{a}_i$ are turned about their points of application the resultant will again pass through O.

The point O is called the *centre of the system* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.

If the coordinates of the initial points A_i are denoted by x_i, y_i, z_i, those of the centre—by x_0, y_0, z_0, then selecting point O' as the origin of the system, we obtain by (2)

$$x_0 = \frac{\Sigma a_i x_i}{\Sigma a_i}, \quad y_0 = \frac{\Sigma a_i y_i}{\Sigma a_i}, \quad z_0 = \frac{\Sigma a_i z_i}{\Sigma a_i}. \tag{4}$$

§ 19. Elementary transformations of a system. The following transformations of a system of vectors are termed *elementary*:

(a) adding to the system (or removing from it) two vectors equal in magnitude, opposite in sense and lying on the same line;

(b) adding to the system (or removing from it) several vectors having a common origin and a sum equal to zero.

Elementary transformations obviously do not change the sum or the moment of the system. Therefore, if we apply elementary transformations to a system, we always obtain systems equipollent to it. Elementary transformations play an important role in the theory of rigid bodies.

It is easy to show that by means of elementary transformations we can:

1. *translate the point of application of a vector to an arbitrarily chosen point of the line on which the vector lies;*

2. *replace several vectors having a common origin by their sum having the same origin;*

3. *replace one vector by several vectors having the same origin as the given vector and having a sum equal to that of the given vector.*

Proof. 1. Suppose that among the vectors of the given system there is a vector $\mathbf{a}$ whose initial point is A.

Select an arbitrary point B of the line l on which $\mathbf{a}$ lies. Introduce to the system two vectors $\mathbf{a}$ and $-\mathbf{a}$ whose initial points are at B. We have

thus carried out elementary transformation (a). Remove now from the system the vectors: $\mathbf{a}$ (whose origin is A) and $-\mathbf{a}$. This will be elementary transformation (b). The operations which we have carried out on the system are equivalent to the translation of the point of application of vector $\mathbf{a}$ from A to B (Fig. 38).

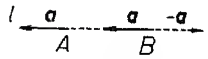

Fig. 38.

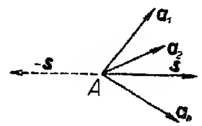

Fig. 39.

2. Suppose that the point A is the origin of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Add to the system two vectors whose common origin is A: $\mathbf{s} = \mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n$, and $-\mathbf{s}$ (elementary transformation (a)). Now remove the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, $-\mathbf{s}$ (elementary transformation (b)). The operations which we have performed are equivalent to the replacement of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ by their sum $\mathbf{s}$ (Fig. 39).

3. is proved similarly.

We shall now prove the following theorems:

Theorem I. *By means of elementary transformations every system of vectors can be reduced to a system of three vectors equipollent to the given system.*

Proof. Suppose that we have a system of vectors $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ whose points of application are $A_1, A_2, \ldots, A_n$, respectively. Select three non-collinear points L, M, N in such a way that none of the points $A_1, A_2, \ldots, A_n$ will lie on the plane passing through L, M, N.

Since lines A_1L, A_1M and A_1N do not lie in the same plane, therefore the vector $\mathbf{a}_1$, can be replaced by three vectors $\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1$ with common origin A_1 lying on A_1L, A_1M, A_1N, while obviously $\mathbf{a}_1 = \mathbf{u}_1 + \mathbf{v}_1 + \mathbf{w}_1$ (Fig. 40). The vectors $\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1$ can be translated along the lines on which they lie to the points L, M, N, respectively. In this way we have replaced the vector $\mathbf{a}_1$ by the vectors $\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1$ whose points of application are at L, M, N. Similarly, we replace each one of the vectors $\mathbf{a}_2, \ldots, \mathbf{a}_n$ by three vectors whose points of application are at L, M, N.

We now replace the vectors with origin at L by their sum $\mathbf{u}$ with origin also at L. Similarly, vectors with origins at M and N are replaced by sums $\mathbf{v}$ and $\mathbf{w}$ whose origins are M and N, respectively.

In this manner, by means of the elementary transformations, we

have reduced the given system to a system consisting of three vectors, q. e. d.

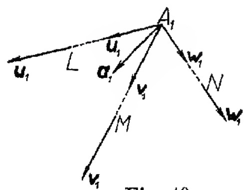

Fig. 40.

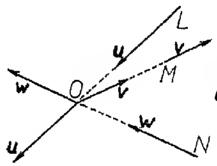

Fig. 41.

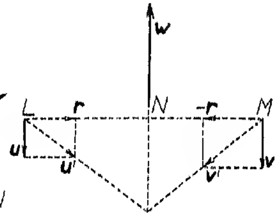

Fig. 42.

Theorem 2. *By means of elementary transformations a system equipollent to zero can be reduced to a zero vector.*

Proof. Assume that the system $(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)$ is equipollent to zero. According to theorem 1 it can be replaced, by means of elementary transformations, by a system consisting of the three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ with points of application at L, M, N, respectively. The system $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is equipollent to zero because it is equipollent to the given system (for elementary transformations do not alter the sum or moment).

According to the theorem on p. 22, the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are either parallel or their prolongations are concurrent at O (Fig. 41). In the second case we can translate the points of application of the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ to O and then remove these vectors since their sum is zero.

Assume then that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are parallel (Fig. 42). Were $\boldsymbol{u} + \boldsymbol{v} = 0$, then obviously $\boldsymbol{w} = 0$. The system would then be reduced to the couple $\boldsymbol{u}, \boldsymbol{v}$. Since the moment is zero, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ would lie on the same line; since, besides $\boldsymbol{u} + \boldsymbol{v} = 0$, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ could be removed. Therefore, let $\boldsymbol{u} + \boldsymbol{v} \neq 0$. Add two vectors $\boldsymbol{r}$ and $-\boldsymbol{r}$ lying on the line LM and having points of application at L and M, respectively. The vectors $\boldsymbol{u}$ and $\boldsymbol{r}$ with origin at L can be replaced by their sum $\boldsymbol{u}'$ with its point of application also at L. Similarly, the vectors $\boldsymbol{v}$ and $-\boldsymbol{r}$ can be replaced by their sum $\boldsymbol{v}'$ whose point of application is at M. The vectors $\boldsymbol{u}'$ and $\boldsymbol{v}'$ are not parallel; hence the vectors $\boldsymbol{u}', \boldsymbol{v}'$ and $\boldsymbol{w}$ can be removed as before. Thus the system equipollent to zero has been reduced to a zero vector by means of the elementary transformations, q. e. d.

Theorem 3. *If two systems of vectors are equipollent, then by means of elementary transformations one system can be transformed into the other.*

Proof. Suppose that the system of vectors $(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)$ with points of application at $A_1, A_2, \ldots, A_n$ is equipollent to the system of vectors $(\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_r)$ with points of application at $B_1, B_2, \ldots, B_r$.

To the first system add the vectors $\boldsymbol{b}_1, -\boldsymbol{b}_1$ with origin at B_1, the vectors $\boldsymbol{b}_2, -\boldsymbol{b}_2$ with origin at B_2 etc. Since the vectors

$$\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n, -\boldsymbol{b}_1, -\boldsymbol{b}_2, \ldots, -\boldsymbol{b}_r$$

form a system equipollent to zero, therefore by theorem 2 this system can be removed by means of elementary transformations, i. e. replaced by a zero vector. After the removal there remains the system $(\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_r)$.

CHAPTER II

KINEMATICS OF A POINT

I. MOTION RELATIVE TO A FRAME OF REFERENCE

§ 1. Time. In kinematics, in addition to known geometric concepts, there arises the concept of time. For purposes of theoretical kinematics it is sufficient to assume that to each moment there is assigned a certain number t, and that there are assigned smaller numbers for moments before t than for moments after t. Conversely, to each ordering of numbers t there should correspond a certain moment: to a larger number a later moment.

In theoretical kinematics it is entirely immaterial in what way the above ordering of time was defined. In any concrete problem we proceed in the following manner. We select an arbitrary *unit of time*, e. g. a second, and an arbitrary moment which we call the *initial moment*. To the initial moment we assign the number 0. Every other moment is represented by a number t whose absolute value is the number of seconds that elapsed between the initial and given moments. The number t is positive for moments after, and negative for moments before the initial moment.

§ 2. Frame of reference. In kinematics we assume that a certain system of coordinates, called a *frame of reference*, is given.

A body moves relative to a frame of reference if the coordinates of the points of the body change. The problem of kinematics is to describe the motion of the body relative to a frame of reference when the coordinates of the points of that body are given at each moment of time.

In kinematics it is a matter of indifference how a frame of reference was selected. In concrete problems we select a frame of reference attached to certain bodies like the earth, the sun, the fixed stars, etc.

The motion of a body depends on the frame of reference. Relative to one frame a body may be at rest, but relative to another frame it may

be in motion. A passenger sitting in a car of a moving train is at rest relative to a frame attached to the car, but in motion relative to the earth.

In concrete preblems a question arises as to whether it is not possible to examine the motion of a body independent of other bodies. Such a motion would be the so-called absolute motion. It appears, however, by observing that points in space are indistinguishable, that by measuring distances and appealing to theorems of geometry, it is not possible to prove in any way whether a body examined at two different moments has, or has not, altered its position. The concept of absolute motion is therefore useless. Hence we must confine ourselves to the study of relative motion, i. e. to the motion of a body relative to other bodies.

§ 3. Motion of a point. We shall concern ourselves at first with the motion of one point because the description ef the motion of a body is reduced to the description of the motion of its points. Moreover, in many cases the discription of the motion of a body is reduced in practice to that of the motion of one of its points, e. g. if the dimensions of the body are small in comparison with the path traversed (the motion of the earth around the sun, the motion of a bullet) or if the motion of one point determines the motion of the entire body (e. g. the motion of a car).

Let us denote the coordinates of a moving point M relative to a certain frame of reference by x, y, z. The coordinates x, y, z depend on the time and are thus functions of the variable t:

$$x = f(t), \quad y = \varphi(t), \quad z = \psi(t).$$

These functions give us a description of the motion of the point M relative to the chosen frame of reference. Knowing them, we can obtain the coordinates x, y, z of the point M at any time t.

We assume that the functions f, φ and ψ are continuous together with their first and second derivatives in the interval $[t_0, t_1]$ during which the motion is being examined.

The motion of the point can be characterized by means of one vector function. Let us put $\mathbf{r} = \overline{OM}$ (O being the origin of the reference frame). Hence

$$\mathbf{r} = \mathbf{F}(t).$$

The above vector function describes the motion in its entirety, giving at each moment a vector $\mathbf{r}$ and consequently the position of the point M.

Let us note that the functions f, φ and ψ give the components of the vector $\boldsymbol{r}$.

The curve that is described by the point during its motion is called a *path* or a *trajectory*.

Suppose that the path of the point is the arc L. Let us give this arc a certain sense and select on it an arbitrary point O, which we shall call the *initial point* (Fig. 43). The position of the point M on the arc L will be determined by giving a number s whose absolute value is equal to the length of the arc OM, and which is positive or negative depending on whether the sense of OM agrees, or does not agree, with the sense originally selected. The number s is called the *arc coordinate* of the point M on the arc L.

The motion of the point M along the arc L will also be determined by the function

$$s = f(t),$$

which gives the arc coordinate s of the point M at each moment t.

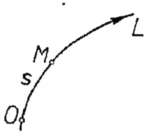

Fig. 43.

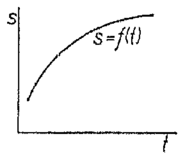

Fig. 44.

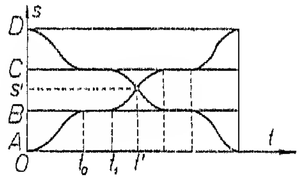

Fig. 45.

§ 4. Graph of a motion. Let the motion of a point along a curve L be defined by the function $s = f(t)$. Select two perpendicular axes s and t.

The graph of the function $s = f(t)$ is called the *graph* or the *diagram* of the motion (Fig. 44).

In Fig. 45 we have a graph of the motion of two trains, one running from station A to station D (through stations B, C), and the other running from D to A. From the diagram we read, for instance, that at $t = 0$ the train departed from station A and arrived at B at $t = t_0$. It left the station B at $t = t_1$, etc. The coordinates (t', s') of the point of intersection of both graphs represent the time and place at which both trains meet.

§ 5. Velocity. Suppose that a point moving along a curve L is at the point A at the time t, and at the point B at the time $t + \Delta t$.

The vector $\overline{AB}$ is called the *displacement* of the moving point during the time Δt. The quotient

$$\overline{AB} \,/\, \Delta t = \overline{AC} \tag{1}$$

represents the displacement per unit of time. The above quotient is also called the *average velocity vector*, or briefly, the *average velocity* during the time Δt.

Let us assume that the limit of the ratio (1) exists as $\Delta t \to 0$. Denote this limit by $\boldsymbol{v}$. Then

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\overline{AB}}{\Delta t}. \tag{I}$$

The vector $\boldsymbol{v}$ is called the *velocity vector*, or briefly, the *velocity* at the time t.

As $\Delta t \to 0$ the secant AB tends to the tangent. Therefore: *the velocity vector is tangent to the path.*

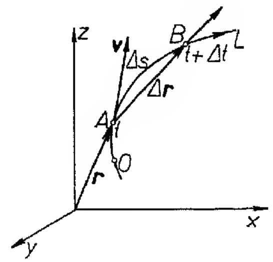

Fig. 46.

Let the motion be defined by the functions $x = f(t)$, $y = \varphi(t)$, $z = \psi(t)$. Denote the coordinates of the point A by x, y, z and those of the point B by $x + \Delta x$, $y + \Delta y$, $z + \Delta z$. The projections of $\overline{AB}$ are Δx, Δy, Δz. The projections of the quotient $\overline{AB} \,/\, \Delta t$ will therefore be the ratios $\Delta x : \Delta t$, $\Delta y : \Delta t$, $\Delta z : \Delta t$.

It follows from this that the projections of the velocity $\boldsymbol{v}$ are expressed by the formulae:

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = f'(t), \quad v_y = \frac{dy}{dt} = \varphi'(t), \quad v_z = \frac{dz}{dt} = \psi'(t).$$

In mechanics the derivate with respect to time is denoted by a dot above, after the dependent variable. Thus

$$v_x = x^{\cdot}, \quad v_y = y^{\cdot}, \quad v_z = z^{\cdot}. \tag{II}$$

Hence: *the projections of the velocity vector on the coordinate axes are equal to the derivatives (with respect to time) of the coordinates of the moving point.*

Let the motion be defined now by the vector function $\boldsymbol{r} = \boldsymbol{F}(t)$. Setting $\overline{AB} = \Delta \boldsymbol{r}$, we obtain

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\overline{AB}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{d\boldsymbol{r}}{dt} = \boldsymbol{r}^{\cdot}.$$

Therefore

$$\boldsymbol{v} = \boldsymbol{r}^{\cdot}. \tag{III}$$

Velocity as a derivative of the path. Finally, let the motion of the point along the path L be defined by the function $s = f(t)$, where s denotes the arc coordinate. Since the velocity is tangent to the path, it

is sufficient to give its magnitude and sense in order to determine it at the point A. From the definition of velocity it follows that

$$|\mathbf{v}| = \lim_{\Delta t\to 0}\left|\frac{\overline{AB}}{\Delta t}\right| = \lim_{\Delta t\to 0}\left|\frac{\overline{AB}}{\Delta s}\right|\cdot\left|\frac{\Delta s}{\Delta t}\right|,$$

where $|\Delta s|$ denotes the length of the arc AB. Since $\lim\limits_{\Delta s\to 0}\left|\frac{\overline{AB}}{\Delta s}\right| = 1$,

$$|\mathbf{v}| = \lim_{\Delta t\to 0}\left|\frac{\Delta s}{\Delta t}\right| = \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right| = |s^{\cdot}|.$$

Let us draw a tangent at the point A and give to it a sense agreeing with that chosen for the curve L. If $s^{\cdot} > 0$, then $\Delta s > 0$ for small $\Delta t > 0$; therefore the point moves along the path in the positive direction, and hence $\mathbf{v}$ has a sense agreeing with that of the tangent. Similarly, if $s^{\cdot} < 0$, then $\mathbf{v}$ has a sense opposite to that of the tangent. Therefore, if v denotes the component of the velocity vector along the tangent to which we have assigned a sense agreeing with that of the path, then

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = s^{\cdot}. \tag{IV}$$

§ 6. Acceleration. Suppose that at the time t a point was at A and had a velocity $\mathbf{v}$, while at the time $t+\Delta t$ it was at B and had a velocity $\mathbf{v}'$. Put $\Delta\mathbf{v} = \mathbf{v}' - \mathbf{v}$.

The limit $\lim\limits_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t} = \mathbf{p}$ is called the *acceleration vector*, or briefly, the *acceleration* at the time t.

Let the motion be defined by the functions $x = f(t)$, $y = \varphi(t)$, $z = \psi(t)$. We have $p_x = \lim\limits_{\Delta t\to 0}\frac{\Delta v_x}{\Delta t}$. Since $v_x = f'(t)$ and $v_x' = f'(t+\Delta t)$, it follows that $\Delta v_x = v_x' - v_x = f'(t+\Delta t) - f'(t)$. Therefore

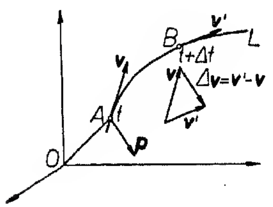

Fig. 47.

$$p_x = \lim_{\Delta t\to 0}\frac{f'(t+\Delta t) - f'(t)}{\Delta t} = f''(t);$$

similarly $p_y = \varphi''(t)$ and $p_z = \psi''(t)$.

The derivatives

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2},\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2},\ \frac{\mathrm{d}^2z}{\mathrm{d}t^2}$$

are denoted by $x^{\cdot\cdot}$, $y^{\cdot\cdot}$, $z^{\cdot\cdot}$. Hence

$$p_x = x^{\cdot\cdot},\qquad p_y = y^{\cdot\cdot},\qquad p_z = z^{\cdot\cdot}. \tag{I}$$

Therefore: *the projections of the acceleration on the coordinate axes are equal to the second derivatives of the coordinates of the moving point.*

If the motion is defined by the vector function

$$\mathbf{r} = \mathbf{F}(t),$$

then — as follows from the definition of the second derivative — we have

$$\mathbf{p} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \mathbf{r}^{\cdot\cdot}. \tag{II}$$

Example 1. A point moves in a plane in such a way that its coordinates at the time t are expressed by the following equations:

$$x = a\cos kt,\qquad y = b\sin kt\qquad (a>0,\ b>0,\ k>0). \tag{1}$$

Determine the velocity, acceleration and the path.

We have

$$x^{\cdot} = -ak\sin kt,\qquad y^{\cdot} = bk\cos kt,$$
$$x^{\cdot\cdot} = -ak^2\cos kt,\qquad y^{\cdot\cdot} = -bk^2\sin kt,$$

and hence the absolute value of the velocity will be

$$|\mathbf{v}| = k\sqrt{a^2\sin^2 kt + b^2\cos^2 kt},$$

and the absolute value of the acceleration

$$|\mathbf{p}| = k^2\sqrt{a^2\cos^2 kt + b^2\sin^2 kt}.$$

In order to determine the path it is necessary to find a relation between x and y, i. e. it is necessary to eliminate t. Dividing the equations (1) by a and b respectively, squaring and adding, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence the path is an ellipse with axes $2a$ and $2b$. The velocity vector is obviously tangent to the ellipse. If we make use of equations (1), then the components of the acceleration vector can be written in the form:

$$x^{\cdot\cdot} = -k^2x,\qquad y^{\cdot\cdot} = -k^2y,$$

whence

$$|\mathbf{p}| = k^2\sqrt{x^2+y^2},\qquad y^{\cdot\cdot}/x^{\cdot\cdot} = y/x.$$

The acceleration vector is therefore proportional to the distance from the origin of the coordinate system and always directed toward it.

Example 2. Determine the velocity, the acceleration and the path of a point whose motion is defined by the equations:

$$x = \tfrac{1}{2}a(1+\cos t),\qquad y = \tfrac{1}{2}a\sin t,\qquad z = a\sin\tfrac{1}{2}t\qquad (a>0). \tag{2}$$

Differentiating, we obtain

$$\dot{x} = -\tfrac{1}{2}a \sin t, \quad \dot{y} = \tfrac{1}{2}a \cos t, \quad \dot{z} = \tfrac{1}{2}a \cos \tfrac{1}{2}t,$$
$$\ddot{x} = -\tfrac{1}{2}a \cos t, \quad \ddot{y} = -\tfrac{1}{2}a \sin t, \quad \ddot{z} = -\tfrac{1}{4}a \sin \tfrac{1}{2}t,$$

whence

$$|\mathbf{v}| = \tfrac{1}{2}a\sqrt{1+\cos^2\tfrac{1}{2}t}, \quad \text{and} \quad |\mathbf{p}| = \tfrac{1}{2}a\sqrt{1+\tfrac{1}{4}\sin^2\tfrac{1}{2}t}.$$

In order to determine the path of the point it is necessary to eliminate t from the equations (2), which will give us two relations between x, y, z defining the space curve along which the point moves. This elimination in general presents great computational difficulties; in this example, however, it is easy to accomplish. Squaring the equations (2) and adding, we obtain

$$x^2 + y^2 + z^2 = a^2.$$

Similarly, from the first two equations (2) it follows that

$$(x - \tfrac{1}{2}a)^2 + y^2 = (\tfrac{1}{2}a)^2.$$

From the equations obtained we see that the path of the point is the curve of intersection of a sphere and a circular cylinder.

Example 3. Uniform straight line motion. A point moves in such a way that the acceleration is always zero. We are assuming then that $\mathbf{p} = 0$. Hence

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} = 0.$$

After integrating, we get

$$\dot{x} = c_1, \quad \dot{y} = c_2, \quad \dot{z} = c_3, \tag{3}$$

where c_1, c_2, c_3 are certain constants. Integrating once more, we obtain

$$x = c_1 t + d_1, \quad y = c_2 t + d_2, \quad z = c_3 t + d_3. \tag{4}$$

Here d_1, d_2, d_3 also denote certain constants. The equations (4) represent the parametric equations of a straight line. From (3) it follows that the velocity vector is constant. Therefore, the motion takes place along a straight line with a constant velocity. Such a motion is called a *uniform straight line motion.*

Let us note that setting $\mathbf{p} = 0$ is equivalent to the assumption that the velocity vector is constant. For we have $\mathbf{p} = \dot{\mathbf{v}}$. Therefore, if $\mathbf{v} = \text{const}$, then $\mathbf{p} = 0$ and conversely, if $\mathbf{p} = 0$, then $\dot{\mathbf{v}} = 0$ and hence $\mathbf{v} = \text{const}$.

Hodograph. Let a point move along the curve L (Fig. 48). Choose an arbitrary point O. At each moment t draw from O a velocity vector which

the moving point has at the given moment (Fig. 49). The end points of these velocity vectors describe a curve H which is called a *hodograph.*

On the hodograph we denoted the end points of the velocity vectors which a moving point along the curve L has at A, B, C by A', B', C'. If the point moves along the path L, then the end point of the corresponding velocity vector moves along the hodograph.

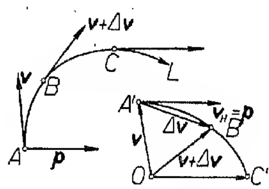

Fig. 48. Fig. 49.

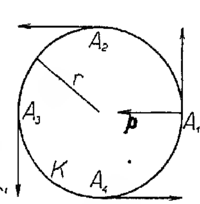

Fig. 50.

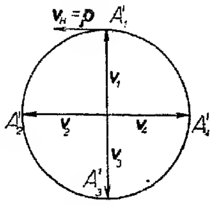

Fig. 51.

Denote by $\mathbf{v}_H$ the velocity of the point on the hodograph at A'. From the definition of velocity we have $\mathbf{v}_H = \lim\limits_{\Delta t\to 0} \dfrac{\overline{A'B'}}{\Delta t}$. But $\overline{A'B'} = \Delta\mathbf{v}$; hence

$$\mathbf{v}_H = \lim_{\Delta t\to 0} \frac{\Delta\mathbf{v}}{\Delta t} = \mathbf{p}.$$

Therefore: *the acceleration of a moving point is equal to the velocity of the corresponding point on the hodograph.*

Example 4. If a point moves along a straight line, then the direction of the velocity is constant. Therefore the hodograph is also a straight line.

Example 5. Let us assume that a point moves along a circle K with a velocity which is constant in magnitude (Fig. 50).

The hodograph will be a circle (Fig. 51). The point will move along the hodograph with a velocity which is constant in magnitude.

Since the velocity of the point A_1' on the hodograph is tangent to the hodograph, it is perpendicular to $\mathbf{v}_1$. It follows from this that the acceleration of a point moving along the circle K is directed towards the centre of the circle and is constant in magnitude.

§ 7. Resolution of the acceleration along a tangent and a normal.

Motion along a plane curve. Let the motion of the point A along the path L be defined by the function $s = f(t)$, where s denotes the arc coordinate. Draw a tangent t and a normal n at the point A. Give the tangent a sense agreeing with that of the curve L, and the normal — a sense towards the centre of curvature.

The projection of the acceleration $\boldsymbol{p}$ on the tangent is called the *tangential acceleration* $\boldsymbol{p}_t$; the projection of the acceleration on the normal is called the *normal acceleration* $\boldsymbol{p}_n$. Obviously

$$\boldsymbol{p} = \boldsymbol{p}_t + \boldsymbol{p}_n. \tag{I}$$

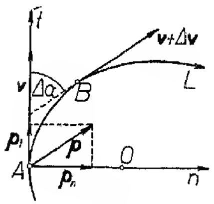

Fig 52.

Having been given the tangential and normal accelerations, we can therefore determine the acceleration $\boldsymbol{p}$. The tangential acceleration will be defined by giving the component p_t along the tangent. Similarly, the component p_n along the normal defines the normal acceleration.

Let us denote by $\Delta\alpha$ the acute angle between the tangents at the point A whose arc coordinate is s, and the neighbouring point B whose arc coordinate is $s + \Delta s$. Assume that $\Delta s > 0$. We then have

$$\lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta s} = \frac{1}{\varrho},$$

where ϱ denotes the radius of curvature. As we know,

$$\boldsymbol{p} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t};$$

therefore

$$p_t = \lim_{\Delta t \to 0} \frac{\mathrm{Proj}_t \Delta \mathbf{v}}{\Delta t},$$

where $\mathrm{Proj}_t \Delta\mathbf{v}$ denotes the component of $\Delta\mathbf{v}$ along the tangent t. But $\mathrm{Proj}_t\Delta\mathbf{v} = \mathrm{Proj}_t(\mathbf{v} + \Delta\mathbf{v}) - \mathrm{Proj}_t\mathbf{v}$. Hence $\mathrm{Proj}_t\Delta\mathbf{v} = (v + \Delta v)\cos\Delta\alpha - v$. Therefore

$$p_t = \lim_{\Delta t \to 0} \frac{(v + \Delta v)\cos\Delta\alpha - v}{\Delta t} = v \lim_{\Delta t \to 0} \frac{\cos\Delta\alpha - 1}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \cos\Delta\alpha. \tag{1}$$

But

$$\lim_{\Delta t \to 0} \frac{\cos\Delta\alpha - 1}{\Delta t} = \lim_{\Delta t \to 0} \frac{\cos\Delta\alpha - 1}{\Delta\alpha} \cdot \frac{\Delta\alpha}{\Delta s} \cdot \frac{\Delta s}{\Delta t}. \tag{2}$$

From known rule for evaluating indeterminate forms we obtain

$$\lim_{\Delta t \to 0} \frac{\cos\Delta\alpha - 1}{\Delta\alpha} = \lim_{\Delta\alpha \to 0} \frac{\cos\Delta\alpha - 1}{\Delta\alpha} = \lim_{\Delta\alpha \to 0} \frac{-\sin\Delta\alpha}{1} = 0.$$

Therefore in virtue of (2)

$$\lim_{\Delta t \to 0} \frac{\cos\Delta\alpha - 1}{\Delta t} = 0 \cdot \frac{1}{\varrho} \cdot v = 0,$$

whence by (1):

$$p_t = \mathrm{d}v \,/\, \mathrm{d}t = s^{\cdot\cdot}.$$

Let us now evaluate p_n. We have

$$p_n = \lim_{\Delta t \to 0} \frac{\mathrm{Proj}_n \Delta\mathbf{v}}{\Delta t}.$$

But $\mathrm{Proj}_n\Delta\mathbf{v} = \mathrm{Proj}_n(\mathbf{v} + \Delta\mathbf{v}) - \mathrm{Proj}_n\mathbf{v} = (v + \Delta v)\sin\Delta\alpha$. Hence

$$p_n = \lim_{\Delta t \to 0} (v + \Delta v) \frac{\sin\Delta\alpha}{\Delta t}.$$

Since

$$\lim_{\Delta t \to 0} \frac{\sin\Delta\alpha}{\Delta t} = \lim_{\Delta t \to 0} \frac{\sin\Delta\alpha}{\Delta\alpha} \cdot \frac{\Delta\alpha}{\Delta s} \cdot \frac{\Delta s}{\Delta t} = 1 \cdot \frac{1}{\varrho} \cdot v,$$

it follows that

$$p_n = v^2 / \varrho.$$

Therefore the tangential acceleration p_t and the normal acceleration p_n are expressed by the formulae

$$p_t = \mathrm{d}v / \mathrm{d}t = s^{\cdot\cdot}, \qquad p_n = v^2 / \varrho, \tag{II}$$

where ϱ is the radius of curvature.

Since the tangential acceleration is perpendicular to the normal acceleration, it follows that

$$|\boldsymbol{p}| = \sqrt{p_t^2 + p_n^2}. \tag{III}$$

From formula (1) we see that $p_n \geqq 0$. Therefore the normal acceleration is always directed towards the centre of curvature.

Let us note that the tangential acceleration depends only on the change of the absolute value of the velocity and not on the change of its direction. The tangential acceleration is constantly zero if and only if $v = \mathrm{const.}$

The normal acceleration depends on the radius of curvature ϱ and therefore on the change of the direction of the velocity. The normal acceleration is constantly zero if constantly $v = 0$ or $1/\varrho = 0$. In the first case the point is at rest, and in the second case the motion is along a straight line.

Example. A point moves along a circle of radius r with a velocity v whose absolute value is constant. Therefore by (I)

$$p_t = \mathrm{d}v \,/\, \mathrm{d}t = 0, \quad p_n = v^2 / r = \mathrm{const.}$$

Hence the acceleration is constantly directed towards the centre of the circle and its absolute value is constant (cf. p. 39, example 5).

Motion along a space curve. Let a point move along a space curve L. Denote the velocity of the point at A by $\mathbf{v}$ and its velocity at B by $\mathbf{v} + \Delta\mathbf{v}$.

Pass a plane Π through the tangent at A parallel to the tangent at B. The vectors $\mathbf{v}$ and $\mathbf{v} + \Delta\mathbf{v}$ drawn from the point A lie in this plane. Therefore the vector $\Delta\mathbf{v}/\Delta t$ lies in the plane Π. As the point B tends to the point A, the plane Π tends to the so-called *osculating plane* at A. It therefore follows that the acceleration $\mathbf{p} = \lim\limits_{\Delta t \to 0} \Delta\mathbf{v} / \Delta t$ lies in the osculating plane.

Hence: *the acceleration vector lies in the osculating plane.*

The tangent lies in the osculating plane. The line perpendicular to the tangent and lying in the osculating plane is called the *principal normal*. The centre of curvature lies on the principal normal. Forming the projections of the acceleration on the tangent and the normal, we obtain

$$\mathbf{p} = \mathbf{p}_t + \mathbf{p}_n,$$

which is analogous to formula (I) obtained in connection with plane motion.

Giving the tangent a sense agreeing with that of the curve, and the normal a sense towards the centre of curvature and proceeding as before, we obtain:

$$p_t = \mathrm{d}v / \mathrm{d}t = s^{\cdot\cdot} \quad \text{and} \quad p_n = v^2 / \varrho,$$

where ϱ denotes the radius of curvature.

The above relations are identical with those of (II) in the case of plane motion.

Example 1. U n i f o r m m o t i o n. Let a point A move along a curve L on which an initial point O and a sense have been chosen. Assume that the velocity of the point A is constant in magnitude (i. e. in absolute value). Such a motion is called a *uniform motion* along the curve L.

We are assuming, therefore, that $v = s^{\cdot} = \text{const}$. Integrating, we obtain

$$s = vt + s_0. \tag{3}$$

Substituting $t = 0$, we get $s = s_0$. Hence the constant s_0 denotes the arc coordinate of the point A at the time $t = 0$.

Uniform motion is defined by a function of the first degree with respect to t. Conversely, an arbitrary function of the first degree $s = at + b$ defines a uniform motion with the velocity $v = s^{\cdot} = a$. In addition $s_0 = b$.

By (3) we have

$$p_t = v^{\cdot} = s^{\cdot\cdot} = 0, \quad p_n = v^2/\varrho.$$

Since the tangential acceleration is zero, the acceleration is constantly directed towards the centre of curvature. The magnitude of the acceleration is p_n. Therefore the magnitude of the acceleration is inversely proportional to the radius of curvature ϱ (or directly proportional to the curvature $K = 1/\varrho$).

In particular, if a point moves uniformly along a circle of radius r, then $\varrho = r$, and hence

$$p_n = v^2 / r.$$

Therefore: *if a point moves along a circle uniformly, then the acceleration is of constant magnitude and it is directed towards the centre of the circle.*

Example 2. U n i f o r m l y a c c e l e r a t e d m o t i o n. A point moving along a curve L has a constant tangential acceleration. Such a motion is called a *uniformly accelerated motion* along the curve L.

Assuming that $p_t = p = \text{const}$, we get $s^{\cdot\cdot} = p_t = p$; hence

$$s^{\cdot} = v = pt + c_1, \quad s = \tfrac{1}{2}pt^2 + c_1 t + c_2. \tag{4}$$

Uniformly accelerated motion is defined by the function $s = f(t)$ of the second degree in t. Conversely, an arbitrary function of the second degree $s = at^2 + bt + c$ defines a uniformly accelerated motion because after differentiating twice:

$$p_t = s^{\cdot\cdot} = 2a = \text{const}.$$

Let us assume that in a uniformly accelerated motion defined by function (4), a point had a velocity $v = v_0$ and an arc coordinate $s = s_0$ at $t = 0$.

Putting $t = 0$, we obtain from (4) $v_0 = c_1$, $s_0 = c_2$. Substituting in equations (4), we obtain

$$v = pt + v_0, \quad s = \tfrac{1}{2}pt^2 + v_0 t + s_0.$$

In particular, if $v_0 = 0$ and $s_0 = 0$, we get

$$v = pt \quad \text{and} \quad s = \tfrac{1}{2}pt^2.$$

Example 3. **Motion along a cycloid.** Let a circle of radius r roll along a straight line. Examine the motion of an arbitrary point on the circumference of the circle.

Assume that at the beginning a given point P on the periphery of the circle is a point of tangency of the circle and the straight line. Let us

select this point as the origin of the coordinate system with the line as the x-axis. If the circle turns through an angle φ, the point will occupy a new position $P'(x, y)$. In order to express the coordinates x, y as functions of the angle φ, let us note that the new position of the point can be obtained

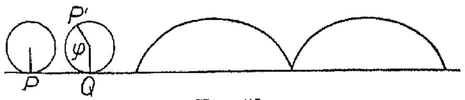

Fig. 53.

by first turning the circle clockwise about the centre through the angle φ and then translating it along the x-axis through the segment PQ which is equal to the arc $P'Q$ subtending the angle φ. Since the length of the arc $P'Q$ is $r\varphi$, it follows that $x = -r\sin\varphi + r\varphi$, $y = r - r\cos\varphi$, or:

$$x = r(\varphi - \sin\varphi), \qquad y = r(1 - \cos\varphi).$$

After one revolution of the circle, i. e. for $\varphi = 2\pi$, the point P is again a point of tangency, after which the motion repeats itself. The resulting curve consisting of congruent arcs is called the *cycloid*.

Assume that the circle revolves uniformly, i. e. that the angle φ is proportional to the time: $\varphi = \omega t$ (where ω is a constant). The equations of the motion of the point P are then

$$x = r(\omega t - \sin\omega t), \quad y = r(1 - \cos\omega t).$$

Differentiating them twice with respect to time, we obtain the components of velocity and acceleration

$$x^{\cdot} = r\omega(1 - \cos\omega t), \quad y^{\cdot} = r\omega\sin\omega t,$$
$$x^{\cdot\cdot} = r\omega^2\sin\omega t, \qquad y^{\cdot\cdot} = r\omega^2\cos\omega t.$$

Hence

$$|\mathbf{v}| = \sqrt{r^2\omega^2(2 - 2\cos\omega t)} = 2r\omega|\sin\tfrac{1}{2}\omega t|, \qquad |\mathbf{p}| = r\omega^2. \tag{5}$$

Because of the fact that the motion repeats itself after one complete revolution, we can confine ourselves to the time interval $0 \leq t \leq 2\pi/\omega$. From the above equations we see that the magnitude of the acceleration of the point P is constant, but the magnitude of the velocity changes. At $t = 0$ or $t = 2\pi/\omega$, i. e. when the point is on the line, the velocity is zero, whereas at $t = \pi/\omega$, i. e. when the point attains its highest position, the velocity is greatest and equal to $2r\omega$.

Let us give the cycloid a sense agreeing with that of the motion of the point. During the time $0 \leq t \leq 2\pi/\omega$ we have $v = |\mathbf{v}|$ and hence by (5)

$$v = 2r\omega\sin\tfrac{1}{2}\omega t.$$

The tangential acceleration will hence be

$$p_t = v^{\cdot} = r\omega^2\cos\tfrac{1}{2}\omega t.$$

In order to determine the normal acceleration, we calculate the curvature from a well-known formula

$$\frac{1}{\varrho} = \frac{|x^{\cdot}y^{\cdot\cdot} - y^{\cdot}x^{\cdot\cdot}|}{(x^{\cdot 2} + y^{\cdot 2})^{\frac{3}{2}}} = \frac{r^2\omega^3(1 - \cos\omega t)}{8r^3\omega^3\sin^3\omega t/2} = \frac{1}{4r\sin\omega t/2}.$$

Hence

$$p_n = v^2/\varrho = r\omega^2\sin\tfrac{1}{2}\omega t.$$

Since $\mathrm{d}s = v\,\mathrm{d}t$, the distance covered by the point up to the time t is

$$s = \int_0^t \mathrm{d}s = \int_0^t v\,\mathrm{d}t = \int_0^t 2\omega r\sin\tfrac{1}{2}\omega t\,\mathrm{d}t = 4r(1 - \cos\tfrac{1}{2}\omega t).$$

In particular, for $t = 2\pi/\omega$ we obtain the length of the path of the cycloid for one revolution of the circle. The length turns out to be $8r$.

§ 8. Angular velocity and acceleration. Let a point A move along a circle of radius r and centre M. Let us select a sense on the circle and an initial point O (Fig. 54). Denote by φ the angle between the radii MA and MO assumed to agree with the sense selected. We have $s = r\varphi$; therefore

$$s^{\cdot} = r\varphi^{\cdot} \quad \text{and} \quad s^{\cdot\cdot} = r\varphi^{\cdot\cdot}. \tag{1}$$

The derivative $\varphi^{\cdot}$, which is usually denoted by ω, is called the *angular velocity*.

The derivative $\varphi^{\cdot\cdot}$ is called the *angular acceleration* and it is denoted by ε. We obviously have $\varphi^{\cdot} = \omega$ and $\omega^{\cdot} = \varepsilon$. Therefore by (1)

$$v = r\omega \quad \text{and} \quad p_t = r\varepsilon, \tag{I}$$

and since $p_n = v^2/r$, it follows that

$$p_n = r\omega^2. \tag{II}$$

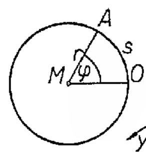

Fig. 54.

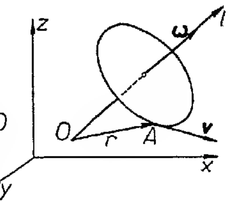

Fig. 55.

Angular velocity vector. Let a point rotate about a certain axis l. This means that it moves along a circle lying in a plane perpendicular to l, whose centre is the point of intersection of this plane and the axis l (Fig. 55).

Let the point have an angular velocity ω at the time t. Denote the vector lying on l and having a length $|\boldsymbol{\omega}|$ by $\boldsymbol{\omega}$. Select the sense of the

vector $\boldsymbol{\omega}$ in such a way that a person, having his head at the terminal point and his feet at the initial point of the vector, sees the motion proceeding from his right hand to his left hand.

The vector $\boldsymbol{\omega}$ is called the *angular velocity vector*.

It is easy to verify that the velocity $\mathbf{v}$ of the point A is equal to the moment of $\boldsymbol{\omega}$ with respect to A:

$$\mathbf{v} = \mathrm{Mom}_A \boldsymbol{\omega}. \tag{2}$$

If the vector $\overline{OA}$ is denoted by r, where O is an arbitrary point on the line l, then $\mathbf{v} = \boldsymbol{\omega} \times \overline{OA}$ and hence

$$\mathbf{v} = \mathbf{r} \times \boldsymbol{\omega}. \tag{III}$$

Denoting the projections of the vector $\boldsymbol{\omega}$ on the coordinate axes by $\omega_x, \omega_y, \omega_z$, the coordinates of the point A by x, y, z and the coordinates of the point O by x_0, y_0, z_0, we obtain

$$\begin{aligned} v_x = \omega_z(y - y_0) - \omega_y(z - z_0), \quad v_y &= \omega_x(z - z_0) - \omega_z(x - x_0), \\ v_z = \omega_y(x - x_0) &- \omega_x(y - y_0). \end{aligned} \tag{IV}$$

If, in particular, l passes through the origin of the system of coordinates, then, setting $x_0 = y_0 = z_0 = 0$, we obtain

$$v_x = \omega_z y - \omega_y z, \quad v_y = \omega_x z - \omega_z x, \quad v_z = \omega_y x - \omega_x y. \tag{V}$$

§ 9. Plane motion in a polar coordinate system. If a point moves in the xy-plane, then its position is completely determined by the length of the segment $OA = r$, called the *radius vector*, and the angle φ which OA makes with the x-axis. The motion of the point will therefore be defined by two functions

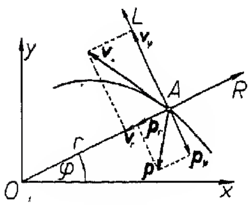

Fig. 56.

$$r = F(t), \quad \varphi = f(t).$$

Since $x = r\cos\varphi$, $y = r\sin\varphi$, taking derivatives with respect to t, we obtain:

$$x^{\cdot} = r^{\cdot}\cos\varphi - r\varphi^{\cdot}\sin\varphi, \quad y^{\cdot} = r^{\cdot}\sin\varphi - r\varphi^{\cdot}\cos\varphi, \tag{1}$$

$$\begin{aligned} x^{\cdot\cdot} &= r^{\cdot\cdot}\cos\varphi - 2r^{\cdot}\varphi^{\cdot}\sin\varphi - r\varphi^{\cdot 2}\cos\varphi - r\varphi^{\cdot\cdot}\sin\varphi, \\ y^{\cdot\cdot} &= r^{\cdot\cdot}\sin\varphi + 2r^{\cdot}\varphi^{\cdot}\cos\varphi - r\varphi^{\cdot 2}\sin\varphi + r\varphi^{\cdot\cdot}\cos\varphi. \end{aligned} \tag{2}$$

From equations (1) and (2) we can determine $x^{\cdot}, x^{\cdot\cdot}, y^{\cdot}, y^{\cdot\cdot}$ if we know $r^{\cdot}, r^{\cdot\cdot}, \varphi^{\cdot}, \varphi^{\cdot\cdot}$, and conversely. From (1) we obtain

$$v^2 = x^{\cdot 2} + y^{\cdot 2} = r^{\cdot 2} + r^2\varphi^{\cdot 2}, \tag{3}$$

$$r^{\cdot} = \frac{xx^{\cdot} + yy^{\cdot}}{\sqrt{x^2 + y^2}}, \qquad \varphi^{\cdot} = \frac{xy^{\cdot} - yx^{\cdot}}{x^2 + y^2}. \tag{4}$$

It is often convenient to resolve the velocity and acceleration not in the directions of the coordinate axes, but in the direction of the radius vector and a direction perpendicular to it, while the positive sense along these directions is chosen as in the figure. These components are called the *radial* and *transverse* components, respectively.

If a_x, a_y are the components of an arbitrary vector $\boldsymbol{a}$ beginning at the point $A(r, \varphi)$, then projecting this vector on the axis AR as well as on AL, we obtain for the radial component a_r and the transverse component a_φ:

$$a_r = a_x\cos\varphi + a_y\sin\varphi, \qquad a_\varphi = -a_x\sin\varphi + a_y\cos\varphi.$$

Applying these formulae to the velocity and acceleration vectors, we obtain by equations (1) and (2):

$$v_r = r^{\cdot}, \qquad v_\varphi = r\varphi^{\cdot}, \tag{I}$$

$$p_r = r^{\cdot\cdot} - r\varphi^{\cdot 2}, \quad p_\varphi = r\varphi^{\cdot\cdot} + 2r^{\cdot}\varphi^{\cdot} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\varphi^{\cdot}). \tag{II}$$

Example. A point moves along the spiral $r = a + b\varphi$ in such a way that the angle φ is proportional to the time t. Hence $\varphi = \omega t$, where ω is the factor of proportionality. Then $\varphi^{\cdot} = \omega$, $\varphi^{\cdot\cdot} = 0$, $r^{\cdot} = b\varphi^{\cdot} = b\omega$ and $r^{\cdot\cdot} = 0$, whence

$$v_r = b\omega, \qquad p_r = -(a + b\omega t)\,\omega^2, \qquad v_\varphi = (a + b\omega t)\,\omega, \qquad p_\varphi = 2b\omega^2.$$

§ 10. Areal velocity. Let a motion take place in the xy-plane. Denote the area of the region swept out by the radius vector during the time from t to $t + \Delta t$ by ΔS. From the formula for calculating areas in polar coordinates we obtain

$$\Delta S = \tfrac{1}{2}\int_{\varphi}^{\varphi+\Delta\varphi} r^2\,\mathrm{d}\varphi,$$

whence, on the basis of the mean value theorem,

$$\Delta S = \tfrac{1}{2}r_s^2\,\Delta\varphi, \tag{1}$$

where r_s denotes the mean value between the maximum and minimum of the radius r during the time from t to $t + \Delta t$.

The $\lim\limits_{\Delta t \to 0} \dfrac{\Delta S}{\Delta t}$ is called the *areal velocity* and it is denoted by A. Therefore by (1)

$$A = \tfrac{1}{2}r^2\varphi^{\cdot}. \tag{I}$$

From formulae (4) on p. 47 we obtain $\dot\varphi = \frac{1}{r^2}(x\dot y - y\dot x)$. Hence by (I)

$$A = \tfrac{1}{2}(x\dot y - y\dot x). \tag{II}$$

Example. Determine the velocity and acceleration of a point moving along the circle $x^2 - 2ax + y^2 = 0$ with a constant areal velocity h.

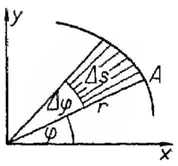

Fig. 57.

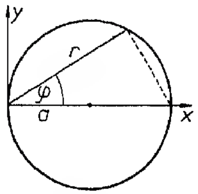

Fig. 58.

The equation of the given circle in polar coordinates is $r = 2a\cos\varphi$. The condition that the areal velocity be constant is expressed by

$$\tfrac{1}{2}r^2\dot\varphi = h, \quad \text{or} \quad \dot\varphi = 2h/r^2. \tag{2}$$

Differentiating the equation of the circle, we obtain

$$\dot r = -2a\dot\varphi\sin\varphi = -\frac{4ah}{r^2}\sin\varphi.$$

Therefore
$$v_r = -\frac{4ah}{r^2}\sin\varphi, \quad v_\varphi = \frac{2h}{r}.$$

As the areal velocity is constant, we have from formula (II), p. 47,

$$p_\varphi = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot\varphi),$$

it follows that $p_\varphi = 0$. Therefore the acceleration always has the direction of the radius vector.

Differentiating the first of the relations (2), we obtain $2r\dot r\dot\varphi + r^2\ddot\varphi = 0$ and hence

$$\ddot\varphi = -\frac{2\dot r\dot\varphi}{r} = \frac{16ah^2}{r^5}\sin\varphi;$$

differentiating $\dot r$, we get

$$\ddot r = -2a\ddot\varphi\sin\varphi - 2a\dot\varphi^2\cos\varphi = -2a\sin\varphi\frac{16ah^2}{r^5}\sin\varphi -$$
$$-2a\cos\varphi\frac{4h^2}{r^4} = -\frac{32a^2h^2}{r^5}\sin^2\varphi - 4a^2\cos^2\varphi\frac{4h^2}{r^5} = -\frac{16a^2h^2}{r^5}(1 + \sin^2\varphi).$$

Since

$$r\dot\varphi^2 = r\frac{4h^2}{r^4} = r^2\frac{4h^2}{r^5} = 4a^2\cos^2\varphi\frac{4h^2}{r^5} = \frac{16a^2h^2}{r^5}\cos^2\varphi,$$

from the formula $p_r = \ddot r - r\dot\varphi^2$ we obtain

$$p_r = -32a^2h^2/r^5.$$

§ 11. Dimensions of kinematic magnitudes. The measure of velocity, acceleration, etc. depends on the units of length and time. We shall concern ourselves with the question of how these measures change when the units of length and time undergo changes.

Let us select an arbitrary unit of length L and a unit of time T. Assume that a point in uniform motion traversed a path of length sL in the time tT (e. g. if L denotes cm, and T sec, then $sL = s$ cm, $tT = t$ sec). The measure of the velocity v using units L and T is

$$v = s/t.$$

Let us now select new units of length and time L', T'. Assume that the new units and the preceding are connected by the relations

$$L = \lambda L', \qquad T = \tau T', \tag{1}$$

where λ and τ indicate how many new units are contained in the old. Denoting the measures of length, time and velocity in the new units by s', t' and v', we obtain

$$v' = s'/t'.$$

Since $sL = s\lambda L'$ and $tT = t\tau T'$, it follows that $s' = s\lambda$, $t' = t\tau$, whence

$$v' = \frac{s\lambda}{t\tau} = \frac{s}{t}\cdot\frac{\lambda}{\tau}$$

and hence

$$v' = v\frac{\lambda}{\tau} \quad (\text{or } v' = v\lambda\tau^{-1}).$$

The unit of velocity (i. e. the velocity whose measure is 1) in units L and T is denoted by

$$\frac{L}{T} \text{ or } LT^{-1}.$$

The velocity whose measure is v we denote by

$$v\frac{L}{T} \quad \text{or} \quad vLT^{-1}.$$

For example, 5 cm · sec^{-1} denotes a velocity whose measure is 5, if the unit of length is the cm and the unit of time the sec.

Let us now suppose that we have chosen new units of length and time L' and T' which are connected with the old units by equations (1). Substitute in vLT^{-1} for L and T, $\lambda L'$ and $\tau T'$ respectively, and then transform the resulting expression as if the letters L' and T' denoted numbers. We obtain

$$vLT^{-1} = v\lambda L'\tau^{-1}T'^{-1} = v\lambda\tau^{-1}L'T'^{-1}.$$

Substitute $v' = v\lambda\tau^{-1}$. Then

$$vLT^{-1} = v'L'T'^{-1}.$$

Let us note that, according to the definition, v' is the measure of velocity in units L' and T'; consequently, $v'L'T'^{-1}$ represents the velocity in units of length and time L' and T'.

We see, therefore, that the symbol LT^{-1} enables us to determine the measure of the velocity in changed units by formal calculation.

Example. Determine the measure of the velocity 12 cm $\cdot$ sec^{-1} in units m and min.

We have cm $= 0.01$ m, sec $= \frac{1}{60}$ min. Calculating formally, we obtain 12 cm $\cdot$ sec^{-1} $= 12 \cdot 0.01$ m $\cdot$ $(\frac{1}{60}$ min$)^{-1} = 12 \cdot 0.01 \cdot 60$ m $\cdot$ min$^{-1} = 7.2$ m $\cdot$ min^{-1}.

Therefore 7.2 is the measure of the given velocity in units m and min.

The expression LT^{-1}, in which L and T do not denote particular units, but are symbols representing arbitrary units of length and time, is called the *dimension of velocity.*

General definition of dimension. The notion of dimension given above for velocity can be generalized to other magnitudes such as acceleration, angular velocity and acceleration, etc.

We shall call the *dimension* of any magnitude A the expression

$$L^\alpha T^\beta,$$

where the exponents α, β are numbers satisfying the condition: if a is the measure of the magnitude A in units of length and time L, T, and a' its measure in units L', T' connected with L and T by equations (1), then

$$a' = a\lambda^\alpha\tau^\beta. \tag{2}$$

The dimension of the magnitude A is denoted by $[A]$. Hence

$$[A] = L^\alpha T^\beta.$$

The unit of the magnitude A in units of length and time L, T is denoted by $L^\alpha T^\beta$. Therefore, if a is the measure of the magnitude A in terms of the unit $L^\alpha T^\beta$, then this magnitude is denoted by

$$aL^\alpha T^\beta.$$

Suppose now that we have introduced new units of length and time L', T' connected with the preceding by equations (1), p. 49. Calculating formally, we obtain $aL^\alpha T^\beta = a(\lambda L')^\alpha(\tau T')^\beta = a\lambda^\alpha L'^\alpha\tau^\beta T'^\beta$, and hence $aL^\alpha T^\beta = (a\lambda^\alpha\tau^\beta)\,L'^\alpha T'^\beta$. Therefore, denoting the measure of the given magnitude in units L' and T' by a', we obtain from (2)

$$aL^\alpha T^\beta = a'L'^\alpha T'^\beta. \tag{3}$$

Thus formal calculation permits us to determine the measure in terms of the changed units.

Example. The dimension of acceleration is LT^{-2} (as can be verified by employing the same method as used in connection with the velocity). Represent an acceleration of 5 cm $\cdot$/sec^{-2} in units m and min.

Since 1 cm $= 0.01$ m, 1 sec $= \frac{1}{60}$ min, it follows that $\lambda = 0.01$, $\tau = \frac{1}{60}$, $\alpha = 1$, $\beta = -2$, and hence by (3), 5 cm $\cdot$ sec^{-2} $= 5(0.01$ m$)(\frac{1}{60}$ min$)^{-2} =$ $= 5 \cdot 0.01 \cdot 60^2$ m $\cdot$ min^{-2}, or 5 cm $\cdot$ sec $= 180$ m $\cdot$ min^{-2}.

Determination of dimension. The following theorem is useful for the determination of dimensions:

Let there be given the magnitudes A and B for which

$$[A] = L^\alpha T^\beta, \quad [B] = L^\gamma T^\delta, \tag{4}$$

as well as a third magnitude C depending on A and B in such a manner that if we denote by a, b, c the measures of the magnitudes A, B, C expressed in arbitrary units L and T, we always have

$$c = ga^pb^q, \tag{5}$$

where the numbers g, p, q do not depend on the units L and T.

From these assumptions follows

$$[C] = L^{p\alpha+q\gamma}T^{p\beta+q\delta}. \tag{I}$$

This formula can be written in still another way. Dimension $[C]$ can be obtained if we reckon formally as follows:

$$[C] = (L^\alpha T^\beta)^p(L^\gamma T^\delta)^q = L^{p\alpha}T^{p\beta}L^{q\gamma}T^{q\delta} = L^{p\alpha+q\gamma}T^{p\beta+q\delta};$$

we can therefore give formula (I) in the form

$$[C] = [A]^p[B]^q.$$

Proof. Let us select new units L' and T' connected with the preceding by equations (1), p. 49. Denote the measures of the magnitudes A, B, C in terms of the new units by a', b', c'. According to assumption (5) we have $c' = ga'^pb'^q$. By (4), $a' = \lambda^\alpha\tau^\beta a$ and $b' = \lambda^\gamma\tau^\delta b$ hence $c' =$ $= g(\lambda^\alpha\tau^\beta a)^p(\lambda^\gamma\tau^\delta b)^q = \lambda^{p\alpha+q\gamma}\tau^{p\beta+q\delta}c$, and by the definition of dimension follows formula (I), q. e. d.

From the above theorem we obtain the following corollaries:

Corollary I. *If formula* (5) *has the form*

$c = ab$, then $[C] = [A][B] = L^{\alpha+\gamma}T^{\beta+\delta}$,
$c = a/b$, then $[C] = [A]/[B] = L^{\alpha-\gamma}T^{\beta-\delta}$,
$c = a^p$, then $[C] = [A]^p = L^{\alpha p}T^{\beta p}$.

Examples: 1. Velocity is expressed by the formula $v = s/t$. Therefore

$$[v] = [s]/[t] = L/T = LT^{-1}.$$

2. Acceleration in uniformly accelerated motion is given by the formula $p = v/t$. Hence

$$[p] = [v]/[t] = LT^{-1}/T = LT^{-2}.$$

A similar result is obtained if we use formula $s = \frac{1}{2}pt^2$. We calculate from it $p = 2s/t^2$. Therefore

$$[p] = [s]/[t]^2 = L/T^2 = LT^{-2}.$$

3. Angular velocity is given by $\omega = \mathrm{d}\varphi/\mathrm{d}t$. The dimension of the angle φ is L^0T^0 because the measure of φ is independent of the units of length. Hence

$$[\omega] = 1/T = T^{-1}.$$

4. Angular acceleration is given by $\varepsilon = \mathrm{d}^2\varphi/\mathrm{d}t^2$. Therefore

$$[\varepsilon] = 1/T^2 = T^{-2}.$$

Corollary 2. Certain constants can depend on the choice of the units of length and time. We can therefore speak of the *dimension* of these *constants*.

Example. In a certain motion the absolute value of the acceleration is proportional to the square of the velocity. Denoting the constant of proportionality by k, we have

$$p = kv^2.$$

In units of cm and sec, $k = 2$. Calculate k in units of m and min. We have $k = p/v^2$, from which $[k] = [p]/[v]^2 = LT^{-2}/(LT^{-1})^2$; hence $[k] = L^{-1}$ and

$$2\ \mathrm{cm}^{-1} = 2(\tfrac{1}{100}\ \mathrm{m})^{-1} = 200\ \mathrm{m}^{-1}.$$

Therefore $k = 200$ in a system of units m and min.

II. CHANGE OF FRAME OF REFERENCE

§ 12. Relation among coordinates. The velocity and acceleration of a point depend on the frame of reference relative to which the motion of the point is being examined. The motion of one and the same point will therefore be described differently by two observers moving relative to each other.

If we are travelling in a train, for instance, then the passengers travelling with us are at rest relative to us. To an observer standing near the tracks

the passengers move with the velocity of the train. We can express this in the following way: relative to a frame attached to the train the passengers are at rest, and relative to a frame attached to the earth the passengers move with the velocity of the train.

The motions of the planets and sun relative to a frame of reference attached to the earth are very complicated. Copernicus discovered that the motions of the planets are represented much more simply if we choose as a frame of reference a frame attached to the sun.

Let there be given a frame of reference $O(x, y, z)$ and a second frame $M(\xi, \eta, \zeta)$ moving relative to the former (Fig. 59). In order to differentiate one frame from the other, we shall call (x, y, z) a fixed frame and (ξ, η, ζ) a moving frame. The motion of one and the same point will be represented differently in both frames.

We shall be concerned with the problem of representing the motion of the point A relative to one frame when this motion is known relative to another frame.

This problem is very important and we shall meet it in many situations.

Denote by x_0, y_0, z_0 the coordinates of the origin M in the frame $O(x, y, z)$, and by ξ_0, η_0, ζ_0 the coordinates of the origin O in the frame $M(\xi, \eta, \zeta)$. Let $\alpha_1, \alpha_2, \ldots, \gamma_3$ be the angles between the axes of both frames as indicated in the table:

axes	ξ	η	ζ
x	α_1	α_2	α_3
y	β_1	β_2	β_3
z	γ_1	γ_2	γ_3

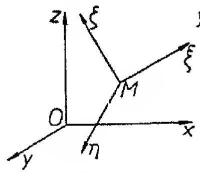

Fig. 59.

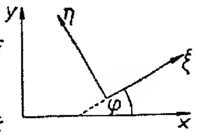

Fig. 60.

If x, y, z and ξ, η, ζ are the coordinates of the point A in the first and second frames, respectively, then, as it is known from analytic geometry,

$$\begin{aligned} x &= x_0 + \xi\cos\alpha_1 + \eta\cos\alpha_2 + \zeta\cos\alpha_3, \\ y &= y_0 + \xi\cos\beta_1 + \eta\cos\beta_2 + \zeta\cos\beta_3, \\ z &= z_0 + \xi\cos\gamma_1 + \eta\cos\gamma_2 + \zeta\cos\gamma_3, \end{aligned} \tag{I}$$

$$\begin{aligned} \xi &= \xi_0 + x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1, \\ \eta &= \eta_0 + x\cos\alpha_2 + y\cos\beta_2 + z\cos\gamma_2, \\ \zeta &= \zeta_0 + x\cos\alpha_3 + y\cos\beta_3 + z\cos\gamma_3. \end{aligned} \tag{I'}$$

The motion of the frame $M(\xi, \eta, \zeta)$ relative to $O(x, y, z)$ will be known if the coordinates x_0, y_0, z_0 and the angles $\alpha_1, \alpha_2, \ldots, \gamma_3$ are given for each moment t. Therefore x_0, y_0, z_0 and $\alpha_1, \alpha_2, \ldots, \gamma_3$ are functions of the time t. If the motion of A relative to $M(\xi, \eta, \zeta)$ is defined by the functions $\xi = f(t)$, $\eta = \varphi(t)$, $\zeta = \psi(t)$, then the motion relative to $O(x, y, z)$ is obtained from formulae (I)

$$x = x_0 + f(t)\cos\alpha_1 + \varphi(t)\cos\alpha_2 + \psi(t)\cos\alpha_3$$

and similarly for y and z.

If the motion of A takes place in the plane Π, then selecting axes x, y and ξ, η in this plane and denoting by φ the angle between the axes x and ξ (Fig. 60), we obtain

$$x = x_0 + \xi\cos\varphi - \eta\sin\varphi, \qquad y = y_0 + \xi\sin\varphi + \eta\cos\varphi, \tag{II}$$

$$\xi = \xi_0 + x\cos\varphi + y\sin\varphi, \qquad \eta = \eta_0 - x\sin\varphi + y\cos\varphi. \tag{II'}$$

If the directions of the axes of the moving frame $M(\xi, \eta, \zeta)$ do not change, then this frame is said to move with an *advancing* motion.

In this case the angles $\alpha_1, \alpha_2, \ldots, \gamma_3$ are constant.

We say that a moving frame revolves about the axis l with an angular velocity ω, if the points lying on the axes ξ, η, ζ revolve about the axis l with an angular velocity ω.

Let the frame $M(\xi, \eta, \zeta)$ revolve about the ζ-axis with an angular velocity ω. Let us assume that the fixed frame $O(x, y, z)$ coincided with the moving frame $M(\xi, \eta, \zeta)$ at the time $t = 0$. We then have $x_0 = y_0 = z_0 = \xi_0 = \eta_0 = \zeta_0 = 0$. Denote the angle between the x and ξ axes by φ. Obviously $\varphi = \omega t$. Since the axes x, y and ξ, η constantly lie in one plane, by (II) and (II'):

$$x = \xi\cos\omega t - \eta\sin\omega t, \qquad y = \xi\sin\omega t + \eta\cos\omega t, \qquad z = \zeta, \tag{III}$$

$$\xi = x\cos\omega t + y\sin\omega t, \qquad \eta = -x\sin\omega t + y\cos\omega t, \qquad \zeta = z. \tag{III'}$$

Example 1. The motion of a point relative to a fixed frame is defined by the equations

$$x = a\cos\omega t, \quad y = b\sin\omega t, \tag{1}$$

and hence the path of the point is an ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

How is the motion of this point represented in a moving frame with the same origin, if this frame revolves in a positive direction with an angular velocity ω?

We assume that both frames are coincident at the moment $t = 0$.

Denoting by ξ, η the coordinates of an arbitrary point relative to the moving frame, we have

$$\xi = x\cos\omega t + y\sin\omega t, \qquad \eta = -x\sin\omega t + y\cos\omega t,$$

for by hypothesis the angle φ between the x and ξ axes is equal to ωt. Substituting on the right side of these formulae the expressions (1), we obtain the equations of the path described by the point in the moving frame

$$\xi = a\cos^2\omega t + b\sin^2\omega t, \qquad \eta = (b - a)\sin\omega t\cos\omega t,$$

or, making use of the identities $\cos^2\omega t = \frac{1}{2}(1 + \cos 2\omega t)$, $\sin^2\omega t = \frac{1}{2}(1 - \cos 2\omega t)$ and $\sin\omega t\cos\omega t = \frac{1}{2}\sin 2\omega t$:

$$\xi = \frac{a+b}{2} + \frac{a-b}{2}\cos 2\omega t, \qquad \eta = \frac{b-a}{2}\sin 2\omega t,$$

whence

$$\left[\xi - \frac{a+b}{2}\right]^2 + \eta^2 = \left(\frac{a-b}{2}\right)^2.$$

Hence: *relative to the moving frame the point describes a circle when $a \neq b$, and it remains at rest when $a = b$.*

Example 2. Motion along a helix. An important example of the motion of a point along a space curve is *helical motion* which arises in the following manner. The moving frame (ξ, η, ζ) revolves with a constant angular velocity ω about the ζ-axis, while relative to the moving frame, the point moves uniformly with a velocity c along the line $\xi = r$, $\eta = 0$ (i. e. along the line parallel to the ζ-axis and cutting the ξ-axis in the point $\xi = r$).

Such a motion arises, for example, when a circular cylinder rotates about its axis with an angular velocity ω, while a point moves along the generatrix of the cylinder with uniform motion.

At $t = 0$ let the fixed frame (x, y, z) be coincident with the moving frame (ξ, η, ζ), while the moving point has the coordinates $(r, 0, 0)$. By (III)

$$x = r\cos\omega t, \qquad y = r\sin\omega t, \qquad z = ct. \tag{2}$$

The above equations can also be obtained directly from the figure. They represent the parametric equations of a helix. The motion

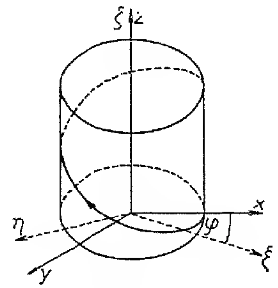

Fig. 61.

of the point therefore takes place along a helix. Differentiating (2) we obtain

$$x^{\cdot} = -r\omega \sin \omega t, \quad y^{\cdot} = r\omega \cos \omega t, \quad z^{\cdot} = c,$$
$$x^{\cdot\cdot} = -r\omega^2 \cos \omega t, \quad y^{\cdot\cdot} = -r\omega^2 \sin \omega t, \quad z^{\cdot\cdot} = 0.$$

Consequently

$$|\mathbf{v}| = \sqrt{r^2\omega^2 + c^2}, \quad |\mathbf{p}| = r\omega^2.$$

Hence: *in motion along a helix the absolute values of the velocity and acceleration are constant.*

§ 13. Relation among velocities. The velocity of a point A relative to a fixed frame (x, y, z) is called *absolute velocity*; we denote it by $\mathbf{v}_a$.

The velocity of a point relative to a moving frame is called *relative velocity*; we denote it by $\mathbf{v}_r$.

Let us imagine that the point A, whose motion we are investigating, is attached *rigidly* to the moving frame (ξ, η, ζ), i. e. that its coordinates ξ, η, ζ do not change. Under this assumption the point A connected with the moving frame would possess a certain velocity relative to the fixed frame. This velocity is called the *velocity of transport* and we denote it by $\mathbf{v}_t$.

We can also say that the velocity of transport of the point A at a given moment is the velocity of a point attached to the moving frame and coinciding at the given moment with the point A.

Let us suppose, for example, that a passenger is running along the aisle of a train. As the fixed frame let us take the frame attached to the earth, as the moving frame — the frame attached to the train.

A person standing near the track will observe the motion of the passenger relative to the fixed frame, and a person sitting in the car — relative to the moving frame.

The velocity of the passenger, as observed by the person near the track, will be *absolute velocity*. The velocity, as observed by the passenger sitting in the car, will be *relative velocity*. The *velocity of transport* will be the velocity of that point on the floor of the aisle which is touched at the given moment by the passenger running along the aisle.

The velocity of transport in this case will therefore be the velocity of the train. The absolute velocity will be greater or smaller than the velocity of transport depending on whether the passenger runs in the same or opposite direction of the motion of the train.

We shall now concern ourselves with the relations that obtain among the absolute, relative and transport velocities.

The point A has the coordinates x, y, z relative to the fixed frame.

Consequently the projections of the absolute velocity on the axes of the fixed frame will be:

$$v_{a_x} = x^{\cdot}, \quad v_{a_y} = y^{\cdot}, \quad v_{a_z} = z^{\cdot}. \tag{1}$$

Similarly, the projections of the relative velocity on the axes of the moving frame will be:

$$v_{r_\xi} = \xi^{\cdot}, \quad v_{r_\eta} = \eta^{\cdot}, \quad v_{r_\zeta} = \zeta^{\cdot}. \tag{2}$$

In order to equate the absolute velocity with the relative velocity, let us form the projections of the relative velocity on the axes of the fixed frame. We obtain

$$v_{r_x} = \xi^{\cdot} \cos \alpha_1 + \eta^{\cdot} \cos \alpha_2 + \zeta^{\cdot} \cos \alpha_3, \qquad \text{etc.} \tag{3}$$

By (I), p. 53, the coordinates x, y, z relative to the fixed frame are

$$x = x_0 + \xi \cos \alpha_1 + \eta \cos \alpha_2 + \zeta \cos \alpha_3, \qquad \text{etc.}$$

Differentiating the above expression, we get from (1)

$$v_{a_x} = x^{\cdot} = x_0^{\cdot} + \xi \frac{d \cos \alpha_1}{dt} + \eta \frac{d \cos \alpha_2}{dt} + \zeta \frac{d \cos \alpha_3}{dt} + \xi^{\cdot} \cos \alpha_1 + \\ + \eta^{\cdot} \cos \alpha_2 + \zeta^{\cdot} \cos \alpha_3. \tag{4}$$

The velocity of transport is obtained by supposing that the point A is attached rigidly to the moving frame, i. e. that the coordinates ξ, η, ζ are constant, or that $\xi^{\cdot} = 0$, $\eta^{\cdot} = 0$, $\zeta^{\cdot} = 0$. Therefore in virtue of (4) the projections of the velocity of transport on the axes of the frame (x, y, z) are

$$v_{t_x} = x_0^{\cdot} + \xi \frac{d \cos \alpha_1}{dt} + \eta \frac{d \cos \alpha_2}{dt} + \zeta \frac{d \cos \alpha_3}{dt}, \qquad \text{etc.} \tag{5}$$

By (3) and (5) we obtain from (4) $v_{a_x} = v_{t_x} + v_{r_x}$ and similarly $v_{a_y} = = v_{t_y} + v_{r_y}$, $v_{a_z} = v_{t_z} + v_{r_z}$, or

$$\mathbf{v}_a = \mathbf{v}_t + \mathbf{v}_r. \tag{I}$$

We have thus proved that *the absolute velocity is equal to the sum of the velocity of transport and the relative velocity.*

When the moving frame moves with an advancing motion the angles $\alpha_1, \alpha_2, \ldots, \gamma_3$ are constant; hence the derivatives

$$\frac{d \cos \alpha_1}{dt}, \frac{d \cos \alpha_2}{dt}, \ldots, \frac{d \cos \gamma_3}{dt}$$

are zero. Therefore from formula (5) we obtain

$$v_{t_x} = x_0^{\cdot}, \quad v_{t_y} = y_0^{\cdot}, \quad v_{t_z} = z_0^{\cdot}.$$

Hence, if $\mathbf{v}_0$ is the velocity of the origin of the moving frame, then

$$\mathbf{v}_t = \mathbf{v}_0.$$

Therefore: *if a frame moves with an advancing motion, then the velocity of transport is the same for all points and equal to the velocity of the origin of the frame.*

Remark. We say that the point A executes two motions simultaneously: one with relative velocity, the other with velocity of transport. The motion relative to a fixed frame is termed a *compound motion* of both component motions or their *resultant* motion.

The velocity of the resultant motion is therefore the sum of the velocities of the component motions. In order that the velocity of the resultant motion be defined, it is sufficient to give the velocities of the component motions; it is not necessary to say, in addition to this, which of the velocities is relative and which is the velocity of transport.

Example 1. A train is moving with a velocity $\mathbf{u}$; along the floor of a car a point A rolls with a velocity $\mathbf{v}$ relative to the floor. What is the velocity of the point A relative to the earth?

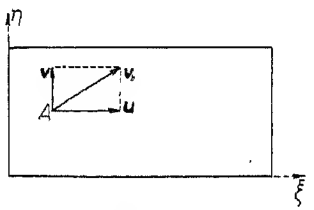

Fig. 62.

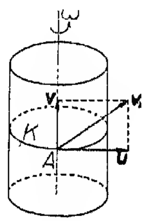

Fig. 63.

Let us assume that the axes of the frame (x, y, z) are attached to the earth and the axes ξ, η, ζ to the car (Fig. 62). The velocity of transport of the point A is therefore $\mathbf{u}$, because this is the velocity A would have relative to the earth were it at rest relative to the car. Let the relative velocity of A be $\mathbf{v}$. Hence its absolute velocity $\mathbf{v}_a$ (i. e. the velocity relative to the earth) is

$$\mathbf{v}_a = \mathbf{u} + \mathbf{v}.$$

Example 2. A cylinder revolves about its axis with an angular velocity ω. A point A moves along the generatrix with a velocity $\mathbf{v}$ (relative to the generatrix). What is the absolute velocity of the point A?

The relative velocity is $\mathbf{v}$. In order to determine the velocity of

transport, let us note that if the point A were attached to the cylinder, then it would move along the circle K with an angular velocity ω (Fig. 63). Therefore, if r denotes the radius of the base of the cylinder, then the velocity of transport $\mathbf{u}$ is tangent to the circle K and $|\mathbf{u}| = r\omega$. The absolute velocity is then $\mathbf{v}_a = \mathbf{v} + \mathbf{u}$, and since $\mathbf{v} \perp \mathbf{u}$,

$$|\mathbf{v}_a| = \sqrt{v^2 + u^2} = \sqrt{v^2 + r^2\omega^2} \qquad (v = |\mathbf{v}|, \quad u = |\mathbf{u}|).$$

§ 14. Relations among accelerations. We shall consider now the relations that obtain among the accelerations of a point relative to various frames. Let us assume that we have two frames: a fixed (x, y, z) and a moving (ξ, η, ζ).

The acceleration of the point A relative to a fixed frame is called the *absolute acceleration* $\mathbf{p}_a$.

The acceleration of a point relative to a moving frame is called the *relative acceleration* $\mathbf{p}_r$.

The acceleration which a point A would possess (relative to a fixed frame), were it attached rigidly to a moving frame, is called the *acceleration of transport* $\mathbf{p}_t$.

We can also say that the acceleration of transport is the acceleration of that point attached to a moving frame which coincides with the point A at a given moment.

For example, suppose that a passenger runs along the aisle of a car. If we select as the fixed frame a frame attached to the earth, and as the moving frame a frame attached to the car, then: the absolute acceleration will be the acceleration observed by a person standing near the track, the relative acceleration will be the acceleration observed by the passenger travelling in this car, finally, the acceleration of transport will be the acceleration relative to the earth of that point on the floor which the running passenger touches at a given moment.

Denote by x, y, z the coordinates of the point A relative to the fixed frame, and by ξ, η, ζ those relative to the moving frame.

The projections of the absolute acceleration $\mathbf{p}_a$ on the x, y, z axes are:

$$p_{a_x} = x^{\cdot\cdot}, \qquad p_{a_y} = y^{\cdot\cdot}, \qquad p_{a_z} = z^{\cdot\cdot}. \tag{1}$$

The projections of the relative acceleration $\mathbf{p}_r$ on the ξ, η, ζ axes are:

$$p_{r_\xi} = \xi^{\cdot\cdot}, \qquad p_{r_\eta} = \eta^{\cdot\cdot}, \qquad p_{r_\zeta} = \zeta^{\cdot\cdot}. \tag{2}$$

Let us form the projections of the vector $\mathbf{p}_r$ on the axes of the fixed frame. We obtain

$$p_{r_x} = \xi^{\cdot\cdot} \cos\alpha_1 + \eta^{\cdot\cdot} \cos\alpha_2 + \zeta^{\cdot\cdot} \cos\alpha_3, \qquad \text{etc.} \tag{3}$$

By (I), p. 53, we have

$$x = x_0 + \xi \cos\alpha_1 + \eta \cos\alpha_2 + \zeta \cos\alpha_3, \qquad \text{etc.} \quad (4)$$

We obtain the acceleration of transport by assuming that the point A is rigidly attached to the moving frame, or that ξ, η, ζ are constants, and therefore that the derivatives $\xi^{\cdot}, \eta^{\cdot}, \zeta^{\cdot}, \xi^{\cdot\cdot}, \eta^{\cdot\cdot}, \zeta^{\cdot\cdot}$ are equal to zero.

The projections of the vector $\mathbf{p}_t$ on the x, y, z axes are obtained by differentiating (4) twice under the assumption that ξ, η, ζ are constants:

$$p_{t_x} = x_0^{\cdot\cdot} + \xi \frac{d^2 \cos\alpha_1}{dt^2} + \eta \frac{d^2 \cos\alpha_2}{dt^2} + \zeta \frac{d^2 \cos\alpha_3}{dt^2}, \qquad \text{etc.} \quad (5)$$

If $\alpha_1, \alpha_2, \ldots$ are constants, then $p_{t_x} = x_0^{\cdot\cdot}$.

Therefore: *if a frame moves with an advancing motion, then the acceleration of transport is for all points equal to the acceleration of the origin of the frame.*

Let us differentiate (4) twice. We obtain

$$\begin{aligned} x^{\cdot\cdot} = x_0^{\cdot\cdot} &+ \xi \frac{d^2\cos\alpha_1}{dt^2} + \eta \frac{d^2\cos\alpha_2}{dt^2} + \zeta \frac{d^2\cos\alpha_3}{dt^2} + \\ &+ \xi^{\cdot\cdot}\cos\alpha_1 + \eta^{\cdot\cdot}\cos\alpha_2 + \zeta^{\cdot\cdot}\cos\alpha_3 + \qquad (6) \\ &+ 2\left(\xi^{\cdot}\frac{d\cos\alpha_1}{dt} + \eta^{\cdot}\frac{d\cos\alpha_2}{dt} + \zeta^{\cdot}\frac{d\cos\alpha_3}{dt}\right), \qquad \text{etc.} \end{aligned}$$

According to (3) and (5) the expressions in the first and second lines denote the projections of $\mathbf{p}_t$ and $\mathbf{p}_r$ on the x, y, z axes.

Denote by $\mathbf{p}_C$ the vector whose projections on the x, y and z axes are expressed by the formulae

$$p_{C_x} = 2\left(\xi^{\cdot}\frac{d\cos\alpha_1}{dt} + \eta^{\cdot}\frac{d\cos\alpha_2}{dt} + \zeta^{\cdot}\frac{d\cos\alpha_3}{dt}\right), \qquad \text{etc.} \quad (7)$$

The vector $\mathbf{p}_C$ is called the *acceleration of Coriolis.*

In virtue of (1), (3), (5) and (7), formula (6) can be written in the form $p_{a_x} = p_{t_x} + p_{r_x} + p_{C_x}$. Similarly, we obtain $p_{a_y} = p_{t_y} + p_{r_y} + p_{C_y}$ and $p_{a_z} = p_{t_z} + p_{r_z} + p_{C_z}$. We may therefore write

$$\mathbf{p}_a = \mathbf{p}_t + \mathbf{p}_r + \mathbf{p}_C. \qquad (I)$$

Hence: *the absolute acceleration is equal to the sum of the accelerations: transport, relative, and Coriolis.*

Acceleration of Coriolis. In order to understand the meaning of the acceleration of Coriolis, draw from the origin M of the moving frame the vector of relative velocity $\overline{MB} = \mathbf{v}_r$. The coordinates of the point B relative to the frame (ξ, η, ζ) are $v_{r_\xi}, v_{r_\eta}, v_{r_\zeta}$ or $\xi^{\cdot}, \eta^{\cdot}, \zeta^{\cdot}$. Let us imagine that the point B is rigidly attached to the frame (ξ, η, ζ). The velocity $\mathbf{u}$ of the point B relative to the fixed frame (x, y, z) is therefore equal to its velocity of transport (because its relative velocity is zero). Writing $\xi^{\cdot}, \eta^{\cdot}, \zeta^{\cdot}$ instead of ξ, η, ζ, we obtain by (5), p. 57:

$$u_x = x_0^{\cdot} + \xi^{\cdot}\frac{d\cos\alpha_1}{dt} + \eta^{\cdot}\frac{d\cos\alpha_2}{dt} + \zeta^{\cdot}\frac{d\cos\alpha_3}{dt}, \qquad \text{etc.}$$

Equating with formula (7), we obtain:

$$u_x = x_0^{\cdot} + \tfrac{1}{2}p_{C_x}, \quad u_y = y_0^{\cdot} + \tfrac{1}{2}p_{C_y}, \quad u_z = z_0^{\cdot} + \tfrac{1}{2}p_{C_z}.$$

Therefore, if we denote by $\mathbf{v}_0$ the velocity of the origin of the moving frame, then $\mathbf{u} = \mathbf{v}_0 + \frac{1}{2}\mathbf{p}_C$, whence

$$\mathbf{p}_C = 2(\mathbf{u} - \mathbf{v}_0). \qquad (II)$$

The difference $\mathbf{u} - \mathbf{v}_0$ is the velocity of the point B relative to the origin M of the moving frame (ξ, η, ζ) (cf. § 15, p. 65, (I)).

Therefore: *in order to obtain the acceleration of Coriolis, draw from the origin of the moving frame the vector of relative velocity $\mathbf{v}_r$ and imagine that this vector is attached rigidly to the moving frame.*

The acceleration of Coriolis is equal to twice the velocity of the end point of the vector $\mathbf{v}_r$ relative to its initial point.

It follows from this that *the acceleration of Coriolis is zero if the relative velocity is zero, or if the moving frame moves with an advancing motion.* For in these cases the origin and the terminus of the vector $\mathbf{v}_r$ have the same velocity. This can also be easily deduced from formula (7) by putting $\xi^{\cdot} = 0$, $\eta^{\cdot} = 0$, $\zeta^{\cdot} = 0$ or $\alpha_1 = \text{const}$, $\alpha_2 = \text{const}$, $\alpha_3 = \text{const}$, etc.

Let the frame (ξ, η, ζ) revolve about a certain axis l with an angular velocity $\boldsymbol{\omega}$ (Fig. 64). Maintaining the previous notation and selecting an arbitrary point O on the axis l, we obtain

$$\mathbf{u} = \overline{OB} \times \boldsymbol{\omega}, \quad \mathbf{v}_0 = \overline{OM} \times \boldsymbol{\omega}.$$

Then $\mathbf{p}_C = 2(\mathbf{u} - \mathbf{v}_0) = 2(\overline{OB} - \overline{OM}) \times \boldsymbol{\omega}$. But $\overline{OB} - \overline{OM} = v_r$. Hence

$$\mathbf{p}_C = 2\mathbf{v}_r \times \boldsymbol{\omega}. \qquad (III)$$

Hence we see that *the acceleration of Coriolis is equal to twice the vector product of the relative velocity vector and the angular velocity vector.*

The acceleration of Coriolis is therefore perpendicular to the axis of revolution and the relative velocity, and its magnitude is

$$|\boldsymbol{p}_C| = 2|\boldsymbol{\omega}||\mathbf{v}_r| \sin \alpha,$$

where α is the angle between the axis of revolution and the relative velocity vector. Hence it follows that the acceleration of Coriolis is zero (in addition to the case when $\mathbf{v}_r = 0$) when $\alpha = 0$, i. e. when $\mathbf{v}_r$ is parallel to $\boldsymbol{\omega}$.

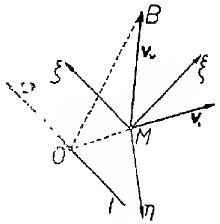

Fig. 64.

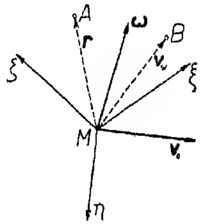

Fig. 65.

We shall show later (in chapter VII) that at every moment the velocities of the points attached rigidly to the moving frame (ξ, η, ζ) are such, as if the frame were executing two motions simultaneously: the first, an advancing motion with a velocity equal to that of the origin of the frame, and the second, a rotation about a certain axis, which passes through the origin of the frame, with an angular velocity $\boldsymbol{\omega}$.

This axis of rotation is called the *instantaneous axis of rotation*; $\boldsymbol{\omega}$ is called the *instantaneous angular velocity*.

At each moment of time there may be a different instantaneous axis of rotation and different $\boldsymbol{\omega}$'s. Therefore, denoting by $\mathbf{v}_0$ the velocity of the origin M of the frame, we obtain for the velocity of transport $\mathbf{v}_t$ of the point A the formula

$$\mathbf{v}_t = \mathbf{v}_0 + \overline{MA} \times \boldsymbol{\omega}.$$

Hence, if we denote by B the end of the relative velocity vector drawn from the origin of the frame, and by $\boldsymbol{u}$ (as before) the velocity of transport of the point B, then $\boldsymbol{u} = \mathbf{v}_0 + \overline{MB} \times \boldsymbol{\omega} = \mathbf{v}_0 + v_r \times \boldsymbol{\omega}$, whence by (II), p. 61

$$\boldsymbol{p}_C = 2\mathbf{v}_r \times \boldsymbol{\omega}. \tag{IV}$$

Formula (IV) represents the acceleration of Coriolis in the general case.

The acceleration of Coriolis is therefore zero: 1° *when* $\boldsymbol{\omega} = 0$ (i. e. when the frame moves with an advancing motion), 2° *when* $\mathbf{v}_r = 0$, 3° *when* $\boldsymbol{\omega} \parallel \mathbf{v}_r$.

Example 1. A train moves along a straight track with an acceleration $\boldsymbol{p}$. A point moves along the floor of the carriage with an acceleration $\boldsymbol{a}$ relative to the carriage. Determine the acceleration of the point relative to the earth.

The relative acceleration is $\boldsymbol{p}_r = \boldsymbol{a}$ and the acceleration of transport is $\boldsymbol{p}_t = \boldsymbol{p}$. Since the frame of reference attached to the carriage moves with an advancing motion, the acceleration of Coriolis $\boldsymbol{p}_C = 0$. Therefore the absolute acceleration (i. e. the acceleration relative to the earth) is

$$\boldsymbol{p}_a = \boldsymbol{a} + \boldsymbol{p}.$$

Example 2. A cylinder of radius r rotates about an axis with an angular velocity ω. A point A moves along the generatrix of the cylinder with a constant velocity $\mathbf{v}$ relative to the generatrix. Determine the acceleration of the point A relative to the fixed frame.

Taking the axis of the cylinder as the ζ-axis, let us select a moving frame attached to the cylinder. If the point A were attached rigidly to the frame (ξ, η, ζ), then it would move along the circle K with an angular velocity ω. The acceleration of transport $\boldsymbol{p}_t$ is therefore directed towards the centre of the circle K and $|\boldsymbol{p}_t| = r\omega^2$. By hypothesis, the relative acceleration $\boldsymbol{p}_r = 0$. Since the relative velocity $\mathbf{v}_r = \mathbf{v}$ is parallel to the axis of rotation, the acceleration of Coriolis $\boldsymbol{p}_C = 0$. Hence

$$\boldsymbol{p}_a = \boldsymbol{p}_t.$$

Example 3. A horizontal plane revolves about a vertical axis with an angular velocity ω. A point A moves along the plane and at a certain moment has a velocity $\mathbf{v}_r$ and an acceleration $\boldsymbol{p}_r$ relative to the plane. Determine the acceleration relative to the fixed frame at that moment.

Let O be the point of intersection of the axis of revolution with the moving plane. If the point A were attached to the moving plane, then it would move along a circle with centre at O and radius $OA = r$ with an angular velocity ω. Hence the acceleration of transport $\boldsymbol{p}_t$ is directed towards O and $|\boldsymbol{p}_t| = r\omega^2$.

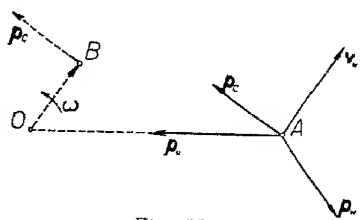

Fig. 66.

In order to determine the acceleration of Coriolis $\boldsymbol{p}_C$, select a point O as the origin of the moving frame attached to the moving plane and draw from O the vector $\overline{OB} = \mathbf{v}_r$. Since the velocity of the point O is zero, $\frac{1}{2}\boldsymbol{p}_C$ is equal to the velocity of the point B. Therefore $\boldsymbol{p}_C \perp \mathbf{v}_r$ and $|\boldsymbol{p}_C| = 2\,OB\,\omega = 2|\mathbf{v}_r|\omega$.

We should obviously have obtained the same result if we had used the formula $\boldsymbol{p}_C = 2\mathbf{v}_r \times \boldsymbol{\omega}$, and the fact that the angular velocity vector $\boldsymbol{\omega}$ has the direction of the axis of revolution and is therefore perpendicular to the moving plane.

The absolute acceleration is obtained by adding together vectors $\boldsymbol{p}_r$, $\boldsymbol{p}_t$ and $\boldsymbol{p}_C$.

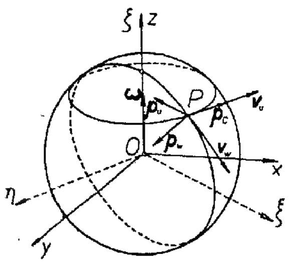
Fig. 67.

Example 4. A sphere of radius r revolves about a fixed axis with a constant angular velocity $\boldsymbol{\omega}$. A point P moves with a constant velocity $\boldsymbol{c}$ along a great circle passing through the axis of revolution. Determine the velocity and acceleration of this point relative to the fixed frame (x, y, z).

Let (ξ, η, ζ) denote the moving frame whose ζ-axis is the same as that of the fixed frame, while the plane $\xi\zeta$ is the plane of the meridian along which the point P moves. This frame revolves along with the sphere about the z-axis with a constant angular velocity $\boldsymbol{\omega}$ (which in the figure is represented by the vector drawn on the z-axis). The velocity of the point P relative to the moving frame, i. e. the relative velocity $\mathbf{v}_r$, is a vector of length c tangent to the meridian (along which the point P moves). The velocity of transport $\mathbf{v}_t$ is equal to the velocity of a point of the parallel of latitude passing through P. Since this point moves along a circle of radius $\varrho = r\cos\varphi$, where φ denotes the latitude of this parallel of latitude (i. e. the angle between OP and the equatorial plane), the velocity $\mathbf{v}_t$ is tangent to the parallel of latitude and is equal to $\varrho\omega = r\omega\cos\varphi$. The velocities $\mathbf{v}_r$ and $\mathbf{v}_t$ are perpendicular to each other; hence $|\mathbf{v}_a| = \sqrt{c^2 + r^2\omega^2\cos^2\varphi}$. The absolute velocity $\mathbf{v}_a$ forms with the meridians an angle Θ defined by the formula $\tan\Theta = |\mathbf{v}_t| / |\mathbf{v}_r| = (r\omega / c)\cos\varphi$. Since the point P moves along the meridian with a constant velocity c, its relative acceleration $\boldsymbol{p}_r$ is directed towards the centre of the sphere and $|\boldsymbol{p}_r| = c^2/r$. Similarly, the acceleration of transport $\boldsymbol{p}_t$ is directed towards the centre of the parallel of latitude (passing through P) and $|\boldsymbol{p}_t| = \varrho\omega^2 = r\omega^2\cos\varphi$. The acceleration of Coriolis $\boldsymbol{p}_C = 2\mathbf{v}_r \times \boldsymbol{\omega}$ is perpendicular to $\boldsymbol{\omega}$ and $\mathbf{v}_r$ and hence perpendicular to the equatorial plane and has the same sense as $\mathbf{v}_t$. Since, as is easily seen, the angle between $\boldsymbol{\omega}$ and $\mathbf{v}_r$ is $\pi - \varphi$, $|\boldsymbol{p}_C| = 2|\mathbf{v}_r||\boldsymbol{\omega}|\sin\varphi = 2c\,\omega\sin\varphi$. Adding together the vectors $\boldsymbol{p}_r$, $\boldsymbol{p}_t$ and $\boldsymbol{p}_C$, we obtain the absolute acceleration $\boldsymbol{p}_a$.

§ 15. Determination of relative motion. Up to this time we have been concerned with the determination of the motion relative to a fixed frame, having been given the motion relative to a moving frame. We frequently encounter the converse problem, i. e. we have to find the motion relative to a moving frame, knowing this motion relative to a fixed frame.

From formulae (I), pp. 60 and 57, we obtain for the relative velocity and relative acceleration:

$$\mathbf{v}_r = \mathbf{v}_a - \mathbf{v}_t, \quad \boldsymbol{p}_r = \boldsymbol{p}_a - \boldsymbol{p}_t - \boldsymbol{p}_C. \tag{I}$$

Therefore: *the relative velocity is obtained by adding to the absolute velocity the velocity of transport with an opposite sense. The relative acceleration is obtained by adding to the absolute acceleration the acceleration of transport and Coriolis with opposite senses.*

Motion relative to a point. Let the points A_1 and A_2 move relative to a certain fixed frame (x, y, z) with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. Let us place the origin of the moving frame (ξ, η, ζ) at A_2, and let us assume that this frame moves with an advancing motion.

The motion of the point A_1 relative to the frame (ξ, η, ζ) will be called *relative motion with respect to the point A_2*.

Such a motion would be observed by a person moving with an advancing motion together with the point A_2.

Let us denote by $\mathbf{v}_{1,2}$ the velocity of the point A_1 relative to the point A_2, i. e. relative to the frame (ξ, η, ζ). Since the velocity of transport is equal to the velocity $\mathbf{v}_2$ of the point A_2, and the absolute velocity is equal to $\mathbf{v}_1$, it follows $\mathbf{v}_1 = \mathbf{v}_{1,2} + \mathbf{v}_2$, whence $\mathbf{v}_{1,2} = \mathbf{v}_1 - \mathbf{v}_2$, or

$$\mathbf{v}_{1,2} = \mathbf{v}_1 + (-\mathbf{v}_2). \tag{II}$$

Denoting by $\boldsymbol{p}_{1,2}$ the acceleration of the point A_1 relative to A_2, i. e. relative to the frame (ξ, η, ζ), and observing that the acceleration of Coriolis is zero, we obtain $\boldsymbol{p}_{1,2} = \boldsymbol{p}_1 - \boldsymbol{p}_2$, or

$$\boldsymbol{p}_{1,2} = \boldsymbol{p}_1 + (-\boldsymbol{p}_2). \tag{III}$$

Therefore: *the velocity (acceleration) of a point A_1 relative to A_2 is obtained by adding to the velocity (acceleration) of A_1 the velocity (acceleration) of the point A_2 with an opposite sense*

Example 1. The points A_1 and A_2 move uniformly along the x and y axes, respectively, with the velocities c_1 and c_2. Determine the velocity of the point A_1 relative to A_2.

The sought for velocity $\mathbf{v}_{1,2}$ is the difference between the velocities

of A_1 and A_2. Hence $\mathbf{v}_{1,2} = \mathbf{v}_1 - \mathbf{v}_2$. The projections of $\mathbf{v}_{1,2}$ on the x and y axes are c_1 and $-c_2$. Therefore

$$|\mathbf{v}_{1,2}| = \sqrt{c_1^2 + c_2^2}.$$

Example 2. A point A_1 moves along a circle of radius r with uniform motion, while the point A_2 moves in such way that it is always at the other end of the diameter passing through A_1. Determine the velocity and acceleration of the point A_1 relative to A_2.

Obviously, the velocity and acceleration of both points are equal in absolute value and have opposite senses. Denoting by $\mathbf{v}$ and $\mathbf{p}$ the velocity and acceleration of the point A_1, we obtain

$$\mathbf{v}_{1,2} = \mathbf{v} - (-\mathbf{v}) = 2\mathbf{v}, \quad \text{and} \quad \mathbf{p}_{1,2} = \mathbf{p} - (-\mathbf{p}) = 2\mathbf{p}.$$

Since $|\mathbf{v}_{1,2}| = \text{const}$, the tangential acceleration of the relative motion is zero; consequently $\mathbf{p}_{1,2}$ is the normal acceleration. Hence $|\mathbf{p}_{1,2}| = |\mathbf{v}_{1,2}|^2 / \varrho = 4|\mathbf{v}|^2 / \varrho$, and since $|\mathbf{p}_{1,2}| = 2|\mathbf{p}| = 2|\mathbf{v}|^2 / r$,

$$4|\mathbf{v}|^2 / \varrho = 2|\mathbf{v}|^2 / r, \quad \text{or} \quad \varrho = 2r.$$

Therefore the motion of the point A_1 relative to A_2 takes place along a circle with centre at A_2 and radius $2r$, with a velocity twice as large as the velocity of the point A_2.

Example 3. A body A moves along the x-axis with a constant velocity u, and emits every T seconds small particles which move with uniform motion along the x-axis with a velocity c. Let ν denote the frequency of emission (i. e. the number of particles emitted per second), and λ the distance between two successively emitted particles. We obviously have $\nu = 1 / T$.

Since the relative velocity of an emitted particle with respect to A is $c - u$, the distance of the particle from A after the time T is $\lambda = (c - u)\,T$. Consequently

$$\nu = (c - u) / \lambda. \tag{1}$$

Let us suppose now that an observer B moves along the x-axis with a constant velocity v. Let us denote by ν' the relative frequency of emission (i. e. the number of particles per second met by an observer), and by T' the time between the meetings of two successive particles. Since the velocity of the particles relative to B is $c - v$, $\lambda = (c - v)\,T'$. Hence

$$\nu' = (c - v) / \lambda. \tag{2}$$

In virtue of (1) and (2) we obtain

$$\nu' = \nu(c - v) / (c - u) = \nu(1 - v / c) / (1 - u / c). \tag{3}$$

Let us assume that the velocity c is large as compared with u and v. Since for small x's we have $1 / (1 - x) = 1 + x$, from (3)

$$\nu' = \nu\left(1 - \frac{v}{c}\right)\left(1 + \frac{u}{c}\right) = \nu\left[1 - \frac{v - u}{c} - \frac{vu}{c^2}\right].$$

Neglecting the last term enclosed by the brackets as very small compared with what remains, we finally have

$$\nu' = \nu[1 - (v - u) / c]. \tag{4}$$

In particular, if $u = 0$, we get

$$\nu' = \nu(1 - v / c). \tag{5}$$

Example 4. A swarm of small particles is moving through space with a constant velocity $\mathbf{v}$. A body A moves through this swarm with a velocity $\mathbf{u}$. The relative velocity of the particles with respect to A is therefore $\mathbf{w} = \mathbf{v} - \mathbf{u}$. Let us denote by w, v and u the absolute values of these velocities, by φ the angle between $\mathbf{w}$ and $\mathbf{v}$, and by α the angle between $\mathbf{w}$ and $\mathbf{u}$. From the triangle with sides $\mathbf{v}$, $-\mathbf{u}$, $\mathbf{w}$ we obtain

$$\sin\varphi = \frac{u}{v}\sin\alpha. \tag{6}$$

We shall give an application of the above formula.

As the fixed frame let us choose a frame attached to the sun and the fixed stars. A certain star G sends to the earth rays of light travelling with a velocity $\mathbf{v}$ ($v = 300\,000$ km/sec). The earth moves with a velocity $\mathbf{u}$ ($u = 30$ km/sec). Therefore the rays of light have a relative velocity $\mathbf{w}$ with respect to the earth. An observer on the earth who wants to see the star G must set his telescope in the direction of the relative velocity $\mathbf{w}$. He will therefore see G seemingly at the place G'. The angle φ, denoting the deviation from the true direction, can be calculated from (6).

Since v is large as compared with u, the angle φ is very small. From formula (6) we obtain

$$\sin\varphi = \frac{\sin\alpha}{10\,000}.$$

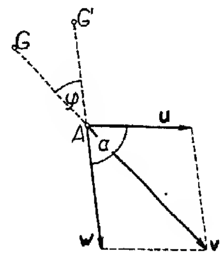

Fig. 68.

For $\alpha = \frac{1}{2}\pi$ (i. e. $\sin\alpha = 1$) we obtain for the angle φ the maximum value $\varphi = 22''$.

Example 5. On the periphery of a circle with centre O the points A_1 and A_2 are moving with the angular velocities ω_1 and ω_2 relative to a cer-

tain fixed frame. Choose the ξ and η axes of the moving frame in the plane of the circle; let O be the origin and the line OA_2 the ξ-axis.

The angular velocity of the point A_1 relative to the chosen moving frame is called the (relative) *angular velocity of the point A_1 with respect to the point A_2*; we denote it by $\omega_{1,2}$.

It is easy to show that

$$\omega_{1,2} = \omega_1 - \omega_2. \tag{7}$$

Let us assume that the points A_1 and A_2 move with constant angular velocities. Denoting by T_1, T_2 the periods of revolutions of the corresponding points A_1, A_2 in the fixed frame, and by $T_{1,2}$ the period of revolution of the point A_1 relative to A_2 (i. e. the period of revolution of the point A_1 in the moving frame), we obtain: $T_1 = 2\pi / \omega_1, T_2 = 2\pi / \omega_2, T_{1,2} = 2\pi / \omega_{1,2}$. Hence by (7)

$$1 / T_{1,2} = 1 / T_1 - 1 / T_2. \tag{8}$$

The period of revolution of the minute hand of a clock is $T_1 = 1$ h, that of the hour hand $T_2 = 12$ h. From formula (8) we get: $1/T_{1,2} = 1 - \frac{1}{12}$, whence $T_{1,2} = \frac{12}{11}$ h $= 1$ h, 5 min, 27 sec. Therefore the hands coincide every 1 h, 5 min, 27 sec.

To a traveller circling the globe from west to east it seems that the journey lasted n mean solar days, because during the journey there were n days and n nights. However, returning to the place from which he started, he finds that the journey did not last n, but n' mean solar days. What is the relation between n and n'?

Let us denote by T_1 the time of the journey, and by T_2 the time taken by the sun to complete an apparent revolution about the earth. Consequently $T_1 = n'$, and $T_2 = -1$ (since the sun seemingly revolves about the earth from east to west, that is, in the direction opposite to that of the journey). The traveller assumed as the apparent mean solar day the period of time between two successive passings of the sun across the changing meridian on which he was. Since in the interval of n apparent days there were n' real days, the apparent mean solar day is equal to n' / n real days. Hence $T_{1,2} = n' / n$. Therefore by (8)

$$n / n' = 1 / n' + 1, \quad \text{or} \quad n' = n - 1.$$

Therefore the number of real days elapsed was one less than the number of apparent days.

If the traveller had gone from east to west, then (as is easily seen) the number of real days elapsed would be one greater than the number of apparent days.

CHAPTER III

DYNAMICS OF A MATERIAL POINT

I. DYNAMICS OF AN UNCONSTRAINED POINT

§ 1. Basic concepts of dynamics. The subject of dynamics is concerned with the investigations of the motion of bodies under the influence of forces which cause this motion.

In kinematics all frames of reference are, as we already know, equally valid; it is a matter of indifference how we measure time (i. e. what intervals of time we consider as equal). The laws of dynamics stated by Newton, however, are not valid for every frame of reference and every measure of time.

Inertial frame, absolute time. A frame of reference for which, along with a certain measure of time, the Newtonian laws of dynamics hold, is called an *inertial frame*, the corresponding measure of time — the measure of *absolute time*, and the motion of the body relative to the inertial frame — *absolute motion*.

Strictly speaking, we do not know at present of any example of either an inertial frame or of absolute time. Nevertheless, in a great number of problems we can select frames of reference and methods of measuring time in such a way, that the application of the laws of dynamics leads to results differing sufficiently little from experience, so that for all practical purposes the errors can be neglected.

For instance, if we are investigating the motion of small particles near the earth during a short interval of time, the results will be sufficiently accurate on the whole, if we take as an inertial frame the frame of reference attached to the earth, and if we base the measurement of absolute time on the assumption that the earth, relative to the fixed stars, revolves about its axis so as to make equal angles in equal times.

In other problems, however (such as Foucault's pendulum, the gyroscope, the motion of planets) the application of the laws of dynamics to a frame of reference attached to the earth does not lead to equally good results. Considerably better results are obtained here if we select for the inertial frame, a frame of reference whose origin is situated within the sun and whose axes point to the fixed stars.

tain fixed frame. Choose the ξ and η axes of the moving frame in the plane of the circle; let O be the origin and the line OA_2 the ξ-axis.

The angular velocity of the point A_1 relative to the chosen moving frame is called the (relative) *angular velocity of the point A_1 with respect to the point A_2*; we denote it by $\omega_{1,2}$.

It is easy to show that

$$\omega_{1,2} = \omega_1 - \omega_2. \tag{7}$$

Let us assume that the points A_1 and A_2 move with constant angular velocities. Denoting by T_1, T_2 the periods of revolutions of the corresponding points A_1, A_2 in the fixed frame, and by $T_{1,2}$ the period of revolution of the point A_1 relative to A_2 (i. e. the period of revolution of the point A_1 in the moving frame), we obtain: $T_1 = 2\pi / \omega_1, T_2 = 2\pi / \omega_2, T_{1,2} = 2\pi / \omega_{1,2}$. Hence by (7)

$$1 / T_{1,2} = 1 / T_1 - 1 / T_2. \tag{8}$$

The period of revolution of the minute hand of a clock is $T_1 = 1$ h, that of the hour hand $T_2 = 12$ h. From formula (8) we get: $1/T_{1,2} = 1 - \frac{1}{12}$, whence $T_{1,2} = \frac{12}{11}$ h $= 1$ h, 5 min, 27 sec. Therefore the hands coincide every 1 h, 5 min, 27 sec.

To a traveller circling the globe from west to east it seems that the journey lasted n mean solar days, because during the journey there were n days and n nights. However, returning to the place from which he started, he finds that the journey did not last n, but n' mean solar days. What is the relation between n and n'?

Let us denote by T_1 the time of the journey, and by T_2 the time taken by the sun to complete an apparent revolution about the earth. Consequently $T_1 = n'$, and $T_2 = -1$ (since the sun seemingly revolves about the earth from east to west, that is, in the direction opposite to that of the journey). The traveller assumed as the apparent mean solar day the period of time between two successive passings of the sun across the changing meridian on which he was. Since in the interval of n apparent days there were n' real days, the apparent mean solar day is equal to n' / n real days. Hence $T_{1,2} = n' / n$. Therefore by (8)

$$n / n' = 1 / n' + 1, \quad \text{or} \quad n' = n - 1.$$

Therefore the number of real days elapsed was one less than the number of apparent days.

If the traveller had gone from east to west, then (as is easily seen) the number of real days elapsed would be one greater than the number of apparent days.

CHAPTER III

DYNAMICS OF A MATERIAL POINT

I. DYNAMICS OF AN UNCONSTRAINED POINT

§ 1. Basic concepts of dynamics. The subject of dynamics is concerned with the investigations of the motion of bodies under the influence of forces which cause this motion.

In kinematics all frames of reference are, as we already know, equally valid; it is a matter of indifference how we measure time (i. e. what intervals of time we consider as equal). The laws of dynamics stated by Newton, however, are not valid for every frame of reference and every measure of time.

Inertial frame, absolute time. A frame of reference for which, along with a certain measure of time, the Newtonian laws of dynamics hold, is called an *inertial frame*, the corresponding measure of time — the measure of *absolute time*, and the motion of the body relative to the inertial frame — *absolute motion*.

Strictly speaking, we do not know at present of any example of either an inertial frame or of absolute time. Nevertheless, in a great number of problems we can select frames of reference and methods of measuring time in such a way, that the application of the laws of dynamics leads to results differing sufficiently little from experience, so that for all practical purposes the errors can be neglected.

For instance, if we are investigating the motion of small particles near the earth during a short interval of time, the results will be sufficiently accurate on the whole, if we take as an inertial frame the frame of reference attached to the earth, and if we base the measurement of absolute time on the assumption that the earth, relative to the fixed stars, revolves about its axis so as to make equal angles in equal times.

In other problems, however (such as Foucault's pendulum, the gyroscope, the motion of planets) the application of the laws of dynamics to a frame of reference attached to the earth does not lead to equally good results. Considerably better results are obtained here if we select for the inertial frame, a frame of reference whose origin is situated within the sun and whose axes point to the fixed stars.

In addition to the previously mentioned method of measuring time, which is based on the assumption that the earth revolves about its axis uniformly, there exist still other methods of measuring time which are given in astronomy.

In dynamics we assume that an *inertial frame and absolute time are given.*

Mass and force. Dynamics gives rise to new concepts such as mass and force. We assume that they are known to the reader from physics and we shall not enlarge upon their definition. We shall give only those of their properties which we assume about them in dynamics.

The mass of a body is expressed by a positive number which depends on the choice of the units of mass, i. e. on the choice of an arbitrary body whose mass is denoted by the number 1.

The ratio of two masses does not depend on the choice of the unit.

Therefore, if we denote by m_1 and m_2 the masses of two bodies expressed in terms of a certain unit, and by m_1' and m_2' the masses of these bodies expressed in terms of another unit, then

$$m_1 / m_2 = m_1' / m_2'.$$

For example, let the mass of a certain body A be m, if we take as the unit the mass of the body B. Let us assume, in addition, that the mass of the body A is m' and the mass of B is m'', if we select as the unit the mass of another body C. The ratio of the masses of the bodies A and B is $m / 1$, if the unit is the mass of the body B, and m' / m'', if the unit is the mass of the body C. Therefore, by hypothesis, $m / 1 = m' / m''$, whence

$$m' = mm''.$$

Hence, knowing the mass of a body in terms of a certain unit, we can compute it in terms of any other unit.

The mass of a body is independent of the time, i. e. the given body has the same mass at each moment.

We consider a *force* to be determined if its *magnitude* (absolute value), *direction, sense and point of application* are given. A force acting on a body can be thought of as a taut string or a stretched spring fastened to the body (see figure).

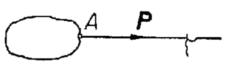

Fig. 69.

A force whose magnitude is expressed by the number 0 is called a *zero force*. We do not distinguish a direction or sense in connection with zero forces.

The magnitude of a non-zero force is expressed by a positive number which depends on the unit of force, i. e. on the choice of an arbitrary (non-zero) force whose magnitude we denote by the number 1.

The ratio of the magnitudes of two (non-zero) *forces does not depend on the choice of the unit.*

On the basis of this, from the magnitude of a force given in terms of a certain unit, we can determine its magnitude in terms of a different unit.

We represent the force acting on a body as a *vector*. With this in view, we select an arbitrary unit of length and an arbitrary unit of force. The given force is represented by a vector whose length is expressed by the same number as the magnitude of the force, while the origin, direction and sense are the same as the origin, direction and sense of the force. For example, a vector having five units of length represents a force having five units of force. A zero force represents a zero vector.

Operations on forces are defined as operations on vectors which represent these forces. For instance, if $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ represent certain forces, then the sum of these forces is the force which is represented by the vector $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$.

The moment of a force (represented by the vector $\mathbf{P}$) with respect to a point O is defined as the moment of the vector $\mathbf{P}$ with respect to O.

Material point. In dynamics we shall be concerned at first with the motion of points, and afterwards with the motion of bodies. As in kinematics, we shall sometimes regard a point as a model of the body (e. g. in the case when the dimensions of the body are small in comparison with the path).

The mass of a point is defined as the mass of the body which the given point represents; the point itself is then termed a *material point.*

If a force acts on a body whose image is the material point A, then this force is represented in the form of a vector whose initial point is at A.

The force acting on a material point can change with time in magnitude as well as in direction and sense.

§ 2. Newton's laws of danymics. The laws of dynamics stated by Newton give the relations that obtain in absolute motion among the mass, acceleration and forces that act upon a material point.

Laws of motion. Let the frame of reference (x, y, z) be an inertial frame, and let t denote absolute time. Under these assumptions the laws of motion can be stated as follows:

I. *If m denotes the mass of a material point, $\boldsymbol{p}$ the acceleration at the moment t, $\boldsymbol{P}$ the sum of the forces acting on the material point at the moment t, then*

$$m\boldsymbol{p} = K\boldsymbol{P}, \tag{1}$$

where K denotes a certain number (positive), depending *only on the choice of units* of length, time, mass and force (and hence independent of the time, mass and force).

From equation (1) it follows that

$$m|\boldsymbol{p}| = K|\boldsymbol{P}|. \tag{2}$$

Therefore, if of the units mentioned, three are selected arbitrarily, the fourth can be so chosen that $K = 1$. For instance, select arbitrary units of time, length and mass, and for the unit of force select a force which gives a point of mass 1 an acceleration 1. For $m = 1$ and $|\boldsymbol{P}| = 1$ we have with these units $|\boldsymbol{p}| = 1$, and hence from formula (2) we obtain $K = 1$. Relation (1) then assumes the following form

$$m\boldsymbol{p} = \boldsymbol{P}. \tag{I}$$

Henceforth we shall always assume that the units are so chosen that $K = 1$. Newton's law will therefore always be taken in the form (I).

Forming the projections on the axes of the frame of reference, we obtain in virtue of (I):

$$mp_x = P_x,\ mp_y = P_y,\ mp_z = P_z. \tag{II}$$

Equations (I) and (II) are obviously equivalent.

Since $\boldsymbol{p} = \mathrm{d}\boldsymbol{v} / \mathrm{d}t$ and m is a constant, $m\boldsymbol{p} = \mathrm{d}(m\boldsymbol{v}) / \mathrm{d}t$. Hence relation (I) can also be written in the form

$$\mathrm{d}(m\boldsymbol{v}) / \mathrm{d}t = \boldsymbol{P}. \tag{III}$$

The vector $m\boldsymbol{v}$ is called the *momentum* (*quantity of motion*).

Therefore: *the derivative of the momentum* (with respect to time) *is equal to the sum of the forces acting on a material point.*

On a material point of mass m let there act forces whose sum $\boldsymbol{P}$ is constantly zero during a certain period of time. Then $m\boldsymbol{p} = 0$; hence the acceleration $\boldsymbol{p} = 0$, and consequently the point moves with uniform motion along a straight line (or is at rest). We therefore have the following law, known as *Newton's law of inertia:*

II. *If forces whose sum is zero act on a material point during a certain period of time, then the point is either at rest or in uniform motion along a straight line.*

Conversely, if a point is at rest or in uniform motion along a straight line, then the acceleration $\boldsymbol{p} = 0$, and since by (I) $m\boldsymbol{p} = \boldsymbol{P}$, it follows that the sum of the acting forces $\boldsymbol{P}$ is zero.

The forces acting on a material point usually arise from the action of other material points on the given point. For such forces Newton gave the following law, known as the *law of action and reaction:*

III. *If a material point A acts on a material point B with a certain force, then the point B also acts on the point A with a force equal in magnitude and direction, but opposite in sense; the forces which the points A and B exert on each other are always directed along the straight line AB joining these points.*

If the force which the point A exerts on the point B has a sense towards A (and hence the force which the point B exerts on A has a sense towards B), then we say that the points A and B *attract each other*; in the opposite case we say that the points *repel each other.*

Equilibrium of a point and forces. If a material point is at rest, then it is said to be in *equilibrium.* The forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$ are said to be in *equilibrium*, or to *balance each other*, if their sum is zero, i. e. if $\boldsymbol{P}_1 + \boldsymbol{P}_2 + \ldots + \boldsymbol{P}_n = 0$.

Therefore, if a material point is in equilibrium, then the forces acting on this point are also in equilibrium. On the other hand, if the forces acting on a material point during a certain period of time are in equilibrium, then in this period of time the acceleration $\boldsymbol{p} = 0$, and hence the point is either in equilibrium or in uniform motion along a straight line.

Force of inertia. D'Alembert's principle. Law (I) can also be written in the form

$$\boldsymbol{P} + (-m\boldsymbol{p}) = 0. \tag{IV}$$

A vector $-m\boldsymbol{p}$ whose origin is at the point m is called a *force of inertia.*

One must not suppose that the vector $-m\boldsymbol{p}$ represents the force acting on the material point m. It is only for the sake of convenience that we call this vector a force (of inertia). Only forces whose sum is $\boldsymbol{P}$ act on the point m.

Relation (IV) can be stated as follows:

The forces acting on a material point are in equilibrium with the force of inertia.

The above formulation is equivalent to Newton's law I and is called *d'Alembert's principle.* This principle is very useful in connection with the investigation of the motion of the so-called constrained points.

§ 3. Systems of dynamical units. The fundamental units used in dynamics are the units of length, time and mass. By means of these units we define the unit of force. As a unit of force we select a force which gives to a mass 1 an acceleration 1.

C. g. s. system. In this system the unit of length is the *centimeter* (cm), the unit of mass the *gram* (g), the unit of time the *second* (sec) and that of force the *dyne*.

At first the meter (m = 100 cm) was to represent one forty millionth part of the earth's meridian. A small error, however, was made in the calculations. Today a meter is defined as the length of a certain bar preserved in Paris. Similarly, the unit of mass 1 g was at first to be the mass of 1 cm^3 of chemically pure water at 4° C under a pressure of 760 mm of mercury. At present, however, we take as 1 kilogram (kg = 1000 g) the mass of a certain block of platinum preserved in Paris.

The unit of time 1 sec is defined by means of the so-called mean solar day whose determination belongs to the subject of astronomy. The mean solar day = 24 hours (h), an hour = 60 minutes (min), a minute = = 60 sec.

The unit of force 1 dyne is the force which will impart an acceleration of 1 cm/sec^2 to a mass of 1 g.

The system of fundamental units (centimeter, gram, second) is called briefly the *c. g. s. system*.

Measurement of masses and forces. In the vicinity of the earth small freely falling bodies drop to the earth vertically with a uniformly accelerated motion (if air resistance is neglected). This acceleration (termed *gravitational*) is the same for all bodies at a given place on earth, but it changes with latitude. It is denoted by g. We shall take the gravitational acceleration to be in average $g = 981$ cm/sec^2.

Let m denote the mass of a small body. The force directed vertically downwards and of magnitude $Q = mg$ is called the *weight* of this body.

Weight is therefore proportional to the mass of the body; bodies having equal weights (at the same place on earth) have equal masses and conversely.

By means of an instrument called a *balance* (with whose principle we shall be acquainted in chapter VI) we can compare the weights of two bodies. Since the equality of masses follows from the equality of weights, the masses of bodies can be compared indirectly by means of a balance. Hence it follows that with the aid of a balance we can *measure*, i. e. we can determine the *masses* of bodies.

Forces are measured by *dynamical* or *statical* methods.

The dynamical method rests on Newton's first law ($\boldsymbol{P} = m\boldsymbol{p}$). From this formula we can determine the force $\boldsymbol{P}$ if we know the mass of the body m and the acceleration $\boldsymbol{p}$ which is imparted to it by $\boldsymbol{P}$.

The statical method is based on the fact that bodies change their shape (become deformed) when acted upon by forces. From a knowledge of the deformations we can infer in certain cases the magnitudes of the forces which cause these deformations. For instance, if a force directed vertically downwards acts at the lower end of a spring hanging vertically, then the spring becomes elongated. When the forces are small the elongation is proportional to the magnitude of the acting force. Instruments which serve to measure forces by statical methods are called *dynamometers*.

Metric gravitational system of units. In engineering the so-called *metric gravitational system of units* is generally used. In this system we assume as fundamental units the units of length, time and force. The unit of length is 1 m, of time 1 sec, and of force 1 kilogram (kg). This is the weight of 1 dcm^3 of water (under normal conditions) at a latitude of 45° north (where the gravitational acceleration $g = 981$ cm/sec^2 = = 9.81 m/sec^2).

If in the formula $|\boldsymbol{P}| = m|\boldsymbol{p}|$ we put $|\boldsymbol{P}| = 1$ and $|\boldsymbol{p}| = 1$, we obtain $m = 1$. Therefore the unit of mass will be a mass to which a force of 1 kg imparts an acceleration of 1 m/sec^2.

Let m be the mass of a body, Q its weight (at a latitude of 45° N) and let $g = 9.81$ m/sec^2. Then $Q = mg$, and therefore

$$m = Q \,/\, g = Q \,/\, 9.81. \tag{1}$$

From the above formula we can determine the mass of a body in terms of metric gravitational units when we know the weight of the body. Since the weight of 9.81 dcm^3 of water (at a latitude of 45° N) is 9.81 kg, the unit of mass in the metric gravitational system represents the mass of 9.81 dcm^3 of water.

In the c. g. s. system the mass of 9.81 dcm^3 of water is 9.81 kg (of mass) = 9810 g (of mass). Hence:

The unit of mass in the metric gravitational system = 9.81 kg (of mass) = 9810 g (of mass). (2)

In order to find the relation between the unit of force (kg) in the metric gravitational system and the unit of force (dyne) in the c. g. s. system, let us note that 1 dcm^3 of water (i. e. a mass of 1000 g) falls to the earth under the influence of its own weight of 1 kg with an accelera-

tion of 981 cm/sec^2. Consequently 1 kg (of force) $=$ 1000 $\cdot$ 981 dynes, whence

$$1 \text{ kg (of force)} = 981\,000 \text{ dynes}. \tag{3}$$

Dimensions of dynamical magnitudes. In dynamics there occur still other magnitudes (e. g. work, kinetic energy, etc.) whose units are defined in the same way as the unit of force by means of the fundamental units, i. e. length, mass and time. Similarly as for kinematic magnitudes (cf. Chap. II, § 11), we can introduce the notion of dimension for dynamical magnitudes. A knowledge of the dimension of a given magnitude enables one to determine easily the measure of this magnitude when the fundamental units are changed.

Suppose that we have chosen two systems of units of length, mass and time, which we denote respectively by L, M, T and L', M', T' and that these units are related as follows

$$L = \lambda L', \quad M = \mu M', \quad T = \tau T'. \tag{4}$$

Let the measure of some dynamical magnitude A bo a in terms of the units L, M, T; and a' in terms of the units L', M', T'.

If it is possible to choose numbers α, β, γ such that for every two systems of units L, M, T and L', M', T' satisfying relations (4) there exists a relation of the form

$$a' = a\lambda^\alpha \mu^\beta \tau^\gamma, \tag{5}$$

then the *dimension* of the magnitude A is defined by the expression

$$L^\alpha M^\beta T^\gamma. \tag{6}$$

The dimension of the magnitude A is denoted by $[A]$, and the unit of the magnitude A in terms of the units L, M, T is represented by the symbol $L^\alpha M^\beta T^\gamma$.

The magnitude of A is therefore $aL^\alpha M^\beta T^\gamma$ in terms of the units L, M, T, and $a'L'^\alpha M'^\gamma T'^\beta$ in terms of the units L', M', T', whence

$$aL^\alpha M^\beta T^\gamma = a'L'^\alpha M'^\beta T'^\gamma. \tag{7}$$

Making use of formulae (4) and calculating formally, we obtain $aL^\alpha M^\beta T^\gamma = a(\lambda L')^\alpha (\mu M')^\beta (\tau T')^\gamma$, whence

$$MaL^{\alpha\beta} T^\gamma = (a\lambda^\alpha \mu^\beta \tau^\gamma)\, L'^\alpha M'^\beta T'^\gamma. \tag{8}$$

By equating (7) and (8) we get (5). In this manner by means of formal reckoning we can, if we know the dimension of the magnitude A, obtain its measure when the units of length, mass and time are changed.

It is easy to generalize the theorem given on the page 51, which is very useful for the determination of the dimension.

Example 1. A force of magnitude (absolute value) P acting on a material point of mass m imparts to it an acceleration of magnitude p. Therefore $P = mp$, whence $[P] = [m] \cdot [p]$. Since $[m] = M$, and $[p] =$ $= LT^{-2}$,

$$[\text{force}] = LMT^{-2}. \tag{I}$$

The unit of force in the c. g. s. system is the dyne. Hence

$$\text{dyne} = \text{cm} \cdot \text{g} \cdot \text{sec}^{-2}. \tag{II}$$

Example 2. Represent a force of magnitude 6 m $\cdot$ kg $\cdot$ min^{-2} in the c. g. s. system.

We have

$$6 \text{ m} \cdot \text{kg} \cdot \text{min}^{-2} = 6\,(100 \text{ cm})(1000 \text{ g})(60 \text{ sec})^{-2} =$$
$$= (6 \cdot 100 \cdot 1000 \cdot 60^{-2})(\text{cm} \cdot \text{g} \cdot \text{sec}^{-2}) = 166\tfrac{2}{3} \text{ dynes}.$$

§ 4. Equations of motion. One of the principal problems of dynamics is the determination of the motion of a material point when the mass m of this point and the force $\mathbf{P}$ acting on it are given. In the simplest case the force $\mathbf{P}$ can be given as a function of time, i. e. there are given functions:

$$P_x = F(t), \quad P_y = \Phi(t), \quad P_z = \Psi(t), \tag{1}$$

defining at each moment t (of a certain period of time) the projections of the force $\mathbf{P}$ on the coordinate axes.

We shall meet with cases, however, which are more complicated. It may happen that some region D has the property that a certain force $\mathbf{P}$ acts on a given material point situated anywhere within the region D.

If the force $\mathbf{P}$ depends only on the position of the point and does not depend on anything else (e. g. velocity), then the region D is called a *force field*.

An example of a force field is the earth's gravitational field: for on a given material point situated near the earth there acts the force of gravity which depends on the position of this point (and does not depend on the velocity).

In a force field the force $\mathbf{P}$ is therefore a function of the coordinates x, y, z of the given point. A field is defined if there are given functions:

$$P_x = F(x, y, z), \quad P_y = \Phi(x, y, z), \quad P_z = \Psi(x, y, z), \tag{2}$$

determining at each point of the field the projections of the force $\mathbf{P}$. Therefore, if we are investigating the motion of a point in a force field, then we are dealing with a force which depends on the position of the point.

If a material point moves in a certain medium (e. g. in air), then, in addition to other forces acting on the material point, there is also the

force of resistance which the medium offers in opposing the motion. This force depends, among other things, on the velocity of the material point. In this case we therefore have a force depending also on the velocity of the point.

In the most general case we shall assume that the force **P** depends on the time, position and velocity of the point. We shall therefore assume that the force **P** acting on a material point is defined by the functions:

$$P_x = F(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t), \quad P_y = \Phi(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t), \quad P_z = \Psi(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t), \tag{3}$$

whose values are the projections of this force which depend on the coordinates of the position of the point (x, y, z), its velocity $x^{\cdot}, y^{\cdot}, z^{\cdot}$, and on the time t.

Functions (3) *are usually assumed to be continuous and to have continuous partial derivatives in a certain region of the variables* $x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t$.

Obviously, in particular problems the force **P** does not have to depend on all the variables $x, y, \ldots, t$ it can be independent of some of them.

Theoretically, the force **P** can depend on higher derivatives (e. g. on the second, third, etc.) of the variables x, y, z. However, such cases are not encountered in practical problems and we shall not consider them here.

Let the motion of a material point be given by the functions:

$$x = f(t), \quad y = \varphi(t), \quad z = \psi(t). \tag{4}$$

In virtue of equations (II), p. 72, we obtain

$$mx^{\cdot\cdot} = P_x, \quad my^{\cdot\cdot} = P_y, \quad mz^{\cdot\cdot} = P_z. \tag{I}$$

If we assumo that P_x, P_y, P_z are functions of the form (3), then equations (I) become

$$\begin{aligned} mx^{\cdot\cdot} &= F(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t), \\ my^{\cdot\cdot} &= \Phi(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t), \\ mz^{\cdot\cdot} &= \Psi(x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot}, t). \end{aligned} \tag{II}$$

These equations represent a system of differential equations of the second order, while the sought for functions are the functions (4).

Let us suppose that we are investigating a motion in the neighbourhood of a certain moment t_0. Assume that at the moment t_0 the point had coordinates x_0, y_0, z_0, and its velocity had projections $x^{\cdot}_0, y^{\cdot}_0, z^{\cdot}_0$. In addition, let us assume that the functions (3) are continuous, possessing continuous partial derivatives in the neighbourhood of the values $x_0, y_0, z_0, x^{\cdot}_0, y^{\cdot}_0, z^{\cdot}_0, t_0$.

From the theory of differential equations it is known that under the preceding assumptions there exists one and only one set of functions (4) continuous together with its first and second derivatives in the neighbourhood of the moment t_0, satisfying equations (II) as well as the relations:

$$f(t_0) = x_0, \ \varphi(t_0) = y_0, \ \psi(t_0) = z_0; \ f'(t_0) = x^{\cdot}_0, \ \varphi'(t_0) = y^{\cdot}_0, \ \psi'(t_0) = z^{\cdot}_0. \tag{5}$$

This system, as the only system of functions (4) satisfying all the required conditions, therefore determines the motion of a material point having at the moment t_0 the coordinates x_0, y_0, z_0 and a velocity whose projections are $x^{\cdot}_0, y^{\cdot}_0, z^{\cdot}_0$.

We see from this that *the motion is completely determined when the mass of a point, the forces acting on it and the so-called initial conditions* (i. e. its position and velocity at the initial moment t_0) *are given.*

Equations (II) are called *Newton's laws of motion.*

Example. The force depends only on the time. Let the force **P** depend only on the time and be given by functions (1). The equations of motion (II) will therefore have the form:

$$mx^{\cdot\cdot} = F(t), \quad my^{\cdot\cdot} = \Phi(t), \quad mz^{\cdot\cdot} = \Psi(t).$$

Dividing by m and integrating both sides, we obtain for $t_0 = 0$:

$$x^{\cdot} = \frac{1}{m}\int_0^t F(t)\,dt + c_1, \quad y^{\cdot} = \frac{1}{m}\int_0^t \Phi(t)\,dt + c_2, \quad z^{\cdot} = \frac{1}{m}\int_0^t \Psi(t)\,dt + c_3.$$

Let us assume that at $t = 0$, $x^{\cdot} = x^{\cdot}_0$, $y^{\cdot} = y^{\cdot}_0$, $z^{\cdot} = z^{\cdot}_0$ (initial conditions). Substituting $t = 0$ in the above equations, we get $c_1 = x^{\cdot}_0$, $c_2 = y^{\cdot}_0$, $c_3 = z^{\cdot}_0$. Hence

$$x^{\cdot} = F_1(t) + x^{\cdot}_0, \ y^{\cdot} = \Phi_1(t) + y^{\cdot}_0, \ z^{\cdot} = \Psi_1(t) + z^{\cdot}_0, \tag{6}$$

where

$$F_1(t) = \frac{1}{m}\int_0^t F(t)\,dt, \text{ etc.}$$

Integrating equations (6), we obtain:

$$x = \int_0^t F_1(t)\,dt + x^{\cdot}_0 t + c_1', \quad y = \int_0^t \Phi_1(t)\,dt + y^{\cdot}_0 t + c_2',$$

$$z = \int_0^t \Psi_1(t)\,dt + z^{\cdot}_0 t + c_3'. \tag{7}$$

Let us now suppose that at $t = 0$, $x = x_0$, $y = y_0$, $z = z_0$. Therefore from equations (7), putting $t = 0$, we get $c_1' = x_0$, $c_2' = y_0$, $c_3' = z_0$. Hence

$$x = F_2(t) + x\dot{}_0 t + x_0,\quad y = \Phi_2(t) + y\dot{}_0 t + y_0,\quad z = \Psi_2(t) + z\dot{}_0 t + z_0, \tag{8}$$

where

$$F_2(t) = \int_0^t F_1(t)\,dt, \text{ etc.}$$

From equations (8) it follows that the motion will be defined if at the initial moment $t = 0$ the position of the point (i. e. x_0, y_0, z_0) and the initial velocity (i. e. $x\dot{}_0, y\dot{}_0, z\dot{}_0$) are given.

§ 5. Motion under the influence of the force of gravity. Let a force $\boldsymbol{P}$ of constant magnitude, direction and sence act on a material point of mass m.

We have to deal with this situation — when investigating the motion of small bodies near the earth and taking as the inertial frame a frame attached to the earth. If air resistance is neglected, then the only force acting on a projected body is the force of gravity which can be considered constant over a small region.

Let $\boldsymbol{P}$ denote the force of gravity. Then $|\boldsymbol{P}| = mg$ (g is the acceleration due to gravity). Let us select a frame (x, y, z) so that the sense of the z-axis is vertically upward. Then

$$P_x = 0,\quad P_y = 0,\quad P_z = -mg.$$

Newton's laws of motion (p. 72, formulae (II)) become: $mx\ddot{} = 0$, $my\ddot{} = 0$, $mz\ddot{} = -mg$, or

$$x\ddot{} = 0,\quad y\ddot{} = 0,\quad z\ddot{} = -g. \tag{1}$$

Intograting the above equations, we obtain:

$$x\dot{} = c_1,\quad y\dot{} = c_2,\quad z\dot{} = -gt + c_3. \tag{2}$$

Integrating once more, we get:

$$x = c_1 t + c_1',\quad y = c_2 t + c_2',\quad z = \tfrac{1}{2}gt^2 + c_3 t + c_3'. \tag{3}$$

The numbers $c_1, c_2, c_3, c_1', c_2', c_3'$ denote constants of integration which we shall determine from known initial conditions, i. e. the coordinates x_0, y_0, z_0 and the projections $x\dot{}_0, y\dot{}_0, z\dot{}_0$ of the velocity $\boldsymbol{v}_0$ of the moving point at the initial moment t_0. Without loss of generality we can assume that $t_0 = 0$; moreover (by selecting a system of coordinates suitably) we can assume that at $t_0 = 0$ the point was situated at the origin of the system and that the velocity $\boldsymbol{v}_0$ lay in the vertical plane zx.

We are therefore assuming that at $t = 0$, $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ *and* $y\dot{}_0 = 0$.

Substituting $t = 0$ in equations (2) and (3), we obtain

$$c_1 = x\dot{}_0,\ c_2 = y\dot{}_0 = 0,\quad c_3 = z\dot{}_0;\ c_1' = x_0 = 0,\quad c_2' = y_0 = 0,\quad c_3' = z_0 = 0.$$

Equations (2) and (3) hence take on the form:

$$x\dot{} = x\dot{}_0,\quad y\dot{} = 0,\quad z\dot{} = -gt + z\dot{}_0, \tag{2'}$$

$$x = x\dot{}_0 t,\quad y = 0,\quad z = -\tfrac{1}{2}gt^2 + z\dot{}_0 t. \tag{3'}$$

Since $y = 0$ constantly, the motion takes place in the vertical zx-plane.

We shall examine two cases: the so-called vertical projection and the oblique projection.

Vertical projection. Let us assumo that at the moment $t = 0$ the velocity $\boldsymbol{v}_0$ was directed vertically (or was zero), and hence that $x\dot{}_0 = 0$. Putting $z\dot{} = v$ and $z\dot{}_0 = v_0$, we obtain from (2′) and (3′)

$$x\dot{} = 0,\ y\dot{} = 0,\ x = 0,\ y = 0, \tag{4}$$

$$v = -gt + v_0,\ z = -\tfrac{1}{2}gt^2 + v_0 t. \tag{5}$$

Since $x = 0$ and $y = 0$ constantly, the point moves along the z-axis, i. e. along a vertical. Moreover, we have $v\dot{} = p = -g$.

Therefore: *if the initial velocity has a vertical direction* (*or is zero*), *then a point moves with a uniformly accelerated motion along a vertical under the influence of the force of gravity.*

Assume that $v_0 > 0$, i. e. that at the initial moment the velocity had an upward sense (e. g. that we had projected the point vertically upwards with a velocity v_0). Let us denote tho height of the projection by h, i. e. the maximum elevation the point will attain. In order to obtain h it is necessary to calculate the maximum of the function $z = -\frac{1}{2}gt^2 + v_0 t$. We get

$$h = v_0^2 / 2g \text{ at the moment } t = v_0 / g. \tag{6}$$

Oblique projection. Let us assume that the velocity $\boldsymbol{v}_0$ ($\neq 0$) makes an angle $\alpha \neq \pm \frac{1}{2}\pi$ with the x-axis. Setting $|\boldsymbol{v}_0| = v_0$, we get $x\dot{}_0 = v_0 \cos\alpha$, $z\dot{}_0 = v_0 \sin\alpha$. Therefore in virtue of (2′) and (3′)

$$x\dot{} = v_0\cos\alpha,\quad y\dot{} = 0,\quad z\dot{} = -gt + v_0\sin\alpha, \tag{7}$$

$$x = v_0 t\cos\alpha,\quad y = 0,\quad z = -\tfrac{1}{2}gt^2 + v_0 t\sin\alpha. \tag{8}$$

Since $\cos\alpha \neq 0$ and $v_0 \neq 0$, by the first of the equations (8) $t = v / v_0 \cos\alpha$, whence

$$z = -\frac{g}{2v_0^2\cos^2\alpha}x^2 + x\tan\alpha. \tag{9}$$

This equation is that of a parabola.

Hence: *a point projected obliquely moves along a parabola.*

The parabola cuts the x-axis in the points O and D. The length of the segment $OD = d$ is called the *range of the projection.*

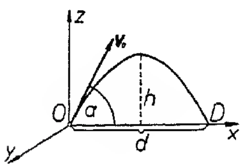

Fig. 70.

In order to calculate d we substitute $z = 0$ in (9). We obtain

$$d = \frac{v_0^2}{g} \sin 2\alpha. \tag{10}$$

Therefore: *the maximum range of a projection with a given velocity v_0 occurs for the angle $\alpha = \frac{1}{4}\pi = 45°$.*

In order to obtain the height h it is necessary to calculate the maximum of the function (9). We get

$$h = v_0^2 \sin^2\alpha \,/\, 2g \quad \text{for} \quad x = v_0^2 \sin 2\alpha \,/\, 2g. \tag{17}$$

§ 6. Motion in a resisting medium. A material point moving in a medium such as air, for instance, encounters resistance. Experiments show that air resistance can be expressed by a force depending only on the velocity of the point (for bodies the resistance also depends on the shape of the body). Resistance has the direction of the velocity, but an opposite sense. The magnitude of the resistance depends on the magnitude of the velocity, but not on its direction. Let us denote the magnitude of the resistance by Γ and the magnitude of the velocity by v. We can therefore write

$$\Gamma = f(v).$$

The function f is an increasing function with $f(0) = 0$. For velocities less than the velocity of sound (which is 333 m/sec in air) we can assume with great accuracy that

$$\Gamma = \lambda v^2, \tag{1}$$

where λ is a factor depending on the temperature and density of the air.

Vertical projection. Let us investigate the case of the falling point. Assume that the point falls along the z-axis to which we give a sense vertically downward. Consequently the component of the velocity $z^\cdot = v > 0$. The resistance is directed upwards and hence its projection on the z-axis is negative. Assuming that the magnitude of the resistance is expressed by formula (1), by putting $\lambda = km$ we obtain:

$$mz^{\cdot\cdot} = m\frac{dv}{dt} = mg - kmv^2. \tag{2}$$

From this

$$\frac{dv}{g - kv^2} = dt,$$

and hence

$$\int \frac{dv}{g - kv^2} = \int dt = t. \tag{3}$$

Setting

$$v_\infty = \sqrt{\frac{g}{k}},$$

we get

$$\int \frac{dv}{g - kv^2} = \frac{1}{k}\int \frac{dv}{v_\infty^2 - v^2} = \frac{1}{2kv_\infty}\ln\frac{v_\infty + v}{v_\infty - v} + c,$$

where c is the constant of integration. Hence by (3)

$$\frac{1}{2kv_\infty}\ln\frac{v_\infty + v}{v_\infty - v} + c = t. \tag{4}$$

Let us assume that at the initial moment $t = 0$ the velocity was $v = 0$. Substituting $v = 0$ and $t = 0$ in formula (4), we get $c = 0$. Consequently

$$v = \frac{e^{2kv_\infty t} - 1}{e^{2kv_\infty t} + 1} v_\infty = \left(1 - \frac{2}{e^{2kv_\infty t} + 1}\right) v_\infty. \tag{5}$$

From formula (5) it follows that v is always less than v_∞.

Hence: *the velocity of a point falling vertically in a resisting medium does not increase infinitely, but is always less than the limiting velocity v_∞.*

Oblique projection. Let us now assume that the point moves in the vertical plane xz. Let the magnitude of the resistance in all generality be given by $\Gamma = f(v)$. Denoting the resistance by Γ, we therefore obtain

$$\Gamma = -\frac{f(v)}{v}\mathbf{v}, \quad \text{whence} \quad \Gamma_x = -\frac{f(v)}{v}v_x \quad \text{and} \quad \Gamma_z = -\frac{f(v)}{v}v_z.$$

Therefore

$$\Gamma_x = -\frac{f(v)}{v}x^\cdot, \quad \Gamma_z = -\frac{f(v)}{v}z^\cdot.$$

The equations of motion will hence have the form

$$mx^{\cdot\cdot} = -\frac{f(v)}{v}x^\cdot, \quad mz^{\cdot\cdot} = mg - \frac{f(v)}{v}z^\cdot, \quad v = \sqrt{x^{\cdot 2} + z^{\cdot 2}}.$$

The science of exterior ballistics is concerned with the solution of the preceding equations. This problem is very difficult because the values of the function $f(v)$ are known only from measurements.

§ 7. Moment of momentum. Let a material point A of mass m move with a velocity $\mathbf{v}$. We have called the vector $m\mathbf{v}$ whose origin is at A the *momentum* or the *quantity of motion* (p. 72). If x, y, z are the coordinates of the point A, then the projections of the momentum on the coordinate axes are $mx^{\cdot}, my^{\cdot}, mz^{\cdot}$, respectively.

Let $\mathbf{K}$ denote the moment of the momentum $m\mathbf{v}$ with respect to the origin of the coordinate system. Then (p. 18, formula (II)):

$$K_x = m(y^{\cdot}z - z^{\cdot}y), \quad K_y = m(z^{\cdot}x - x^{\cdot}z), \quad K_z = m(x^{\cdot}y - y^{\cdot}x). \tag{1}$$

Take the derivative (with respect to time) of the moment of momentum. We obtain:

$$K_x^{\cdot} = m(y^{\cdot\cdot}z - z^{\cdot\cdot}y), \quad K_y^{\cdot} = m(z^{\cdot\cdot}x - x^{\cdot\cdot}z), \quad K_z^{\cdot} = m(x^{\cdot\cdot}y - y^{\cdot\cdot}x). \tag{2}$$

Assume that the frame of reference is an inertial frame and that a force $\mathbf{P}$ acts on the point A. Then $mx^{\cdot\cdot} = P_x$, $my^{\cdot\cdot} = P_y$, $mz^{\cdot\cdot} = P_z$, whence by (2)

$$K_x^{\cdot} = P_y z - P_z y, \quad K_y^{\cdot} = P_z x - P_x z, \quad K_z^{\cdot} = P_x y - P_y x. \tag{3}$$

The expressions on the right hand sides of equations (3) represent the moments of the force $\mathbf{P}$ with respect to the axes of the frame. Therefore, denoting by $\mathbf{M}$ the moment of the force $\mathbf{P}$ with respect to the origin of the frame, we have by (3)

$$K_x^{\cdot} = M_x, \quad K_y^{\cdot} = M_y, \quad K_z^{\cdot} = M_z. \tag{4}$$

The above equations can be written as one vector equation:

$$\mathbf{K}^{\cdot} = \mathbf{M}. \tag{5}$$

The origin of the frame could have been chosen arbitrarily.

Hence: *the derivative of the moment of momentum with respect to an arbitrary fixed point is equal to the moment of the acting force with respect to this point.*

From equations (4) it also follows that *the derivative of the moment of momentum with respect to an arbitrary fixed axis is equal to the moment of the force with respect to this axis.*

Principle of conservation of areas. Let us suppose that the moment of the force $\mathbf{P}$ with respect to a certain axis l is constantly zero; therefore either the line on which the force $\mathbf{P}$ lies cuts the l-axis, or the force $\mathbf{P}$ is parallel to that axis. Let us choose the l-axis as the z-axis. Hence we have $M_z = 0$. In virtue of (4), $K_z^{\cdot} = 0$ or $K_z = \text{const}$. From this and (1) we obtain

$$m(x^{\cdot}y - y^{\cdot}x) = \text{const}, \quad \text{or} \quad x^{\cdot}y - y^{\cdot}x = \text{const}. \tag{6}$$

Let A' be the projection of the point A on the xy-plane. The point A' has the coordinates x, y. Therefore the areal velocity (p. 47) of A' is $-\frac{1}{2}(x^{\cdot}y - y^{\cdot}x)$. From formula (6) it follows that the areal velocity of the point A' is constant.

Therefore: *if the moment of a force with respect to a certain axis is constantly zero, then the moment of momentum of the motion with respect to this axis is constant, and the areal velocity of the projection of the motion on a plane perpendicular to this axis is constant.*

This theorem is called *the principle of conservation of moment of momentum* or *the principle of conservation of areas.*

§ 8. Central motion. If a material point moves in such a manner that its acceleration at each moment is directed along a line passing through a certain fixed point O, then the motion of the point is called a *central motion* and the point O the *centre of motion.*

For instance, the uniform motion of a point along the periphery of a circle is a central motion because the acceleration is constantly directed towards the centre of the circle which in this case is the centre of motion (p. 43).

Since the acceleration has the direction of the force acting on the material point, the line of action of the force in central motion passes through the centre of motion.

A force field in which the lines of action of the forces pass through a certain fixed point O is called a *central field* and the point O the *centre of the field.*

A point of mass M situated motionless at a fixed point O and attracting another point of mass m with a force depending only on mutual distance of these points forms a force field. This field is a central field because the force acting on the point m has — according to the law of action and reaction (p. 73, III) — a line of action passing through the point M.

The material point moves in a central field with central motion; the centre of motion obviously lies at the centre of the field.

Let us choose the origin of the coordinate system at the centre of the field. Since the line of action of the force constantly passes through the origin of the system, its moment with respect to each axis is zero. By the principle of conservation of areas the motion of the projection of the point on each of the coordinate planes has therefore a constant areal velocity, consequently:

$$y^{\cdot}z - z^{\cdot}y = a, \quad z^{\cdot}x - x^{\cdot}z = b, \quad x^{\cdot}y - y^{\cdot}x = c, \tag{1}$$

where a, b, c are certain constants. Multiplying the first equation by x, the second by y, the third by z, and adding, we obtain

$$ax + by + cz = 0. \tag{2}$$

Hence we see that the coordinates of the point constantly satisfy equation (2). This is the equation of a plane passing through the origin of the system (i. e. through the centre of the field). Therefore the point moves in a plane passing through the origin of the system.

If we select x and y axes in this plane, then from (1) it follows that the areal velocity in the plane of motion is constant; the radius vectors consequently sweep out equal areas in equal times.

Therefore: *the path in a central motion is a plane path lying in a plane passing through the centre; the radius vectors emanating from the centre sweep out equal areas in equal times.*

We shall now prove the converse theorem:

If the path of a point is a plane path and the radius vectors emanating from a certain fixed point O (lying in the plane of the path) sweep out equal areas in equal times, then the line of action of the force constantly passes through the point O.

Proof. Choose the origin of the system at O and the x and y axes in the plane of the motion. The point therefore moves in the xy-plane. Since the areal velocity is constant, $x^{\cdot}y - y^{\cdot}x = \text{const}$. Differentiating both sides we get $x^{\cdot\cdot}y - y^{\cdot\cdot}x = 0$; hence $(mx^{\cdot\cdot})\,y - (my^{\cdot\cdot})\,x = 0$, whence

$$P_x y - P_y x = 0. \tag{3}$$

Since $z^{\cdot\cdot} = 0$, $P_z = mz^{\cdot\cdot} = 0$. The force $\mathbf{P}$ therefore lies in the xy-plane; in virtue of (3) the moment of the force $\mathbf{P}$ with respect to O is zero; hence the line of action of the force $\mathbf{P}$ passes through O, q. e. d.

Remark. Let us assume that the areal velocity in a central motion is zero. Then $x^{\cdot}y - y^{\cdot}x = 0$, or (in polar coordinates) $r^2\varphi^{\cdot} = 0$. It follows from this that either $r = 0$ constantly, i. e. that the point is at rest, or $\varphi^{\cdot} = 0$ constantly, or $\varphi = \varphi_0 = \text{const}$, i. e. that the point moves along a line passing through the centre (and inclined at an angle φ_0 with the x-axis).

Hence: *if the areal velocity in a central motion is zero, then the point moves along a straight line passing through the centre.*

On the other hand, if we assume that the areal velocity is different from zero, then $r^2\varphi^{\cdot} \neq 0$, or $r \neq 0$.

Therefore: *if the areal velocity in a central motion is different from zero, then the point never passes through the centre.*

Binet's formula. A point A of mass m moves in a central force field in the xy-plane with an areal velocity different from zero. Let us introduce the polar coordinates r, φ and denote by P the projection of the force $\mathbf{P}$ on the radius vector $\overline{OA}$. Then $P_x = P\cos\varphi$ and $P_y = P\sin\varphi$; hence $P_x\cos\varphi + P_y\sin\varphi = P$, whence

$$P = m(x^{\cdot\cdot}\cos\varphi + y^{\cdot\cdot}\sin\varphi). \tag{4}$$

Since $x = r\cos\varphi$, $y = r\sin\varphi$ (cf. p. 47, formula (2)),

$$x^{\cdot\cdot}\cos\varphi + y^{\cdot\cdot}\sin\varphi = r^{\cdot\cdot} - r\varphi^{\cdot 2},$$

whence in virtue of (4)

$$P = m(r^{\cdot\cdot} - r\varphi^{\cdot 2}). \tag{5}$$

Let us denote the areal velocity by $\frac{1}{2}c$. By assumption $\frac{1}{2}c \neq 0$. Since the areal velocity in polar coordinates is $\frac{1}{2}r^2\varphi^{\cdot}$ (p. 47),

$$r^2\varphi^{\cdot} = c, \quad \text{or} \quad \varphi^{\cdot} = c\,/\,r^2. \tag{6}$$

Suppose that the path has the equation $r = f(\varphi)$. Then

$$r^{\cdot} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{c}{r^2} = -c\frac{\mathrm{d}(1\,/\,r)}{\mathrm{d}\varphi}, \tag{7}$$

and hence

$$r^{\cdot\cdot} = \frac{\mathrm{d}r^{\cdot}}{\mathrm{d}t} = \frac{\mathrm{d}r^{\cdot}}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -c^2\frac{\mathrm{d}^2(1\,/\,r)}{\mathrm{d}\varphi^2}\frac{1}{r^2}. \tag{8}$$

From formula (5) in virtue of (6) and (8) we obtain:

$$P = m\left(-\frac{c^2}{r^2}\frac{\mathrm{d}^2(1\,/\,r)}{\mathrm{d}\varphi^2} - \frac{c^2}{r^3}\right),$$

and therefore

$$P = -\frac{mc^2}{r^2}\left(\frac{\mathrm{d}^2(1\,/\,r)}{\mathrm{d}\varphi^2} + \frac{1}{r}\right). \tag{I}$$

The above formula is called *Binet's formula.*

This formula enables one to determine the force acting in a central motion if one knows the equation of the path. Conversely, knowing the force P as a function of r and φ, we can determine the path.

§ 9. Planetary motions. Kepler's laws. On the basis of observations Kepler gave the following three laws relating to planetary motions:

1. The planets describe ellipses with the sun at one focus.
2. The radius vectors emanating from the sun sweep out equal areas in equal times.
3. The squares of the periods of two planets are proportional to the

third powers of their mean distances from the sun (where by the mean distance is meant the semi-major axis of the ellipse along which a planet moves).

The third law is not quite exact. The reason for this we shall know later. Let us further observe that Kepler's laws are strictly kinematic.

Corollaries from Kepler's laws. From Kepler's laws Newton deduced (by means of dynamics) the law defining the forces which cause the motion of the planets. From the first two of Kepler's laws it follows that the planets move along plane paths with a constant areal velocity. Therefore, in virtue of the converse theorem on p. 86, the forces acting on the planets are central forces whose lines of action pass through the sun.

Let us select axes x, y in the plane of motion of the planet, placing the origin of the frame in the sun as the focus of the ellipse along which the planet revolves. As the direction of the x-axis let us choose the direction of the major axis of the ellipse, and give to the x-axis a senso such that the centre of the ellipse will lie on the negative side of the x-axis. Denote the major axis of the ellipse by $2a$, the minor axis by $2b$, and the distance between the foci by $2e$. The equation of the ellipse in polar coordinates will then have the form

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\varphi}, \qquad (1)$$

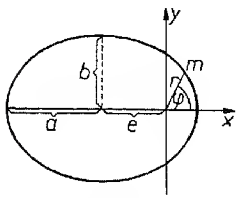

Fig. 71.

where

$$\varepsilon = \frac{e}{a} = \frac{\sqrt{a^2-b^2}}{a}. \qquad (2)$$

From Binet's formula (p. 87, (I)) we can obtain the force acting on the planet. We have in virtue of (1)

$$\frac{1}{r} = \frac{1+\varepsilon\cos\varphi}{a(1-\varepsilon^2)}, \text{ whence } \frac{d^2(1/r)}{d\varphi^2} = -\frac{\varepsilon\cos\varphi}{a(1-\varepsilon^2)},$$

and hence by (2) and Binet's formula

$$P = -\frac{mc^2a}{b^2r^2}. \qquad (3)$$

The area of an ellipse is πab; denoting the period of the planet by T, and noting that $\frac{1}{2}c$ is the areal velocity, we obtain $\frac{1}{2}c = \pi ab / T$, or $c = 2\pi ab / T$. Therefore in virtue of (3) we get

$$P = -\frac{4\pi^2ma^3}{r^2T^2}. \qquad (4)$$

Since $P < 0$, the force is directed towards the sun.

By Kepler's third law we have for two planets $T^2 / T_1^2 = a^3 / a_1^3$, or $a^3 / T^2 = a_1^3 / T_1^2$. The ratio a^3 / T^2 therefore has a constant value for all planets. Putting

$$\mu = a^3 / T^2, \qquad (5)$$

we obtain from (4)

$$P = -\frac{4\pi^2\mu m}{r^2}. \qquad (6)$$

Hence: *the force under whose influence a planet moves is directed towards the sun, and is directly proportional* (*in magnitude*) *to the mass of the planet and inversely proportional to the square of the distance from the sun.*

Law of universal gravitation. The preceding result deduced from Kepler's laws suggested to Newton the supposition that the force acting on the planet is due to the mutual attraction of the planet and the sun. Newton generalized this reflection in the form of the law of *universal gravitation:*

Any two material points attract each other with forces whose magnitude is directly proportional to the product of the masses and inversely proportional to the square of the distance between them.

According to the law of action and reaction, the forces with which the material points attract each other are equal in magnitude, opposite in direction and act along the line joining these points. Denoting by m_1 and m_2 the masses of the points, by r the distance between them, and by P the magnitude of the force with which they attract each other, we therefore obtain

$$P = K\frac{m_1m_2}{r^2}, \qquad (I)$$

where K is a certain constant, the so-called *gravitational constant* which depends only on the units of length, mass, and time.

From equation (I) we have $K = Pr^2 / m_1m_2$; consequently $[K] = [P][r]^2 / [m_1][m_2]$, and hence $[K] = L^3M^{-1}T^{-2}$. Measurements have shown that in the c. g. s. system:

$$K = 6.6 \cdot 10^{-8}\,\mathrm{cm}^3 \cdot \mathrm{g}^{-1} \cdot \mathrm{sec}^{-2}.$$

The gravitational constant can be measured by means of the so-called *Jolly's balance*. It is a balance having an upper and a lower pan on one side and a single pan on the other side. A body a of mass $\dot{m}$ is placed on the upper pan and balanced by a weight of mass m on the opposite pan.

Body a is next transferred to the lower pan; this will not disturb the equilibrium. However, if a body b of mass M is placed under the lower pan, then the balance will tilt. In order to restore equilibrium we must add a mass μ to the mass m.

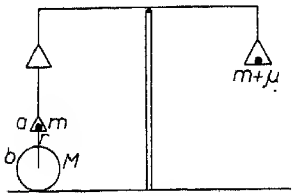

Fig. 72.

In the experiment the body b was a lead sphere. Since, as can be shown, a homogeneous sphere attracts an exterior point as if the entire mass of the sphere were concentrated at its centre, denoting by r the distance from the centre of the sphere to the body a, we have $KmM / r^2 = \mu g$, or

$$K = \mu g r^2 / mM.$$

Mass of the earth. It can be shown that a sphere composed of concentric layers of constant density attracts an exterior point as if the mass of the sphere were concentrated at its centre. Assuming that the earth satisfies the preceding condition, and denoting by M the mass of the earth, by R its radius and by Q the weight of a body of mass m (on the surface of the earth), we obtain $Q = KmM / R^2$. $Q = mg$, therefore $mg = KmM / R^2$, or

$$M = gR^2 / K. \qquad (7)$$

Using $g = 9.81 \text{ m} \cdot \text{sec}^{-2}$, $R = 6300$ km, $K = 6.6 \cdot 10^{-8} \text{ cm}^3 \cdot \text{g}^{-1} \cdot \text{sec}^{-2}$, we obtain (after changing m and km into cm)

$$M = 6 \cdot 10^{27} \text{ g}.$$

The density of the earth is obtained from the formula

$$\varrho = M / \tfrac{4}{3}R^3\pi = 3g / 4KR\pi = 5.6 \text{ g/cm}^3.$$

Kepler's equation. We shall now determine the position of a planet at a given moment of time. Let us choose a system of coordinates in the plane of motion of the planet as on p. 88. In a rectangular coordinate system the ellipse along which a planet moves has the equation

$$(x + e)^2 / a^2 + y^2 / b^2 = 1.$$

Let us introduce an auxiliary angle u defined by the equations:

$$(x + e) / a = \cos u, \quad y / b = \sin u. \qquad (\text{II})$$

The angle u is called the *eccentric anomaly*.

Equations (II) define the angle u unambiguously. From (II) we get

$$x = a(\cos u - e / a), \quad y = b \sin u.$$

Substituting $\varepsilon = e / a$, $b = a\sqrt{1 - \varepsilon^2}$, we obtain

$$x = a(\cos u - \varepsilon), \quad y = a\sqrt{1 - \varepsilon^2} \sin u. \qquad (8)$$

The radius vector r is obtained from the equation

$$r^2 = x^2 + y^2 = a^2(1 - \varepsilon \cos u)^2.$$

Therefore

$$r = a(1 - \varepsilon \cos u). \qquad (\text{III})$$

The angle φ which r makes with the x-axis is called the *true anomaly*.

From the equation of the ellipse in polar coordinates (p. 88, (1)) we get $r\varepsilon \cos \varphi = a(1 - \varepsilon^2) - r$; hence

$$r\varepsilon(1 + \cos \varphi) = (1 - \varepsilon)[a(1 + \varepsilon) - r],$$

whence by (III), $r(1 + \cos \varphi) = a(1 - \varepsilon)(1 + \cos u)$. Since $1 + \cos \varphi = 2 \cos^2 \frac{1}{2}\varphi$ and $1 + \cos u = 2 \cos^2 \frac{1}{2}u$,

$$\sqrt{r} \cos \tfrac{1}{2}\varphi = \sqrt{a(1 - \varepsilon)} \cos \tfrac{1}{2}u, \qquad (\text{IV})$$

and similarly

$$\sqrt{r} \sin \tfrac{1}{2}\varphi = \sqrt{a(1 + \varepsilon)} \sin \tfrac{1}{2}u;$$

whence

$$\tan \tfrac{1}{2}\varphi = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \tan \tfrac{1}{2}u. \qquad (\text{V})$$

Formulae (IV) and (V) determine φ unambiguously in terms of u.

Let us suppose that at the moment $t = 0$, $u = 0$, and hence $\varphi = 0$. The area of an ellipse is πab. If T denotes the period of revolution of a planet, then the areal velocity is $\pi ab / T$. Hence the radius vector sweeps out an area $\frac{\pi ab}{T} t$ during the time from 0 to t. This area can also be represented in the form of an integral

$$\frac{\pi ab}{T} t = \tfrac{1}{2} \int_0^{\Phi} r^2 \, d\varphi. \qquad (9)$$

Differentiating (V) we obtain

$$\frac{d\varphi}{\cos^2 \frac{1}{2}\varphi} = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \, \frac{du}{\cos^2 \frac{1}{2}u}.$$

Therefore by (IV)

$$d\varphi = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \cdot \frac{a(1 - \varepsilon)}{r} du = \frac{a\sqrt{1 - \varepsilon^2}}{r} du.$$

Substituting in (9) we obtain

$$\frac{\pi ab}{T} t = \frac{a\sqrt{1-\varepsilon^2}}{2} \int_0^u r\,du,$$

whence in virtue of (III),

$$\frac{\pi ab}{T} t = \frac{a^2\sqrt{1-\varepsilon^2}}{2} (u - \varepsilon \sin u).$$

From this and the fact that $a^2\sqrt{1-\varepsilon^2} = ab$, we get

$$u - \varepsilon \sin u = 2\pi t / T. \tag{VI}$$

The expression $2\pi t / T$ is called the *mean anomaly*.

Equation (VI) is called *Kepler's equation*.

By means of Kepler's equation we can determine u at each moment t, and then by equations (III), (IV), (V) the radius vector r and angle φ. Astronomy gives numerous methods for solving Kepler's equation.

In astronomy the eccentric anomaly u is usually denoted by the letter E, the true anomaly φ by v, and the mean anomaly $2\pi t / T$ by M.

§ 10. Work. Suppose that a material point was displaced from point A to B and that during this displacement a force $\mathbf{P}$ (there can be other forces besides) acted on it.

Constant force. Let us assume that the force $\mathbf{P}$ acting on a material point during its motion from A to B was constant in magnitude, direction, and sense (even though the motion could take place along a curve).

The work of the force $\mathbf{P}$ through the displacement AB is defined as the scalar product

$$\mathbf{P} \cdot \overline{AB}.$$

If work is denoted by L, then

$$L = \mathbf{P} \cdot \overline{AB}. \tag{I}$$

Let α be the angle between $\mathbf{P}$ and $\overline{AB}$. Consequently

$$L = |\mathbf{P}| \cdot |\overline{AB}| \cos \alpha. \tag{II}$$

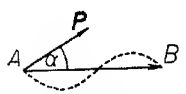

Fig. 73.

Work can be a positive or negative number, or zero. The work of a force $\mathbf{P}$ is zero if $\mathbf{P} = 0$ or $\overline{AB} = 0$ (i. e. when there is no displacement), or when $\alpha = \frac{1}{2}\pi$ (i. e. when the force is perpendicular to the displacement). If $P \neq 0$, $\overline{AB} \neq 0$, and $\cos \alpha \neq 0$, then work is a positive or negative number depending on whether α is an acute or obtuse angle.

If $\alpha = 0$ or $\alpha = \pi$ (i. e. if the force has the direction of the displacement), we have

$$L = \pm |\mathbf{P}| \,.\, |\overline{AB}|,$$

where the sign depends on whether the force and the displacement have the same or opposite senses.

From theorems on a scalar product (Chapt. I, p. 7) it follows that the work of the force $\mathbf{P}$ through the displacement $\overline{AB}$ is equal to the product of the displacement and the projection of the force on the direction of the displacement, or the product of the force and the projection of the displacement on the direction of the force.

It should be noticed that — according to the definition — *work does not depend on the time it takes the material point to be displaced from A to B.*

Let us denote the projections of the displacement $\overline{AB}$ on the coordinate axes by Δx, Δy, Δz. Hence in virtue of (I)

$$L = P_x \,\Delta x + P_y \,\Delta y + P_z \,\Delta z. \tag{I}$$

If the point A has the coordinates x_0, y_0, z_0, and B x_1, y_1, z_1, then $\Delta x = x_1 - x_0$, etc. Consequently

$$L = P_x(x_1 - x_0) + P_y(y_1 - y_0) + P_z(z_1 - z_0). \tag{2}$$

Variable force. Let us now assume that the point moves along a curve C defined parametrically by the functions:

$$x = f(\sigma), \quad y = \varphi(\sigma), \quad z = \psi(\sigma), \qquad (\sigma' \leqq \sigma \leqq \sigma''). \tag{3}$$

Suppose along with this that if $\sigma_1 < \sigma_2$, then the position of the point corresponding to the value σ_1 occurs sooner than the position corresponding to the value σ_2.

Let us further assume that there acts on a material point a variable force $\mathbf{P}$ whose projections at an arbitrary point (x, y, z) of the path are given by the functions:

$$P_x = F(x, y, z), \quad P_y = \Phi(x, y, z), \quad P_z = \Psi(x, y, z). \tag{4}$$

Of course, we assume that the functions F, Φ, Ψ are defined at every point of the path. Let us form an arbitrary subdivision δ of the interval $\sigma'\sigma''$ by means of the points $\sigma' = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma''$. To these values of the parameter σ let there correspond the points

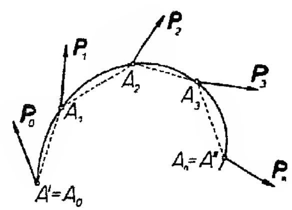

Fig. 74.

$$A' = A_0(x_0, y_0, z_0),\ A_1(x_1, y_1, z_1),\ \ldots,\ A_n(x_n, y_n, z_n) = A'',$$

on the curve C.

In virtue of (3)

$$x_i = f(\sigma_i), \quad y_i = \varphi(\sigma_i), \quad z_i = \psi(\sigma_i) \quad \text{for } i = 0, 1, \ldots, n. \tag{5}$$

Let us put:

$$\Delta x_i = x_{i+1} - x_i, \quad \Delta y_i = y_{i+1} - y_i, \quad \Delta z_i = z_{i+1} - z_i, \quad (i = 0, 1, \ldots, n-1). \tag{6}$$

Finally, let us denote the forces acting at the points $A_0, A_1, \ldots, A_n$ by $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$. By (4)

$$P_{ix} = F(x_i, y_i, z_i), \quad P_{iy} = \Phi(x_i, y_i, z_i), \quad P_{iz} = \Psi(x_i, y_i, z_i). \tag{7}$$

If the force $\mathbf{P}$ acting through the displacements $\overline{A_0A_1}$, $\overline{A_1A_2}$, ... were constant and equal to $\mathbf{P}_0, \mathbf{P}_1, \ldots$ respectively, then the work on these displacements would be expressed by the formulae:

$$L_0 = P_{0x}\,\Delta x_0 + P_{0y}\,\Delta y_0 + P_{0z}\,\Delta z_0,$$
$$L_1 = P_{1x}\,\Delta x_1 + P_{1y}\,\Delta y_1 + P_{1z}\,\Delta z_1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Putting $L' = L_0 + L_1 + L_2 + \ldots$, we therefore obtain

$$L' = \sum_{i=0}^{n-1}(P_{ix}\,\Delta x_i + P_{iy}\,\Delta y_i + P_{iz}\,\Delta z_i). \tag{8}$$

The expression on the right side of the above equality obviously depends on the subdivision δ of the interval $\sigma'\sigma''$.

If L' tends to a certain limit for every normal sequence[1]) of subdivisions $\{\delta_k\}$ of the interval $\sigma'\sigma''$, then this limit is called *the work of the force* $\mathbf{P}$ *along the curve* C (or *along the length of the curve* C).

Expression (8) can be considered as the approximate value of the work L of the force $\mathbf{P}$.

The limit of expression (8) is the so-called line integral along the curve C

$$L = \int_C (P_x\,\mathrm{d}x + P_y\,\mathrm{d}y + P_z\,\mathrm{d}z). \tag{III}$$

The line integral can be reduced to an ordinary definite integral by expressing the variables x, y, z as functions of the parameter σ. Making use of equations (3), we obtain

[1]) i. e. such that the length of the maximum interval of subdivision tends to zero.

$$L = \int_{\sigma'}^{\sigma''}[P_x f'(\sigma) + P_y\,\varphi'(\sigma) + P_z\,\psi'(\sigma)]\,\mathrm{d}\sigma,$$

where $P_x = F(f(\sigma), \varphi(\sigma), \psi(\sigma))$, $P_y = \Phi(f(\sigma), \varphi(\sigma), \psi(\sigma))$, etc. In particular, if σ denotes the time, then $f'(\sigma) = x^{\cdot}$, $\varphi'(\sigma) = y^{\cdot}$, $\psi'(\sigma) = z^{\cdot}$. Consequently

$$L = \int_{t'}^{t''}[P_x x^{\cdot} + P_y y^{\cdot} + P_z z^{\cdot}]\,\mathrm{d}t. \tag{IV}$$

Since $x^{\cdot}, y^{\cdot}, z^{\cdot}$ are the projections of the velocity $\mathbf{v}$, $P_x x^{\cdot} + P_y y^{\cdot} + P_z z^{\cdot} = \mathbf{P}\cdot\mathbf{v}$. Hence

$$L = \int_{t'}^{t''}(\mathbf{P}\cdot\mathbf{v})\,\mathrm{d}t. \tag{V}$$

Remark. Formula (III) is correct when the positions of the moving point on the curve follow each other in the order which corresponds to an increase of the parameter σ. However, if the contrary is true, i. e. if $\sigma_1 > \sigma_2$, then the position corresponding to σ_1 occurs later than that corresponding to σ_2, and hence it is necessary to substitute in formula (III) $-\mathrm{d}x, -\mathrm{d}y, -\mathrm{d}z$ for $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$. We obtain then

$$L = -\int_C (P_x\,\mathrm{d}x + P_y\,\mathrm{d}y + P_z\,\mathrm{d}z).$$

Therefore, if a material point has moved along curve C from A' to A'' and the force $\mathbf{P}$ has done work L, then, if the point moves along the curve C from A'' to A' (in this case the positions will follow each other in an order opposite to that before), the same force $\mathbf{P}$ is going to do work $-L$.

Work of a sum of forces. Let us suppose that a material point moving along a curve C was acted upon by two forces $\mathbf{P}$ and $\mathbf{Q}$. Put $\mathbf{R} = \mathbf{P} + \mathbf{Q}$. Denote by L the work of the force $\mathbf{R}$, by L' the work of the force $\mathbf{P}$, and by L'' the work of the force $\mathbf{Q}$. Then

$$L = \int_C (R_x\,\mathrm{d}x + R_y\,\mathrm{d}y + R_z\,\mathrm{d}z) =$$
$$= \int_C [(P_x + Q_x)\,\mathrm{d}x + (P_y + Q_y)\,\mathrm{d}y + (P_z + Q_z)\,\mathrm{d}z] =$$
$$= \int_C (P_x\,\mathrm{d}x + P_y\,\mathrm{d}y + P_z\,\mathrm{d}z) + \int_C (Q_x\,\mathrm{d}x + Q_y\,\mathrm{d}y + Q_z\,\mathrm{d}z) =$$
$$= L' + L'',$$

and hence

$$L = L' + L''. \tag{VI}$$

We can therefore say that *the work done by a sum of two (or more) forces along a certain curve is equal to the sum of the works done by the separate forces along this curve.*

Dimension and units of work. By (II), p. 92, we have [work] = = [force] · [distance] = $LMT^{-2}L$, and hence

$$[\text{work}] = L^2MT^{-2}.$$

The unit of work in the c. g. s. system is the *erg*. It is the work done by a force of 1 dyne acting through a distance of 1 cm. Consequently

$$\text{erg} = \text{cm}^2 \cdot \text{g} \cdot \text{sec}^{-2}.$$

A greater unit is the *Joule* (J) = 10^7 ergs. In the metric gravitational system the unit of work is the *kilogram-meter* (kgm). It is the work done by a force of 1 kg acting through a distance of 1 m. Since 1 kg (of force) = = 981 000 dynes, and 1 m = 100 cm,

$$\text{kgm} = 9.81 \cdot 10^7 \text{ ergs} = 9.81 \text{ J}.$$

§ 11. Potential force field. We called the region D (p. 77) a *force field* if on a material point, situated anywhere in the region D, there acts a force depending only on the position of that point.

The force field is defined by the given functions:

$$P_x = F(x, y, z), \quad P_y = \Phi(x, y, z), \quad P_z = \Psi(x, y, z), \qquad (1)$$

which determine the projections of the acting force $\mathbf{P}$ at the point with coordinates x, y, z.

Stress field. It can happen that a force $\mathbf{P}$ is proportional to the mass m of a material point. Then the force acting on a unit mass (i. e. the force $\mathbf{P}/m$) at a certain point of the field is called a *stress field* at this point.

An example of such a field is the earth's gravitational field. The weight of a body is proportional to the mass of the body. On the earth's surface the stress field is equal in magnitude to g (gravitational acceleration).

Lines of force. Certain curves in a force field called *lines of force* deserve special consideration. These are curves having the property that a tangent at an arbitrary point has the same direction as the force acting at that point. For instance, in the earth's gravitational field the lines of force are vertical lines. Lines of force are defined by the system of differential equations:

$$dx / P_x = dy / P_y = dz / P_z. \qquad (2)$$

Definition of a potential field. If a material point in a force field moves from a point A to a point B along some arc AB, then the work done by the acting force $\mathbf{P}$ (p. 94, (III)) is

$$L = \int_{AB} (P_x\,dx + P_y\,dy + P_z\,dz). \qquad (I)$$

The work will in general depend not only on the points A and B, but also on the path described, i. e. on the arc AB. Fields in which the work depends only on the points A and B, and not on the arc AB, play an important role in mechanics. Therefore, if a material point moves from A to B along various paths in such a field, then the force $\mathbf{P}$ always does the same work. Such fields are called *potential* or *conservative fields*.

Therefore: *a potential field is a force field in which the work does not depend on the path, but only on its origin and end points.*

If a point in a potential field has traversed a closed path (or has left the point A and returned to A), then the work done throughout the length of the path is zero. The work in a potential field depends only on the origin and end points, hence, if they coincide, the work is such as if the point had not moved at all.

Conversely, if a force field has the property that the work along every closed path is zero, then the field is a potential field. Let us choose two arbitrary points A, B and arc AMB, ANB. Denote by L' the work along arc AMB and by L'' along arc ANB. By hypothesis, the work along the closed curve $AMBNA$ is zero. This work can be represented as the sum of the works: from A to B along the arc AMB and from B to A along the arc BNA. Since the work along the arc BNA is equal to $-L''$, $L' + (-L'') = 0$, whence $L' = L''$. Therefore the work along both arcs is the same.

We can therefore say that *for a force field to be a potential field, it is necessary and sufficient that the work along every closed curve in the field be zero.*

Potential. Let us select an arbitrary coordinate system (x, y, z) and a point A in a potential field. If we look upon the point A as fixed, then the work L_{AB}, where B is an arbitrary point of the field, will depend only on the coordinates x, y, z of the point B. Therefore the work L_{AB} will be a function of the coordinates x, y, z. Denoting this function by $V(x, y, z)$, we obtain

$$L_{AB} = V(x, y, z). \qquad (3)$$

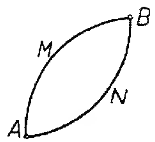

Fig. 75.

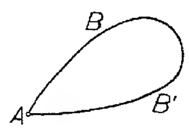

Fig. 76.

The function $V(x, y, z)$ is called a *force function* or a *potential*.

Let us consider some point B' with coordinates x', y', z'. The work along an arbitrary curve $ABB'A$ is zero. Therefore $L_{AB} + L_{BB'} + L_{B'A} = 0$.

But $L_{B'A} = -L_{AB'} = -V(x', y', z')$. Hence by (3) $V(x, y, z) + L_{BB'} - V(x', y', z') = 0$, whence

$$L_{BB'} = V(x', y', z') - V(x, y, z). \tag{II}$$

Formula (II) can be stated as follows:

In the passage from one point to another the work is equal to the difference of potentials between these points.

We have defined the potential as a function depending on the choice of the point A. Had we chosen another point $A'(x', y', z')$, the potential would have been expressed by another function $V'(x, y, z)$.

Since by the definition of a potential we have for an arbitrary point $B(x, y, z)$

$$V'(x, y, z) = L_{A'B} = V(x, y, z) - V(x', y', z'),$$

hence

$$V(x, y, z) - V'(x, y, z) = V(x', y', z') = \text{const.}$$

Therefore the difference of both functions V and V' is constant. We see from this that in a potential force field the function is defined to within a certain constant (as in an indefinite integral). As formula (II) shows, this constant does not play any role, since for the magnitude of the work there enters only the difference of the potentials at the two points.

Dimension of the potential. Since by definition the potential is equal to work, the dimension of the potential is the same as the dimension of work. Therefore

$$[\text{potential}] = L^2MT^{-2}.$$

The units of work are equally units of potential.

Relation between force and potential. Let us move a material point from the point $A(x_0, y, z)$ to the point $B(x, y, z)$ along a line parallel to the x-axis. By (I) and (II) the work is:

$$L_{AB} = V(x, y, z) - V(x_0, y, z), \text{ or } L_{AB} = \int_{AB} (P_x \, dx + P_y \, dy + P_z \, dz).$$

Since the point was translated along a parallel to the x-axis, $dy = 0$ and $dz = 0$. Therefore

$$L_{AB} = \int_{AB} P_x \, dx = \int_{x_0}^{x} P_x \, dx,$$

whence

$$V(x, y, z) - V(x_0, y, z) = \int_{x_0}^{x} P_x \, dx.$$

Taking the partial derivative with respect to x we get $\partial V / \partial x = P_x$; we obtain analogous formulae for the remaining partial derivatives.

Hence: *the partial derivatives of the potential are equal to the corresponding projections of the force on the coordinate axes*, i. e.

$$\frac{\partial V}{\partial x} = P_x, \quad \frac{\partial V}{\partial y} = P_y, \quad \frac{\partial V}{\partial z} = P_z. \tag{III}$$

Conversely, if we assume that in a given force field there exists a function V satisfying relations (III), then the field is a potential field. For let a given function V satisfy relations (III). Then the work from the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$ along an arbitrary arc AB is:

$$L_{AB} = \int_{AB} (P_x \, dx + P_y \, dy + P_z \, dz) = \int_{AB} \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right).$$

Since the expression in the parenthesis of the last integral is the total differential dV, we obtain the formula

$$L_{AB} = \int_{AB} dV = V(x_2, y_2, z_2) - V(x_1, y_1, z_1), \tag{4}$$

expressing the fact that the work does not depend on the path, but only on the end points. In virtue of (4) the function V is therefore a potential.

Hence: *if for a force field there exists a function V satisfying relations* (III), *then the force field is a potential field and the function V is a potential.*

Potential surfaces. If c is an arbitrary constant, the surfaces defined by the equation

$$V(x, y, z) = c \tag{5}$$

are called *potential surfaces.*

Therefore: *a potential surface is one along which the potential has a constant value.*

In differential geometry it is proved that the direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ of the normal to the surface (5) at the point (x, y, z) satisfy the conditions:

$$\cos\alpha : \cos\beta : \cos\gamma = \frac{\partial V}{\partial x} : \frac{\partial V}{\partial y} : \frac{\partial V}{\partial z},$$

whence, by (III),

$$\cos\alpha : \cos\beta : \cos\gamma = P_x : P_y : P_z.$$

Since the direction cosines $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$ of the force $\mathbf{P}$ satisfy similar conditions: $\cos\alpha' : \cos\beta' : \cos\gamma' = P_x : P_y : P_z$, the force $\mathbf{P}$ is normal to the potential surface.

Therefore: *at every point of a potential surface the acting force is perpendicular to this surface.* It follows from this that *the lines of force are perpendicular to potential surfaces.*

Let us select two neighbouring potential surfaces S and S' having potentials c and c', where $c' > c$. From an arbitrary point A of the surface S let us draw a normal to this surface intersecting the surface S' at the point A'.

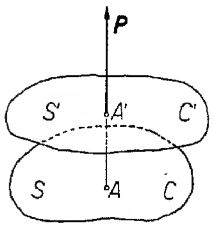

Fig. 77.

The work along the displacement from A to A' is $L_{AA'} = c' - c > 0$. Since the work is positive, the force **P** has the sense of the displacement $\overline{AA'}$.

Hence: *with respect to a potential surface the force points in the direction of increasing potential.*

Approximately we have $L_{AA'} = |\mathbf{P}|\, AA' = c' - c$, or $|\mathbf{P}| = (c' - c) / AA'$.

Hence: *on one and the same potential surface the force is approximately inversely proportional to the segment of the normal enclosed between this surface and a neighbouring potential surface.*

§ 12. Examples of potential fields. Let us now look at several kinds of potential fields which are frequently dealt with in practice.

Constant field. If a force **P** in a certain field is constant in magnitude, direction and sense, then the field is called a *constant field.*

The earth's gravitational field in a small neighbourhood of a given point on the earth-surface is a constant field.

Let us select a coordinate system (x, y, z), giving to the z-axis the direction of the force **P**, but an opposite sense. Putting $|\mathbf{P}| = mg$, we obtain

$$P_x = 0, \quad P_y = 0, \quad P_z = -mg.$$

It is easy to verify that the function

$$V = -mgz$$

is a potential because we have

$$\partial V / \partial x = 0 = P_x, \quad \partial V / \partial y = 0 = P_y, \quad \partial V / \partial z = -mg = P_z.$$

Hence: *a constant field is a potential field.*

The work from the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$ along an arbitrary path is $L_{AB} = -mgz_2 - (-mgz_1)$; hence

$$L_{AB} = mg(z_1 - z_2). \tag{1}$$

By hypothesis, the force **P** is the force of gravity, and it is clear that $z_1 - z_2 = h$ is the difference between the levels at which the points A and B are situated. Therefore, putting $|\mathbf{P}| = Q = mg$, we obtain

$$L_{AB} = Qh.$$

The potential surface has the equation $V = \text{const.}$; hence $-mgz = \text{const.}$, or $z = \text{const.}$ Therefore, potential surfaces are level surfaces (i. e. perpendicular to the direction of the force). Since the lines of force are perpendicular to the potential surfaces, the lines of force are straight lines parallel to the z-axis, i. e. vertical lines.

Central fields. If the direction of a force in a force field always passes through a certain fixed point O, then the field is called a *central field* and the point O the *centre of the field* (p. 85).

Let us assume that in a given central field the magnitude of the force at an arbitrary point A depends only on the distance r of the point A from the centre O. Denote by P the projection of the force **P** acting at A on the direction of $\overline{OA}$. Therefore P is a function of r. Set

$$P = f(r).$$

Let the origin of the coordinate system be at O. Denoting by x, y, z the coordinates of the point A and by α the angle which $\overline{OA}$ makes with the x-axis, we obtain $\cos\alpha = x / r$. Therefore

$$P_x = P\cos\alpha = P\frac{x}{r} \text{ and similarly } P_y = P\frac{y}{r}, \quad P_z = P\frac{z}{r}. \tag{2}$$

Put $V = \int P\, dr = \int f(r)\, dr$. Since $r = \sqrt{x^2 + y^2 + z^2}$, $\partial r / \partial x = x / r$, $\partial r / \partial y = y / r$, and $\partial r / \partial z = z / r$. Therefore

$$\frac{\partial V}{\partial x} = \frac{dV}{dr}\cdot\frac{\partial r}{\partial x} = f(r)\frac{x}{r} = P\frac{x}{r} = P_x,$$

and analogously

$$\frac{\partial V}{\partial y} = P_y, \quad \frac{\partial V}{\partial z} = P_z.$$

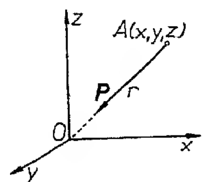

Fig. 78.

Our field is hence a potential field and the function V is a potential.

Therefore: *a central field in which the force depends only on the distance of the point from the centre is a potential field, and the potential is expressed by the formula*

$$V = \int P\, dr. \tag{3}$$

Since the potential at the point A is a function of the distance r of the point A from the centre O, the potential has a constant value on spheres with centre at O. Hence the potential surfaces in this case will be spheres with centre at O. The lines of force are obviously straight lines passing through the point O.

Newtonian gravitational field. Let us suppose that a point of mass m is attracted with a force $\mathbf{P}$ by a fixed point of mass M according to Newton's law of gravitation (p. 89, (I)), i. e. that

$$|\mathbf{P}| = KmM / r^2.$$

Since the force is directed towards the point M, the field is a central field whose centre is the point M. Therefore, according to the definition of the number P, $P = -KmM / r^2$.

We have

$$V = \int P \, dr = -\int KmM \, dr / r^2.$$

Consequently

$$V = KmM / r. \tag{4}$$

Hence the work along an arbitrary arc $A'A$ is

$$L_{A'A} = KmM \left(\frac{1}{r} - \frac{1}{r'} \right),$$

where r and r' denote the distances of the points A and A' from the centre. If, in particular, we select the point A' at infinity i. e. if we put $r' = \infty$, then we shall obtain

$$L_{\infty A} = KmM / r = V. \tag{5}$$

Therefore: *in a Newtonian gravitational field the potential at a point A is equal to the work a force would do in bringing a material point from infinity to A.*

Axial field. A force field having the property that at every point of the field the line of action of the force cuts a certain fixed line l at right angles is called an *axial field*, and the line l is called the *axis of the field*.

Let us assume that the magnitude of the force $\mathbf{P}$ acting at an arbitrary point A depends only on the distance r of the point from the axis of the field. Put $P = -|\mathbf{P}|$ or $P = |\mathbf{P}|$ depending on whether the force $\mathbf{P}$ is pointed towards or away from the axis l. Since the magnitude of the force $\mathbf{P}$ is a function of r (i. e. the distance of the point A from the axis l), we can write

$$P = f(r).$$

Let us select a system of coordinates in which the axis of the field is the z-axis. It is easy to see that the projections of the force $\mathbf{P}$ acting at the

point $A(x, y, z)$ are $P_x = Px / r$, $P_y = Py / r$, and $P_z = 0$, where $r = \sqrt{x^2 + y^2}$. Put

$$V = \int P dr = \int f(r) \, dr.$$

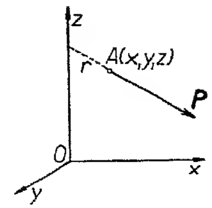

Fig. 79.

Therefore

$$\frac{\partial V}{\partial x} = \frac{dV}{dr} \cdot \frac{\partial r}{\partial x} = P \frac{x}{r} = P_x.$$

Similarly

$$\frac{\partial V}{\partial y} = P_y.$$

Since V does not depend on z (because r does not depend on z),

$$\frac{\partial V}{\partial z} = 0 = P_z.$$

It follows from this that the given field is a potential field and V is the potential.

Hence: *an axial field in which the magnitude of the force depends only on the distance of the point from the axis is a potential field and the potential is*

$$V = \int P \, dr. \tag{6}$$

It is easy to see that in this case the potential surfaces are cylinders whose common axis is the axis of the field. The lines of force are straight lines cutting the axis at right angles.

For instance, if $P = m\omega^2 r$ (ω constant), then $V = \int P \, dr = \int m\omega^2 r \, dr$, and hence $V = \frac{1}{2} m\omega^2 r^2 = \frac{1}{2} m\omega^2 (x^2 + y^2)$. The potential surfaces are obtained by setting $V = \text{const}$. Therefore $\frac{1}{2} m\omega^2 (x^2 + y^2) = \text{const}$, whence $x^2 + y^2 = \text{const}$; this is the equation of a cylinder whose axis is the z-axis.

Sum of potential fields. Let there be given several force fields $\mathbf{P}_1, \mathbf{P}_2, \ldots$ in a certain region D. A force field $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots$ in the region D is called the *sum of the force fields* $\mathbf{P}_1, \mathbf{P}_2, \ldots$

If the force fields $\mathbf{P}_1, \mathbf{P}_2, \ldots$ are potential fields, then — as is easily shown — *the sum of the fields is also a potential field whose potential V is equal to the sum of the potentials $V_1, V_2, \ldots$ of the separate fields.*

For let us put $V = V_1 + V_2 + \ldots$ We have

$$\frac{\partial V}{\partial x} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial x} + \ldots = P_{1x} + P_{2x} + \ldots = P_x,$$

and analogously

$$\frac{\partial V}{\partial y} = P_y, \quad \frac{\partial V}{\partial z} = P_z.$$

Therefore V is the potential of the sum of the given fields.

Let us suppose, for example, that a point of mass m is attracted according to Newton's law by two fixed points of masses m_1 and m_2 with forces $\mathbf{P}_1$ and $\mathbf{P}_2$. The resultant force will therefore be $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$. On p. 103 we have shown that the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ have potentials. Hence according to (4), p. 102, denoting the distances of m from m_1 and m_2 by r_1 and r_2, we obtain:

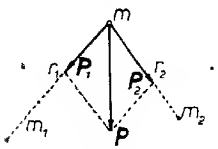

Fig. 80.

$$V_1 = K\frac{mm_1}{r_1} \quad \text{and} \quad V_2 = K\frac{mm_2}{r_2}.$$

The force $\mathbf{P}$ therefore has the potential $V = V_1 + V_2$. Consequently, $V = Km(m_1 / r_1 + m_2 / r_2)$. Similarly, if a point of mass m is attracted by n fixed points of masses $m_1, m_2, \ldots, m_n$ according to Newton's law, then

$$V = Km\left(\frac{m_1}{r_1} + \frac{m_2}{r_2} + \ldots + \frac{m_n}{r_n}\right), \tag{7}$$

where $r_1, r_2, \ldots, r_n$ denote the distances of the point m from the points $m_1, m_2, \ldots, m_n$, respectively.

§ 13. Kinetic and potential energy. Let a force $\mathbf{P}$ act on a material point $A(x, y, z)$ of mass m. Then (p. 78, (I)):

$$m\ddot{x} = P_x, \quad m\ddot{y} = P_y, \quad m\ddot{z} = P_z.$$

Multiply both sides of the first equation by $\dot{x}$, of the second by $\dot{y}$, of the third by $\dot{z}$, and add. We obtain

$$m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = P_x\dot{x} + P_y\dot{y} + P_z\dot{z}. \tag{1}$$

Let us denote the absolute value of the velocity of the point A by v. Then $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, whence $\mathrm{d}(v^2) / \mathrm{d}t = 2(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z})$, and hence $\mathrm{d}(\frac{1}{2}mv^2) / \mathrm{d}t = m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z})$. Substituting this equation in (1), we obtain

$$\mathrm{d}(\tfrac{1}{2}mv^2) / \mathrm{d}t = P_x\dot{x} + P_y\dot{y} + P_z\dot{z}.$$

Integrating both sides (with respect to t) from the initial time t_0 to t, we get

$$\int_{t_0}^{t} \frac{\mathrm{d}(\frac{1}{2}mv^2)}{\mathrm{d}t}\,\mathrm{d}t = \int_{t_0}^{t} [P_x\dot{x} + P_y\dot{y} + P_z\dot{z}]\,\mathrm{d}t. \tag{2}$$

Let v_0 be the absolute value of the velocity at the initial moment t_0; the left hand side of (2) becomes $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$, and the right hand side equals (p. 95, (IV)) the work that the force $\mathbf{P}$ did during the time from t_0 to t. Let us denote this work by L_{t_0t}. Equation (2) can therefore be written in the form

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = L_{t_0t}. \tag{3}$$

The expression $\frac{1}{2}mv^2$ is called the *kinetic energy* of the point.

Putting

$$E = \tfrac{1}{2}mv^2, \quad E_0 = \tfrac{1}{2}mv_0^2 \tag{4}$$

we obtain:

$$E - E_0 = L_{t_0t}. \tag{I}$$

Hence: *the increase in kinetic energy in a certain time is equal to the work of the acting force in this time.*

This theorem is called the *equivalence of work and kinetic energy.*

In particular, if the work of the force $\mathbf{P}$ is constantly zero, then $E - E_0 = 0$, i. e. $E = E_0$, and hence by (4) $v = v_0$. Therefore the point has a velocity which is constant in magnitude. Hence, if the force is e. g. constantly perpendicular to the path, then the point moves with a uniform motion. An example is the uniform motion along a circle of a point under the influence of a force constant in magnitude and directed towards the centre of the circle.

Let the point now move in a potential field. Denote by V and V_0 the potentials the point possesses at the moments t and t_0, respectively. Then $L_{t_0t} = V - V_0$, whence by (I) $E - E_0 = V - V_0$, i. e.

$$E - V = E_0 - V_0. \tag{5}$$

The expression $-V$ is called the *potential energy.*

Setting $-V = U$, and $-V_0 = U_0$, we obtain

$$E + U = E_0 + U_0 = \text{const.} \tag{II}$$

The sum of the kinetic and potential energies, i. e. the expression $E + U$, is called the *total energy.*

Hence: *if a point moves in a potential field, then its total energy is constant.*

The preceding theorem is called the *principle of conservation of total energy.*

Dimension of kinetic and potential energies. By (4) $[E] = [m][v^2]$; hence

$$[\text{kinetic energy}] = L^2MT^{-2}.$$

Therefore kinetic energy has the dimension of work. The units of work are consequently also units of kinetic energy.

By definition, potential energy has the dimension of a potential and therefore also has the dimension of work (p. 98).

§ 14. Motion of a point attracted by a fixed mass. Motion along a curve of the second degree. Let a material point A of mass m be attracted by a fixed point of mass M with a force $\mathbf{P}$ acting according to Newton's law. Let us place the origin O of the coordinate system at the point M. Denoting (as on p. 101) by P the projection of the force on the direction of $\overline{OA}$, we obtain

$$P = -K\frac{mM}{r^2}. \tag{1}$$

Denoting the coordinates of the point A by x, y, z, we obtain (p. 101):

$$P_x = P\frac{x}{r} = -K\frac{mM}{r^2}\cdot\frac{x}{r}, \text{ etc.}$$

Therefore the equations of motion of the point A are:

$$mx^{\cdot\cdot} = -K\frac{mM}{r^2}\frac{x}{r},\quad my^{\cdot\cdot} = -K\frac{mM}{r^2}\frac{y}{r},\quad mz^{\cdot\cdot} = -K\frac{mM}{r^2}\frac{z}{r}. \tag{I}$$

In our case the force $\mathbf{P}$ has the potential $V = KmM / r$ (p. 102), and therefore the potential energy $U = -KmM / r$. By the principle of conservation of total energy $\frac{1}{2}mv^2 - KmM / r = \text{const}$, whence putting $\mu = KM$,

$$v^2 = 2\mu / r + h, \quad \text{where } h = \text{const.} \tag{2}$$

Since the motion in the problem under consideration is central (p. 85), the path is a plane curve. Let us assume, therefore, that the motion takes place in the xy-plane and that the areal velocity is different from zero.

From Binet's formula (p. 87, (I)) we obtain by (1)

$$-\frac{KmM}{r^2} = -\frac{mc^2}{r^2}\left[\frac{d^2(1/r)}{d\varphi^2} + \frac{1}{r}\right], \tag{3}$$

and since $KM = \mu$,

$$\frac{d^2(1/r)}{d\varphi^2} + \frac{1}{r} = \frac{\mu}{c^2}. \tag{4}$$

Let us set $1 / r = u$. We obtain

$$\frac{d^2u}{d\varphi^2} + u = \frac{\mu}{c^2}. \tag{5}$$

A particular solution of the above equation is $u = \mu / c^2$. The general solution of the homogeneous equation

$$\frac{d^2u}{d\varphi^2} + u = 0$$

is — as is easily verified — of the form $u = a\cos\varphi + b\sin\varphi$. The general solution of (5) will therefore be

$$u = \mu / c^2 + a\cos\varphi + b\sin\varphi,$$

where a and b are arbitrary constants. Setting $a = \varrho\cos\varphi_0$, $b = \varrho\sin\varphi_0$ (where ϱ and φ_0 are arbitrary constants) and substituting $1 / r = u$ back again, we obtain the general solution of (4)

$$1 / r = \mu / c^2 + \varrho\cos(\varphi - \varphi_0). \tag{6}$$

Now the general equation of a conic section, if the pole is at a focus, has the form

$$\frac{1}{r} = \frac{1}{p} + \frac{\varepsilon}{p}\cos(\varphi - \varphi_0),$$

where p is a parameter, ε the distance between the foci, and φ_0 the angle the axis of the curve makes with the axis of the coordinate system. Equation (6) is therefore the equation of a conic section. By comparing them we get

$$p = c^2 / \mu \quad\text{and}\quad \varepsilon = \varrho c^2 / \mu. \tag{7}$$

Such a curve is an ellipse, hyperbola or parabola, depending on whether $\varepsilon < 1$, $\varepsilon > 1$ or $\varepsilon = 1$. In order to recognize the type of conic section, we must calculate the constant ϱ. We shall determine it from formula (2).

We have $v^2 = r^{\cdot 2} + r^2\varphi^{\cdot 2}$. Since $\frac{1}{2}c$ is the areal velocity, $\frac{1}{2}c = \frac{1}{2}r^2\varphi^{\cdot}$ i. e. $\varphi^{\cdot} = c / r^2$; therefore (p. 87, formula (7))

$$v^2 = c^2\left(\frac{d(1/r)}{d\varphi}\right)^2 + \frac{c^2}{r^2}.$$

By (6) we obtain

$$v^2 = \mu^2 / c^2 + 2\mu\varrho\cos(\varphi - \varphi_0) + c^2\varrho^2. \tag{8}$$

Determining r in terms of φ from formula (6) and substituting in formula (2), we obtain $v^2 = 2\mu^2 / c^2 + 2\mu\varrho\cos(\varphi - \varphi_0) + h$, whence by comparing with formula (8)

$$\varrho = \sqrt{\frac{\mu^2}{c^4} + \frac{h}{c^2}}, \tag{9}$$

and hence by (7)

$$\varepsilon = \sqrt{1 + \frac{hc^2}{\mu^2}}.$$

Therefore $\varepsilon \gtreqless 1$ depending on whether $h \gtreqless 0$. Setting $t = t_0$, $v = v_0$, $r = r_0$, we obtain from formula (2) $h = v_0^2 - 2\mu / r_0$; consequently:

$$h \gtreqless 0 \textit{ depending on whether } v_0^2 \gtreqless 2\mu / r_0.$$

It follows from this that *the type of conic section does not depend on the direction of the velocity, but only on its magnitude.*

We can therefore determine the type of conic if we know one position of a point and its speed at that position.

Comets, for instance, move within the limits of the solar system under the influence of the sun's attraction and hence move (with respect to the sun) along conics.

Let us now assume that a point moves along an ellipse whose equation is $1/r = 1/p + (\varepsilon/p)\cos(\varphi - \varphi_0)$. From Binet's formula we get

$$-\frac{KMm}{r^2} = -\frac{c^2}{p}\frac{m}{r^2}, \text{ whence } KM = c^2/p.$$

Let a and b be the axes of the ellipse and T the period. The areal velocity will then be $\frac{1}{2}c = ab\pi / T$. Since $p = b^2/a$, $KM = 4\pi^2 a^3 / T^2$, whence

$$a^3 / T^2 = KM / 4\pi^2. \tag{10}$$

It follows from this that *the ratio a^3/T^2 depends only on the mass of the attracting body and not on the mass of the moving point.*

If the sun were at rest, then the ratio a^3/T^2 would be a constant for the planets (such as is required by Kepler's third law). The sun, however, is not at rest, since it is attracted by the planets. This fact accounts for the deviations from Kepler's law.

We shall consider this matter later in connection with the *problem of two bodies* (chap. V).

Motion along a straight line. Let us examine, in addition, the particular case when the areal velocity is zero. The motion in this case takes place along a straight line passing through the centre of the field, i. e. through the point M (p. 86). Since v denotes the absolute value of the velocity,

$$v = |\dot{r}|. \tag{11}$$

Let us suppose that at $t = 0$, $r = r_0$ and $v = v_0$. From equation (2), p. 106, it follows that

$$v^2 = 2\mu / r + h, \tag{12}$$

whence

$$h = v_0^2 - 2\mu / r_0. \tag{13}$$

Let us assume that at $t = 0$ the velocity vector of the moving point was directed away from the point M, that is, that the point was receding from M. Therefore at $t = 0$, $\dot{r} > 0$.

Let us consider the two cases depending on whether $h \geq 0$, or $h < 0$.

1° $h \geq 0$. By (12) $v^2 \geq 2\mu / r$ constantly; hence $v^2 > 0$; therefore $v > 0$ constantly. It follows from this that the point will never stop, but will always move away from M. Hence $\dot{r} > 0$ constantly, whence by (11) $v = \dot{r}$ during the entire time of the motion. From (12) we obtain

$$\dot{r} = v = \sqrt{2\mu / r + h}, \quad \text{whence} \quad \mathrm{d}r / \sqrt{2\mu / r + h} = \mathrm{d}t. \tag{14}$$

Consequently

$$\int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{2\mu r^{-1} + h}} = t. \tag{15}$$

From the above equation it follows that when t tends to ∞, r also tends to ∞, and hence the point recedes to infinity.

2° $h < 0$. In this case there exists an $r = r_1$ for which $v = 0$. We obtain the value of r_1 from (12) by putting $v = 0$ and $r = r_1$. We get

$$r_1 = -2\mu : h. \tag{16}$$

It is easy to show that $r_1 > r_0$. For we have $2\mu > 2\mu - r_0 v_0^2 = r_0(2\mu / r_0 - v_0^2) = r_0(-h)$. Since $h < 0$, $-2\mu / h < r_0$, and therefore by (16) $r_1 > r_0$.

At the beginning of the motion, so long as $r > r_1$, the point will move away from M. During this period $\dot{r} > 0$ constantly; therefore by (11) $\dot{r} = v$ and as a consequence of this, formulae (14) and (15) will hold.

Substituting r_1 for the upper limit of integration in formula (15), we obtain the time t_1 for which $r = r_1$. Therefore

$$\int_{r_0}^{r_1} \frac{\mathrm{d}r}{\sqrt{2\mu r^{-1} + h}} = t_1.$$

At $t = t_1$ we shall have $v = 0$, and for $t > t_1$ the point returns and will come closer to M.

Assuming that the earth is a sphere composed of concentric homogeneous layers (i. e. of constant density), it can be shown that the earth attracts an exterior material point as though the entire mass of the earth were concentrated at its centre O. The results obtained can therefore be applied to the

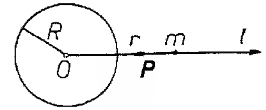

Fig. 81.

motion of bodies attracted by the earth, denoting the mass of the earth (concentrated at its centre O) by M, and assuming that the origin of the coordinate system is at the point O, while the moving point is above the surface of the earth, i. e. that $r \geq R$, where R is the radius of the earth (Fig. 81).

Example. Let us assume that a material point was thrown from the surface of the earth vertically upwards with a velocity v_0. Therefore $r_0 = R$ and by (13) $h = v_0^2 - 2\mu / R$. From formula (7), p. 90, it follows that $\mu = KM = gR^2$, where g denotes the gravitational acceleration. Hence

$$h = v_0^2 - 2gR.$$

If $v_0 < \sqrt{2gR}$, then $h < 0$, and hence the point will return to the earth again. On the other hand, if $v_0 \geq \sqrt{2gR}$, then $h \geq 0$, and hence the point will never return to the earth again.

Assuming that $R = 6300$ km, $g = 9.81$ m/sec^2, we obtain $\sqrt{2gR} = 12$ km/sec. Therefore, if the body is thrown upwards with a velocity $v_0 \geq 12$ km/sec, then it will never return to the earth again. This result does not take into account the resistance of the air.

§ 15. Harmonic motion. Simple harmonic motion. On a material point of mass m in a central field let there act a force $\boldsymbol{P}$ which is always directed towards the centre O, and whose magnitude is proportional to the distance of the point from O.

The force $\boldsymbol{P}$ is called an *elastic force*.

Let us assume for the present that the point moves along the x-axis whose origin is O. Denoting the coordinate of the point m by x, and the component of the force by P, we shall therefore have

$$P = -\lambda^2 x, \tag{1}$$

where λ^2 is the constant of proportionality. Hence $mx^{\cdot\cdot} = -\lambda^2 x$. Putting $k^2 = \lambda^2 / m$, we obtain $x^{\cdot\cdot} = -k^2 x$, whence

$$x^{\cdot\cdot} + k^2 x = 0. \tag{2}$$

From equation (2) it follows that the magnitude of the acceleration of the point is proportional to the distance of the point from O and always directed towards O.

A motion having this property is called a *simple harmonic* (or *oscillatory*) motion.

The differential equation (2) is a linear equation of the second order with constant coefficients. The roots of the characteristic equation $r^2 + k^2 = 0$ are $r_{1,2} = \pm ki$. The general solution of equation (2) is therefore

$$x = c_1 \sin kt + c_2 \cos kt. \tag{3}$$

Writing constants c_1, c_2 in the form $c_1 = a \cos kt_0$, $c_2 = -a \sin kt_0$, (where a and t_0 are arbitrary constants and $a \geq 0$), we obtain

$$x = a \sin k(t - t_0), \tag{4}$$

whence, starting the calculation of time from the moment t_0,

$$x = a \sin kt. \tag{I}$$

The constant a is called the *amplitude*.

Since $|\sin kt| \leq 1$, the amplitude a represents the greatest deviation of the point from O. For $t = \pm \pi / 2k$ we get $x = \pm a$. The path of the point is therefore the line segment from $-a$ to a. Let us put

$$T = 2\pi / k. \tag{5}$$

Then $a \sin k(t + T) = a \sin(kt + 2\pi) = a \sin kt$. Therefore by (I) the point occupies the same position at the times t and $t + T$. The motion is therefore *periodic* of *period* T.

Substituting in (I) for k the value determined from (5), we obtain

$$x = a \sin \frac{2\pi}{T} t. \tag{II}$$

If n denotes the number of periods in 1 second, then $n = 1 / T$. Hence in virtue of (II)

$$x = a \sin 2n\pi t. \tag{III}$$

Differentiating (II), we obtain:

$$x^{\cdot} = v = \frac{2a\pi}{T} \cos \frac{2\pi}{T} t, \quad x^{\cdot\cdot} = p = -\frac{4a\pi^2}{T^2} \sin \frac{2\pi}{T} t. \tag{6}$$

By (II) and (6) we can form the following table giving the position, velocity, and acceleration of the point at $t = 0, \frac{1}{4}T, \frac{1}{2}T, \frac{3}{4}T$, and T:

t	0	$\frac{1}{4}T$	$\frac{1}{2}T$	$\frac{3}{4}T$	T
x	0	a	0	$-a$	0
v	$2a\pi / T$	0	$-2a\pi / T$	0	$2a\pi / T$
p	0	$-4a\pi^2 / T^2$	0	$4a\pi^2 / T^2$	0

From the table we see that during the period T the point moves from the origin of the coordinate system to the point $x = a$, then returns

through the point O and arrives at the point $x = -a$, then returns to O etc. The maximum velocity is at O, whereas at the end points of the path (i. e. at the points $x = \pm a$) the velocity is zero. The acceleration, on the other hand, is greatest at the end points, i. e. for $x = \pm a$; at O the acceleration is zero.

Example. A sphere of mass m is attached at the lower end of a spring hanging vertically (Fig. 82). Let O denote the point at which the mass is at rest (in equilibrium). If the sphere is depressed along the vertical from its position of equilibrium, then the sphere will begin to oscillate vertically. If the mass of the spring is small, then we can assume as an approximation that the spring acts on the sphere with a force $\boldsymbol{P}$ proportional to the extension (or contraction), and is directed constantly towards the point A_0 which was the position of the end of the unstretched spring before the sphere was attached to it.

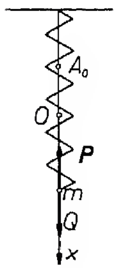

Fig. 82.

Let O be the origin of the x-axis directed vertically downwards. Putting $A_0O = d$, we obtain

$$P = -\lambda^2(x + d),$$

where λ is a constant depending on the spring. Since the sphere is in equilibrium at O, and $P = -\lambda^2 d$ (because $x = 0$), it follows that $-\lambda^2 d + mg = 0$, whence $\lambda^2 = mg / d$. During the motion $mx^{\cdot\cdot} = P + mg = -\lambda^2(x + d) + mg$; hence $mx^{\cdot\cdot} + \lambda^2 x = 0$; therefore $x^{\cdot\cdot} + k^2 x = 0$, where

$$k^2 = \lambda^2 / m = g / d.$$

By (I), p. 111, the solution of the above equation is $x = a \sin kt$; therefore

$$x = a \sin \sqrt{\frac{g}{d}}\, t.$$

The sphere will therefore execute a simple harmonic motion about the point O. By (5) the period of the motion is

$$T = 2\pi / k = 2\pi\sqrt{d / g} = 2\pi\sqrt{m} / \lambda.$$

The period of the motion therefore depends on the mass of the point.

Plane harmonic motion. Let a point move in a central force field in which the force $\boldsymbol{P}$ is directed towards the centre of the field and is (in magnitude) proportional to the distance of the point from the centre.

Let us select the centre of the field as the origin O of the coordinate

system. Since a central motion is a plane motion, we can assume that it takes place in the xy-plane.

According to Newton's law $m\boldsymbol{p} = \boldsymbol{P}$, where $\boldsymbol{p}$ denotes the acceleration. The acceleration is therefore directed towards the centre of the field and is (in magnitude) proportional to the distance of the point from the centre.

A motion having this property is called a *plane harmonic* motion and the force $\boldsymbol{P}$ is called an *elastic* force (cf. p. 110).

By hypothesis, we have

$$P_x = -\lambda^2 x, \quad \text{and} \quad P_y = -\lambda^2 y,$$

where λ is a constant of proportionality. The equations of motion will have the form

$$mx^{\cdot\cdot} = -\lambda^2 x, \quad my^{\cdot\cdot} = -\lambda^2 y.$$

As before, putting $k^2 = \lambda^2 / m$, we obtain

$$x^{\cdot\cdot} = -k^2 x, \quad y^{\cdot\cdot} = -k^2 y. \tag{7}$$

On p. 111, formula (4), we showed that the solution of the above equations is:

$$x = a' \sin k(t - t_0'), \quad y = a'' \sin k(t - t_0''), \tag{8}$$

where a', a'', t_0', t_0'' are arbitrary constants.

As is easily shown, this motion is also periodic of period $T = 2\pi / k$. From equations (8) we obtain:

$$a''x \cos kt_0'' - a'y \cos kt_0' = a'a'' \cos kt \sin k(t_0'' - t_0'),$$
$$a''x \sin kt_0'' - a'y \sin kt_0' = a'a'' \sin kt \sin k(t_0'' - t_0').$$

Squaring each of the equations and adding, we obtain

$$a''^2x^2 + a'^2y^2 - 2a'a''xy \cos k(t_0'' - t_0') = [a'a'' \sin k(t'' - t')]^2. \tag{9}$$

If $a' = 0$, or $a'' = 0$, or $t_0'' - t_0' = n\pi / k$ (where n is an integer), then equation (9) is the equation of a straight line. In the remaining cases (9) is the equation of an ellipse whose centre is at the origin of the coordinate system.

Hence: *a plane harmonic motion takes place along a straight line passing through the centre of the field, or along an ellipse whose centre is the centre of the field.*

A plane harmonic motion along a line is obviously a simple harmonic motion.

Damped harmonic motion. On a material point moving along the x-axis, let there act in addition to an elastic force $\boldsymbol{P}$ (i. e. a force which is

proportional to the distance from the centre and directed towards the centre), another force **Q** (*damping* or retarding the motion) which is in magnitude proportional to the velocity, but directed opposite to it.

The motion which the point will then execute is called a *damped harmonic* motion.

Denoting the components of the forces **P** and **Q** by P and Q, we can write:

$$P = -\lambda^2 x, \quad \text{and} \quad Q = -2\mu x^{\cdot}, \tag{10}$$

where λ^2 and $\mu > 0$ are constants of proportionality. Therefore $mx^{\cdot\cdot} = -\lambda^2 x - 2\mu x^{\cdot}$. Putting

$$\lambda^2 / m = k^2, \quad \text{and} \quad \mu / m = \varepsilon, \tag{11}$$

we therefore obtain

$$x^{\cdot\cdot} + 2\varepsilon x^{\cdot} + k^2 x = 0. \tag{IV}$$

Equation (IV) is a linear differential equation of the second order with constant coefficients. Its characteristic equation is

$$r^2 + 2\varepsilon r + k^2 = 0; \tag{12}$$

hence

$$r_{1,2} = -\varepsilon \pm \sqrt{\varepsilon^2 - k^2}. \tag{13}$$

We shall consider three cases here, depending on whether the discriminant $\varepsilon^2 - k^2$ is negative, positive, or zero.

1° $\varepsilon^2 - k^2 < 0$. This case arises when ε is small, i. e. when the damping force **Q** is small. Let us set

$$\sqrt{k^2 - \varepsilon^2} = k_1. \tag{14}$$

Therefore, by (13) $r_{1,2} = -\varepsilon \pm ik_1 t$. Hence the general solution of equation (IV) in this case is

$$x = e^{-\varepsilon t}(c_1 \sin k_1 t + c_2 \cos k_2 t).$$

Writing constants c_1, c_2 in the form $c_1 = A \cos k_1 t_0$ and $c_2 = -A \, . \sin k_1 t_0$, where $A > 0$ and t_0 are arbitrary constants, we obtain

$$x = Ae^{-\varepsilon t} \sin k_1(t - t_0). \tag{15}$$

Let us select as a new initial time, the time t_0; therefore let us substitute $t - t_0 = t'$. We get $x = Ae^{-\varepsilon(t'+t_0)} \sin k_1 t'$; writing t again instead of t' and putting $Ae^{-\varepsilon t_0} = a$, we obtain

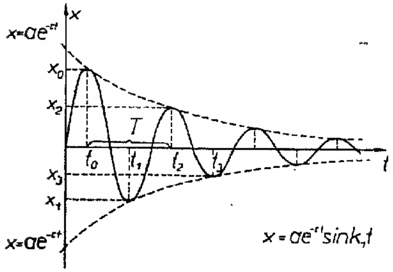

Fig. 83.

$$x = ae^{-\varepsilon t} \sin k_1 t, \qquad \text{where } a > 0. \tag{V}$$

The graph of the above function is shown in the Fig. 83. In order to determine the extrema of this function, it is necessary to determine the places where the derivative $x^{\cdot} = ae^{-\varepsilon t}(k_1 \cos k_1 t - \varepsilon \sin k_1 t)$ is zero. Hence $x^{\cdot} = 0$ for those values of t, for which

$$\tan k_1 t = k_1 / \varepsilon. \tag{16}$$

If t_0 is the smallest positive root of equation (16), then the remaining roots have the form

$$t_n = t_0 + n\pi / k_1, \tag{17}$$

where n is an arbitrary integer. Examining the sign of the second derivative, we establish that a maximum occurs for an even n, and a minimum for an odd n. It follows from this that at the times t_n the derivative $x^{\cdot}$ changes its sign and therefore the velocity changes its sense.

The times t_n are called *times of return*, while the corresponding positions of the moving point — *points of return*.

The points of return occur periodically every π / k_1 seconds, successively, once to the right and once to the left of the origin O.

The time $T_1 = 2\pi / k_1$ is called, as before, the *period of the motion*.

The time $\frac{1}{2}T_1 = \pi / k_1$ between two times of return is called a *period of oscillation*.

Hence by (17) we have

$$t_n = t_0 + \tfrac{1}{2}nT_1. \tag{18}$$

Let us take under consideration two successive points of return x_n, x_{n+1}, which correspond to the times t_n, t_{n+1}. By (V) and (18) we have

$$|x_n| = ae^{-\varepsilon(t_0+\frac{1}{2}nT_1)}|\sin k_1 t_0|, \quad |x_{n+1}| = ae^{-\varepsilon(t_0+\frac{1}{2}(n+1)T_1)}|\sin k_1 t_0|,$$

whence

$$|x_{n+1}| / |x_n| = e^{-\frac{1}{2}\varepsilon T_1}.$$

It follows from this that the coordinates x_n (in absolute value) decrease to zero in geometric progression.

Hence: *if the damping force is small, then the maximum displacements of the points follow each other in equal intervals of time* (period of oscillation) *and decrease to zero in geometric progression.*

2° $\varepsilon^2 - k^2 > 0$. This case arises when the damping force is large. It is easy to verify that the roots of the characteristic equation (13) are in this case negative. Denoting them by $-\varrho_1$ and $-\varrho_2$, we obtain the general solution of the equation (IV) in the form

$$x = Ae^{-\varrho_1 t} + Be^{-\varrho_2 t}, \tag{VI}$$

where A and B are arbitrary constants with $\varrho_1 > 0$ and $\varrho_2 > 0$.

When the time t increases, then x tends to zero rapidly. It is not difficult to verify that there exists at most one point of return. The velocity is therefore zero at most once.

3° $\varepsilon^2 - k^2 = 0$. With this assumption the characteristic equation (13) has a double root $-\varepsilon$. The general solution of (IV) has the form

$$x = e^{-\varepsilon t}(At + B), \tag{VII}$$

where A and B are arbitrary constants.

When the time increases, x tends to zero rapidly. As before, there exists at most one point of return, and therefore the velocity becomes zero at most only once.

Forced harmonic motion. On a material point moving along the x-axis let there act, in addition to an elastic force **P** and a damping force **Q**, a force **R** directed along the x-axis and depending only on time.

The component of the force **R** will therefore be

$$R = mw\, f(t),$$

where w is a constant.

Let us suppose that the force **R** is periodic, e. g. that

$$R = mw \sin(\alpha t + \beta), \tag{19}$$

where α and β are constants. The equation of motion has the form (cf. (IV), p. 114):

$$x^{\cdot\cdot} + 2\varepsilon x^{\cdot} + k^2 x = w \sin(\alpha t + \beta), \tag{20}$$

where the meaning of the constants ε and k is the same as before. In order to obtain the general solution of equation (20), we determine one particular solution of the form

$$x = b \sin(\alpha t + \gamma). \tag{21}$$

Having the determination of b and γ in mind, let us substitute (21) in (20). We get

$$(k^2 - \alpha^2)\, b \sin(\alpha t + \gamma) + 2\alpha\varepsilon b \cos(\alpha t + \gamma) = w \sin(\alpha t + \beta). \tag{22}$$

Setting $\alpha t + \gamma = 0$ the first time, and $\alpha t + \gamma = \frac{1}{2}\pi$ the second time, we get

$$2\alpha\varepsilon b = w \sin(\beta - \gamma), \quad (k^2 - \alpha^2)\, b = w \cos(\beta - \gamma), \tag{23}$$

whence

$$b^2 = \frac{w^2}{(k^2 - \alpha^2)^2 + 4\alpha^2\varepsilon^2}, \quad \tan(\beta - \gamma) = \frac{2\alpha\varepsilon}{k^2 - \alpha^2}, \tag{24}$$

and from these equations we determine b and γ.

On the basis of (24) it is easy to verify that (21) satisfies (20) identically for every t.

Let us consider the case $\varepsilon^2 - k^2 < 0$. The general solution of the homogeneous equation $x^{\cdot\cdot} + 2\varepsilon x^{\cdot} + k^2 x = 0$ is given by formula (15), p. 114. Therefore the general solution of equation (20) is

$$x = Ae^{-\varepsilon t} \sin k_1(t - t_0) + b \sin(\alpha t + \gamma), \text{ where } k_1 = \sqrt{k^2 - \varepsilon^2}. \tag{25}$$

As t increases, the first term tends to zero rapidly, and the motion becomes approximately harmonic with the equation

$$x = b \sin(\alpha t + \gamma).$$

The amplitude of this motion is b. The force **R** is periodic with period $T' = 2\pi / \alpha$. The period of the damped harmonic motion is $T_1 = 2\pi / k_1$. Let us suppose that the periods T' and T_1 differ little from each other, so that α differs little from k_1. If the damping force is small, then ε is small; hence $k_1 = \sqrt{k^2 - \varepsilon^2}$ differs little from k. Therefore k also will differ little from α. By (24), b can therefore be large even when w is small (i. e. when the force **R** is small).

We see from this that *a small periodic force with a period near that of the motion can cause large displacements of the point from the centre if the damping force is small.*

A company of soldiers marching across a bridge will cause it to vibrate. If the periods of the steps and the vibration of the bridge differ little from each other, the displacements of the bridge can become large so rapidly that the bridge will collapse. Similarly, when an automobile experiences bumps on a bad road, even small bumps, but ones whose period is near the natural period of the car springs, then the vibrations can become so large that the car springs will break.

Lissajous' curves. On a material point let there act a force **P** whose projections on the coordinate axes are (in magnitude) proportional to the coordinates of the point and directed towards the origin of the system. We can therefore assume that:

$$P_x = -\lambda_1^2 x, \quad P_y = -\lambda_2^2 y, \quad P_z = -\lambda_3^2 z,$$

where $\lambda_1, \lambda_2, \lambda_3$ are constants. The equations of motion have the form:

$$mx^{\cdot\cdot} = -\lambda_1^2 x, \quad my^{\cdot\cdot} = -\lambda_2^2 y, \quad mz^{\cdot\cdot} = -\lambda_3^2 z.$$

Putting $\lambda_1^2 / m = k_1^2$, $\lambda_2^2 / m = k_2^2$ and $\lambda_3^2 / m = k_3^2$, we obtain:

$$x^{\cdot\cdot} = -k_1^2 x, \quad {}^{\cdot\cdot} = -y\, k_2^2 y, \quad z^{\cdot\cdot} = -k_3^2 z. \tag{26}$$

The solutions of the above equations (cf. p. 111, formula (4)) are the functions:

$$x = a_1 \sin k_1(t - t_0'), \quad y = a_2 \sin k_2(t - t_0''), \quad z = a_3 \sin k_3(t - t_0'''). \quad (27)$$

The periods of these functions are (p. 111, formula (5)):

$$T_1 = 2\pi / k_1, \quad T_2 = 2\pi / k_2, \quad T_3 = 2\pi / k_3. \quad (28)$$

If the motion is periodic of period T, then the ratios $T : T_1$, $T : T_2$, $T : T_3$ must be integers. Therefore the ratios $T_1 : T_2$, $T_1 : T_3$, and $T_2 : T_3$ (or because of (28) the ratios $k_2 : k_1$, $k_3 : k_1$, and $k_3 : k_2$) must be rational numbers. Therefore, if not all of these ratios are rational numbers, then the motion is not a periodic motion.

In the case of motion in the plane, the paths of the motion defined by equations (27) are called *Lissajous' curves*; they play an important role in acoustics.

Example. The motion takes place in the xy-plane. Let $k_2 : k_1 = 2$, and $t_0' = t_0'' = 0$.

Putting $k_1 = k$ and $k_2 = 2k$, we obtain by (27):

$$x = a_1 \sin kt, \quad \text{and} \quad y = a_2 \sin 2kt.$$

Since $y = 2a_2 \sin kt \cos kt$, it follows that $\sin kt = x / a_1$ and $\cos kt = a_1 y / 2a_2 x$, whence $(x / a_1)^2 + (a_1 y / 2a_2 x)^2 = 1$; therefore

$$4a_2 x^4 - 4a_1^2 a_2^2 x^2 + a_1^4 y^2 = 0.$$

The path will therefore be a curve of the fourth degree.

§ 16. Conditions for equilibrium in a force field. If a material point in a certain force field is in equilibrium at the point A, then obviously the force $\boldsymbol{P}$ acting at A is equal to zero. Conversely, if at a certain point $A(x_0, y_0, z_0)$ of the field the force $\boldsymbol{P} = 0$, then the material point situated at A at the time $t = t_0$ without initial velocity (i. e. $\boldsymbol{v}_0 = 0$) will remain constantly at rest, i. e. in equilibrium. This follows from the fact that the initial conditions determine the motion unambiguously, and rest (i. e. motion defined by the equations $x = x_0$, $y = y_0$, $z = z_0$) satisfies the initial conditions and the equation $m\boldsymbol{p} = \boldsymbol{P}$; for, we have constantly $\boldsymbol{p} = 0$ and $\boldsymbol{P} = 0$.

In a potential field the partial derivatives of the potential V are equal, as we know, to the projections of the force on the axes of the coordinate system (§ 11, p. 99). Therefore, if the point A is a position of equilibrium, then at the point A:

$$\partial V / \partial x = 0, \ \partial V / \partial y = 0, \ \partial V / \partial z = 0. \quad (1)$$

The above equations hold in particular at those points for which the maxima or minima of the potential occur.

Hence: *the points at which the extrema of a potential occur are the positions of equilibrium.*

The positions of equilibrium can also arise, however, at such points for which the potential does not have an extremum; for, equations (1) represent only the necessary conditions for the existence of an extremum.

Stable equilibrium. Let a material point be in equilibrium at the point A in a force field.

Equilibrium is said to be *stable* if a material point, after being displaced slightly from the point A and after receiving initially a small amount of kinetic energy, will constantly move at a small distance from A and possess constantly a small amount of kinetic energy. Strictly speaking, equilibrium at A is stable, if for every two numbers $R > 0$ and $\varepsilon > 0$, we can choose numbers $R_0 > 0$ and $\varepsilon_0 > 0$, such that a material point situated anywhere at a distance less than R_0 from A, after receiving initially kinetic energy in amount less than ε_0, will move at distance from A constantly less than R and possess kinetic energy constantly less than ε.

If the equilibrium at the point A is not stable, then this point is said to be in an *unstable equilibrium.*

Dirichlet's theorem. *In a potential field a point at which the potential attains a proper maximum is the position of stable equilibrium.*

Proof. In a certain potential field let the potential V attain a proper maximum at the point A (a function is said to attain a *proper maximum* at the point A if, in a certain region about this point, it assumens its greatest value only at the point A).

Let us assume that the potential has the value zero at A; this we can always obtain by adding a suitable constant, since a potential is defined only to within a certain constant (p. 98).

Let us take arbitrary $R > 0$ and $\varepsilon > 0$. Without any loss of generality of proof we can also choose an R so small that in a sphere K with centre at A and radius R, the potential is negative everywhere outside of A. Let us denote the maximum potential on the surface of the sphere K by L; therefore $L < 0$.

Now let ε_0 be an arbitrary number satisfying the inequalities:

$$\varepsilon_0 > 0, \quad \varepsilon_0 < -\tfrac{1}{2}L, \quad \varepsilon_0 < \tfrac{1}{2}\varepsilon. \quad (2)$$

Since the potential is zero at A, there exists a sphere K_0 with centre at A and radius $R_0 < R$, such that

$$-\varepsilon_0 < V \leqq 0 \text{ in sphere } K_0. \tag{3}$$

Let us place the material point anywhere at a distance $< R_0$ from A (i. e. in sphere K_0) and give it an initial kinetic energy

$$E_0 < \varepsilon_0. \tag{4}$$

By (5), p. 105,

$$E - V = E_0 - V_0 \tag{5}$$

constantly during the motion.

Since by (3) $-\varepsilon_0 < V_0$, we have on account of (5) and (4)

$$E - V < 2\varepsilon_0. \tag{6}$$

As $E \geqq 0$, we obtain $-V \leqq 2\varepsilon_0$, whence by (2) $-V < -L$, so that $V > L$. Therefore the material point never goes outside the surface of the sphere K (because the potential on it is $\leqq L$); its motion will hence take place inside the sphere K, i. e. at a distance from A less than R. In addition, within the sphere K, $V \leqq 0$ constantly, i. e. $-V \geqq 0$; therefore by (6) $E < 2\varepsilon_0$, whence by (1) $E < \varepsilon$. Hence the equilibrium at A is stable, q. e. d.

Example. Let us consider a force field in which $P_x = -k^2x$, $P_y = -k^2y$, $P_z = -k^2z$. The field is hence a potential field with a potential $V = -\frac{1}{2}k^2r^2$, where $r^2 = x^2 + y^2 + z^2$.

The point $A(0, 0, 0)$ is the position of stable equilibrium because at this point the potential attains the largest value zero, and beyond it is negative.

We shall prove now directly the stability of equilibrium at A.

Let $R > 0$ and $\varepsilon > 0$ be given arbitrary numbers. Let us place a material point at a distance r_0 from A and give it a kinetic energy E. Therefore $E + \frac{1}{2}k^2r^2 = E_0 + \frac{1}{2}k^2r_0^2$, whence

$$E \leqq E_0 + \tfrac{1}{2}k^2r_0^2. \tag{7}$$

In addition $\frac{1}{2}k^2r^2 \leqq E_0 + \frac{1}{2}k^2r_0^2$, whence

$$r \leqq \sqrt{\frac{2}{k^2}E_0 + r_0^2}. \tag{8}$$

If we therefore choose ε_0 and R_0 such that

$$\varepsilon_0 + \tfrac{1}{2}k^2R_0^2 < \varepsilon \quad \text{and} \quad \sqrt{\frac{2}{k^2}\varepsilon_0 + R_0^2} < R$$

simultaneously, then we obtain for every $E_0 < \varepsilon_0$ and $r_0 < R_0$ by (7) and (8) $E < \varepsilon$ and $r < R$. Thus we have proved that the equilibrium at A is stable.

II. DYNAMICS OF A CONSTRAINED POINT

§ 17. Equations of motion. So far we have examined the motion of an *unconstrained* material point, i. e. one which could execute arbitrary motions when acted upon by suitable forces. However, we shall also encounter problems in which the motions of the point are subject to certain restraints, e. g., that the point must always remain on a certain line, surface, etc.

Example. Let us imagine that a small sphere is strung on a stiff wire (e. g. in the form of a circle). Whatever the forces acting on the sphere, it can execute only those motions during which it will always remain on the wire. Therefore, the problem in this case is that of investigating the motion of a material point which must always remain on a certain curve.

Such a point is said to be *constrained*, and the restraining conditions which the motions of the constrained point must satisfy are called *constraints*.

Reaction. When inquiring into the motion of constrained points, we shall assume that there acts on the constrained point (besides the given forces) a certain additional force which causes the point to maintain constraints. This additional force is called the *reaction*.

We attribute this reaction to the action on a material point by the bodies causing the constraints. The reaction of the wire is therefore e. g. the force with which the wire resists its being left by the sphere strung on it.

Let a material point A be constrained to remain on the curve C (Fig. 84). Let the reaction at a certain position of the point A be $\mathbf{R}$. The component $\mathbf{N}$ of the reaction, perpendicular to the tangent, is called the *normal reaction*, the tangential component $\mathbf{T}$ is called the *tangential reaction* or *friction*.

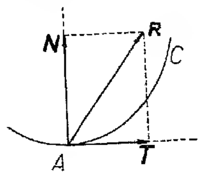

Fig. 84.

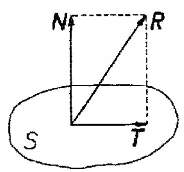

Fig. 85.

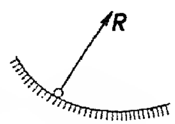

Fig. 86.

Similarly, if a point is constrained to remain on a certain surface S (Fig. 85), then the vector component of the reaction perpendicular to the surface S is called the *normal reaction*, whereas the tangential vector component is called the *tangential reaction* or *friction*.

Therefore, every time we assume that there is no friction, we are assuming equivalently that the reaction is perpendicular to the curve (surface). If there is no friction, the curve (surface) is said to be *smooth*.

If we only assume that a point A lying on a certain side of a surface cannot pass to the other side (even though it can leave this surface), then the reaction is regarded as being directed towards that side of the surface on which the point lies (Fig. 86).

For instance, if a small ball lies on a table, then the reaction of the table is directed upwards.

Equations of motion. We have defined the reaction as an additional force which causes the constrained point to maintain constraints. Therefore, if we add the reaction $\mathbf{R}$ to the acting force $\mathbf{P}$, then we can regard the material point as an unconstrained point. Denoting the mass by m, and the acceleration of the point by $\mathbf{p}$, we therefore obtain

$$m\mathbf{p} = \mathbf{P} + \mathbf{R}. \tag{I}$$

In this manner the investigation of the motion of a constrained point is reduced to the investigation of an unconstrained point. If we assume in addition, that the reaction satisfies certain special conditions, e. g. that there is no friction, then (as we shall show later) equation (I) is sufficient to determine the motion.

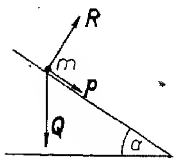

Fig. 87.

Example. Let a point of mass m slide down a plane, inclined at an angle α with the horizontal, under the influence of its weight $Q = mg$.

Let us assume that there is no friction. The reaction $\mathbf{R}$ is therefore perpendicular to the plane (Fig. 87).

Denoting the acceleration of the point by $\mathbf{p}$, we have by (I) $m\mathbf{p} = \mathbf{Q} + \mathbf{R}$. Forming the projections on the inclined plane and putting $p = |\mathbf{p}|$, we obtain $mp = mg \sin \alpha$, whence

$$p = g \sin \alpha.$$

Kinetic energy. The increase in the kinetic energy of a constrained point is equal to the sum of the works of the acting force $\mathbf{P}$ and the reaction $\mathbf{R}$. Under the assumption that there is no friction, the reaction is perpendicular to the path, and therefore the work of the reaction is zero.

It follows from this that, if there is no friction, the increase in kinetic energy is equal only to the work done by the force $\mathbf{P}$.

In particular, *if there is no friction, then the sum of the kinetic and potential energies of a point moving in a potential field is constant.*

§ 18. Motion of a constrained point along a curve. Motion along a plane curve. Let us assume that a point A of mass m is to remain on a plane curve C, and that the force $\mathbf{P}$ acting on the point lies in the plane of the curve C. Let us suppose that there is no friction, i. e. that the reaction $\mathbf{R}$ is perpendicular to the curve.

Denoting the acceleration of the point by $\mathbf{p}$, we have (cf. formula (I), p. 122)

$$m\mathbf{p} = \mathbf{P} + \mathbf{R}. \tag{1}$$

Let us give the tangent t a sense agreeing with that of the curve, and the normal n a sense towards the centre of curvature (Fig. 88). Let p_t, p_n, P_t, P_n be the projections of the acceleration $\mathbf{p}$ and the force $\mathbf{P}$ on the tangent and the normal, and let R be the projection of the reaction $\mathbf{R}$ on the normal. Forming the projections on the tangent and normal, we obtain from equation (1)

$$mp_t = P_t, \quad mp_n = P_n + R. \tag{2}$$

Let v denote the projection of the velocity on the tangent, and ϱ the radius of curvature. Then (p. 41):

$$p_t = v^{\cdot}, \quad p_n = v^2 / \varrho,$$

whence by (2):

$$mv^{\cdot} = P_t, \quad mv^2 / \varrho = P_n + R. \tag{I}$$

The first of the equations (I) enables one to determine the motion if one knows the force $\mathbf{P}$ or its projection P_t. This equation can also be written in another form, namely:

$$ms^{\cdot\cdot} = P_t, \tag{3}$$

where s denotes the arc coordinate on the curve C.

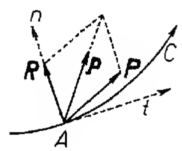

Fig. 88.

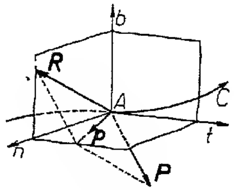

Fig. 89.

The second of the equations (I) enables one to calculate the reaction $\mathbf{R}$ if one knows the velocity v.

Motion along a space curve. Let us assume that the path is a space curve C and that there is no friction.

Let us give the tangent t a sense agreeing with that of the curve, the principal normal n a sense towards the centre of curvature, and finally the binormal b a sense such that the system (t, n, b) has a sense agreeing with that of the coordinate system (Fig. 89).

Let us form projections on the tangent, tho principal normal, and the binormal. Since the projection of the acceleration on the binormal (p. 42) and the projection $\mathbf{R}$ on the tangent are zero, we obtain from the equation $m\mathbf{p} = \mathbf{P} + \mathbf{R}$:

$$mp_t = P_t, \quad mp_n = R_n + P_n, \quad 0 = P_b + R_b,$$

whence

$$mv^{\cdot} = P_t, \quad mv^2 / \varrho = P_n + R_n, \quad P_b + R_b = 0. \tag{II}$$

The first of the equations (II) enables one to determine the motion; from the remaining two equations in (II) one can calculate the components R_n and R_b, and hence the reaction $\mathbf{R}$.

Motion of a heavy constrained point. Let the force of gravity act on a constrained material point of mass m. Let us assume that there is no friction. The potential of the gravitational force is $V = -mgz$ (the z-axis being directed vertically upwards). By the principle of conservation of total energy we therefore obtain $\frac{1}{2}mv^2 + mgz = \text{const}$, or after simplifying

$$v^2 + 2gz = h. \tag{III}$$

Knowing the velocity v_0 and the coordinate z_0 at a certain moment t_0, we can determine the constant h. We get

$$h = v_0^2 + 2gz_0, \quad \text{whence} \quad v^2 + 2gz = v_0^2 + 2gz_0. \tag{4}$$

From (III) it follows that $2gz \leqq h$, and hence $z \leqq h / 2g$. Hence the maximum height to which the point can rise is

$$z_{\max} = \left(\frac{1}{2g}\right) h = \left(\frac{1}{2g}\right) v_0^2 + z_0. \tag{5}$$

If a point is situated several times at the same level $z = z'$ during the motion, then by (III) we have $v^2 = h - 2gz'$.

Hence: *on one and the same level a point has one and the same velocity.*

Example 1. A curve C along which a point falls is situated in the vertical xz-plane. The equation of the curve C is $z = f(x)$. Let us assume that at the time $t = 0$ the point is at $A(x_0, z_0)$ and has a velocity $\mathbf{v}_0 = 0$.

Denoting the arc coordinate by s and noting that $v = s^{\cdot}$, we get by (4):

$$s^{\cdot 2} + 2gz = 2gz_0, \quad \text{whence} \quad s^{\cdot 2} = 2g(z_0 - z).$$

Let us select a sense on the curve C which agrees with the initial motion of the point (i. e. a downward sense). Up to the time when the material point arrives at the point B, situated at the same height as the point A (Fig. 90), we have $s^{\cdot} = \sqrt{2g(z_0 - z)}$, and hence $ds / \sqrt{2g(z_0 - z)} = dt$. Since $ds = \sqrt{1 + f'^2(x)}\, dx$,

$$\int_{x_0}^{\xi} \sqrt{\frac{1 + f'^2(x)}{2g[f(x_0) - f(x)]}}\, dx = t. \tag{6}$$

The above formula gives the time at which a material point arrives at the point D having coordinates $x = \xi$, $y = f(\xi)$. If x_1 denotes the abscissa of the point B, then for $x_0 \leqq \xi < x_1$ integral (6) has a finite value, and hence the time t is finite. For $\xi = x_1$ the integrand becomes infinite because by hypothesis $z_0 = f(x_0) = f(x_1)$. In this case the value of the integral can be finite or infinite. It follows from this that the material point may arrive at the point B or not: this will depend on the shape of the

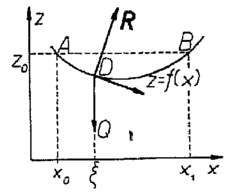

Fig. 90.

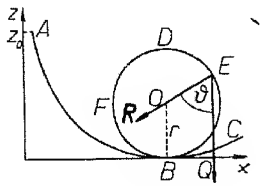

Fig. 91.

curve C. It is easy to show that if the tangent at the point B is not horizontal (i. e. if $f'(x_1) \neq 0$), then the value of (6) is finite, and hence the material point will arrive at the point B.

Example 2. Let a point slide in a vertical plane along a curve C, a portion of which, namely, $BEDF$ is a circle with centre at O and radius r. Let us assume there is no friction. Let us also assume that the point need not always remain on the curve C, just so that it does not go over to the other side; the reaction will therefore be directed towards the side on which the point is situated (Fig. 91).

Let us ask from what height z_0 should a point be released, without initial velocity, in order to traverse the periphery of the circle $BEDF$.

Let us select an arbitrary point E on the circle. Denote by v the

velocity of the point at E and by ϑ the angle which the radius OE makes with the vertical. By (I), p. 123, $mv^2 / r = mg \cos\vartheta + R$, whence

$$R = \frac{m}{r}(v^2 - gr\cos\vartheta).$$

Since $R \geq 0$ (because the reaction must be directed towards the side of the point, i. e. towards the centre of the circle),

$$v^2 - gr\cos\vartheta \geq 0. \tag{7}$$

Since the point was released from a height z_0 without initial velocity, denoting the ordinate of the point E by z, we shall have $v^2 + 2gz = 2gz_0$. Determining v^2 from this equation and substituting in (7), we obtain $2gz_0 - 2gz - gr\cos\vartheta \geq 0$, whence

$$z_0 \geq z + \tfrac{1}{2}r\cos\vartheta. \tag{8}$$

The inequality (8) is the necessary and sufficient condition which must be satisfied by the height z_0 in order that the point traverse the periphery $BDEF$. The right side of this inequality attains its maximum value at the highest point on the circle, at which $z = 2r$ and $\vartheta = 0$. Substituting these values in (8), we obtain

$$z_0 \geq 5r / 2.$$

Hence, if a material point is released from a height $z_0 \geq 5r / 2$, then the point will go completely around the circle.

If, on the other hand, $z_0 < 5r / 2$, then at a certain point of the circle, namely, at that point at which $z_0 = z + \frac{1}{2}r\cos\vartheta$ our material point will leave the circle. This is so, because were the point to move farther along the circle, then, as is easily verified, we should have $R < 0$, which is impossible, since this would mean that the point is pressed to the curve. After leaving the circle the point will obviously fall only under the influence of its weight.

Example 3. A point of mass m moves under the action of the force of gravity along a helix

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = k\varphi. \tag{9}$$

We have

$$x^{\cdot} = -r\varphi^{\cdot}\sin\varphi, \quad y^{\cdot} = r\varphi^{\cdot}\cos\varphi \quad \text{and} \quad z^{\cdot} = k\varphi^{\cdot},$$

hence $v^2 = x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2} = (r^2 + k^2)\varphi^{\cdot 2}$, whence by (III), p. 124, we obtain $(r^2 + k^2)\varphi^{\cdot 2} + 2gk\varphi = h$, and therefore

$$dt / d\varphi = \pm\sqrt{(r^2 + k^2)} / \sqrt{(h - 2gk\varphi)},$$

and finally

$$t = \pm\frac{1}{gk}\sqrt{r^2 + k^2}\sqrt{h - 2gk\varphi} + c.$$

The sign on the right hand side and the constant c depend on the initial conditions. Expressing φ in terms of t and substituting in (9), we obtain the equations of motion.

§ 19. Motion of a constrained point along a surface. Let a force $\mathbf{P}$ act on a material point of mass m. Let us assume that there is no friction and that the point is to remain constantly on the surface S whose equation is

$$F(x, y, z) = 0. \tag{1}$$

The reaction $\mathbf{R}$ is therefore perpendicular to S. From differential geometry it is known that the direction numbers of the normal to the surface are proportional to the partial derivatives $\partial F / \partial x$, $\partial F / \partial y$, $\partial F / \partial z$. Since the reaction $\mathbf{R}$ has the direction of the normal

$$R_x = \lambda\,\partial F / \partial x, \quad R_y = \lambda\,\partial F / \partial y, \quad R_z = \lambda\,\partial F / \partial z, \tag{2}$$

where λ is a factor of proportionality depending on time. Therefore $\lambda = \lambda(t)$.

From the equation $m\mathbf{p} = \mathbf{P} + \mathbf{R}$ we obtain by (2):

$$mx^{\cdot\cdot} = P_x + \lambda\frac{\partial F}{\partial x}, \quad my^{\cdot\cdot} = P_y + \lambda\frac{\partial F}{\partial y}, \quad mz^{\cdot\cdot} = P_z + \lambda\frac{\partial F}{\partial z}. \tag{I}$$

Equations (1) and (I) taken together determine the unknown functions of time $x = f(t)$, $y = \varphi(t)$, $z = \psi(t)$ and $\lambda = \lambda(t)$. After determining these functions we can calculate the reaction $\mathbf{R}$ from equations (2).

Example 1. A heavy point of mass m moves over the surface of a right circular cylinder (the z-axis being directed vertically upwards)

$$x^2 + y^2 = r^2.$$

We have here $F(x, y, z) \equiv x^2 + y^2 - r^2 = 0$, $P_x = 0$, $P_y = 0$, and $P_z = -mg$; hence by (I):

$$mx^{\cdot\cdot} = 2\lambda x, \quad my^{\cdot\cdot} = 2\lambda y, \quad mz^{\cdot\cdot} = -mg. \tag{3}$$

The third of the equations (3) gives after integrating

$$z = -\tfrac{1}{2}gt^2 + at + b, \tag{4}$$

where a and b are constants. Let the initial conditions for $t = 0$ be:

$$x_0 = r, \quad y_0 = 0, \quad z_0 = 0, \quad x_0^{\cdot} = 0, \quad y_0^{\cdot} = u, \quad z_0^{\cdot} = w, \tag{5}$$

where u and w denote certain constants ($x_0^{\cdot} = 0$, because at the time

$t = 0$ the velocity $\mathbf{v}_0$ is tangent to the cylinder, and hence perpendicular to the x-axis). By (4) and (5) we get $b = 0$, and $a = w$; therefore

$$z = -\tfrac{1}{2}gt^2 + wt. \tag{6}$$

Since $v^2 + 2gz = v_0^2 + 2gz_0$, $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2gz = u^2 + w^2$, whence by (6) $\dot{x}^2 + \dot{y}^2 + (-gt + w)^2 + 2wgt - g^2t^2 = u^2 + w^2$, and therefore

$$\dot{x}^2 + \dot{y}^2 = u^2. \tag{7}$$

Hence the projection of the point on the horizontal plane moves along the circle $x^2 + y^2 = r^2$ with a constant velocity u; the angular velocity is therefore $\omega = u / r$. From this $x = r\cos(ut / r + \varphi_0)$, and $y = r\sin(ut / r + \varphi_0)$. Since at $t = 0$, according to (5), $x_0 = r$ and $y_0 = 0$, we can take $\varphi_0 = 0$. We therefore get:

$$x = r\cos\frac{u}{r}t, \quad y = r\sin\frac{u}{r}t. \tag{8}$$

Equations (6) and (8) define the motion of the point. We obtain the factor λ from equations (3) by substituting for $\ddot{x}$ and x the values obtained from (8). We get $\lambda = -mu^2 / 2r^2$, whence by (2):

$$R_x = -\frac{mu^2}{r^2}x, \quad R_y = -\frac{mu^2}{r^2}y, \quad R_z = 0,$$

and finally

$$R = \sqrt{R_x^2 + R_y^2} = \frac{mu^2}{r^2}\sqrt{x^2 + y^2} = \frac{mu^2}{r}.$$

Hence: the reaction is constant in magnitude and always perpendicular to the axis of the cylinder.

Example 2. A point of mass m, under the influence of gravity, moves on a sphere (the z-axis being directed vertically upwards)

$$x^2 + y^2 + z^2 - r^2 = 0. \tag{9}$$

In virtue of (**I**), p. 127:

$$m\ddot{x} = 2\lambda x, \quad m\ddot{y} = 2\lambda y, \quad m\ddot{z} = 2\lambda z - mg. \tag{10}$$

Equations (10) cannot be solved by means of elementary functions. Nevertheless, we can deduce certain consequences without solving these equations.

Let us note that the reaction $\mathbf{R}$ is constantly directed towards the center of the sphere, and hence that its projection $\mathbf{R}'$ on the horizontal plane is constantly directed towards the origin of the coordinate system. Consequently, $\mathbf{R}'$ is a central force.

Since the projection of the force of gravity on the horizontal plane is zero, denoting by $\mathbf{p}'$ the projection of the acceleration of the point on the horizontal plane, we obtain $m\mathbf{p}' = \mathbf{R}'$.

It follows from this (p. 86) that the motion of the projection will be a central motion. The path of projection will therefore be either a straight line l passing through the origin O, or a curve C which will never pass through the origin (p. 86).

In the first case the motion of the point itself will take place in a vertical plane whose trace is l; hence the point will move along a meridian. This case will occur if the point is given an initial velocity tangent to the meridian, because then the projection of the velocity (on the xy-plane) will be directed towards the origin O, the areal velocity of the projection will be zero, and the path of the projection will be a straight line passing through the centre O.

In the second case, when the path of the projection is the curve C never passing through O, we will have, denoting by r_0 and r_1 the smallest and the largest distance of the projection from O, $r_0^2 \leq x^2 + y^2 \leq r_1^2$.

By (9) $z^2 = r^2 - (x^2 + y^2)$; hence $r^2 - r_1^2 \leq z^2 \leq r^2 - r_0^2$, whence

$$\sqrt{r^2 - r_1^2} \leq |z| \leq \sqrt{r^2 - r_0^2}.$$

It follows from this that the point goes around the sphere between two horizontal planes. This case will occur if the initial velocity $\mathbf{v}_0$ of the point is not tangent to the meridian, because then the projection of the velocity $\mathbf{v}_0$ on the xy-plane will not be directed towards O and the areal velocity of the projection will be different from zero.

§ 20. Mathematical pendulum.

A *mathematical pendulum* is a material point m suspended in a gravitational field by a weightless and inextensible string fixed at one end at the point S.

The string acts on the material point only when it is in tension; the reaction $\mathbf{R}$ is directed along the string towards the point S. The distance of the point m from S is constantly not greater than the length l of the string. The point can therefore move within and on the surface of a sphere K with centre at S and radius l.

Let the string be in tension and make an angle $< \frac{1}{2}\pi$ with the vertical SO. If we release the point m freely (i. e. without an initial velocity), then the point will move in a vertical plane passing through S along a circle with centre at S and radius l.

Taking an arbitrary sense on the circle, let us denote the position of the point A (lying on the lower half of the circle) by means of the arc

coordinate s, calculated from the lowest point of the circle O. Let us denote by φ the angle between SO and SA, and let the sign of the angle φ agree with the sense of the arc OA. Therefore

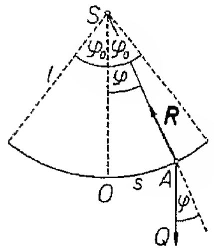

Fig. 92.

$$s = l\varphi. \tag{1}$$

Forming the projections of the force of gravity **Q** and the reaction **R** on the tangent at the point A, we obtain $ms^{\cdot\cdot} = -mg\sin\varphi$, and since by (1) $s^{\cdot\cdot} = l\varphi^{\cdot\cdot}$, it follows that $ml\varphi^{\cdot\cdot} = -mg\sin\varphi$, and hence

$$\varphi^{\cdot\cdot} = -\frac{g}{l}\sin\varphi. \tag{I}$$

Suppose that at $t = 0$, we had $\varphi = \varphi_0 > 0$. During the entire motion obviously $-\varphi_0 \leqq \varphi \leqq \varphi_0$, since the point cannot rise to position higher than the initial position.

If φ_0 is sufficiently small, then we can assume with a good approximation that $\sin\varphi = \varphi$. Therefore by (1) we obtain

$$\varphi^{\cdot\cdot} + \frac{g}{l}\varphi = 0,$$

and since according to (1) $\varphi = s\,/\,l$,

$$s^{\cdot\cdot} + \frac{g}{l}s = 0. \tag{2}$$

Comparing equation (2) with the equation of harmonic motion (p. 110) we see that the point will move with a harmonic motion. In our case $k = \sqrt{g\,/\,l}$, so that the period of motion (by (5), p. 111) is

$$T = 2\pi\sqrt{l\,/\,g}. \tag{3}$$

Formula (3) is an approximate formula derived on the assumption that the angle φ_0 is small. It is interesting to note that the *period T does not depend on the angle of the displacement.*

Let us now discard the assumption that angle φ_0 is small. Let us multiply both sides of equation (I) by $\varphi^{\cdot}$ and integrate. We obtain:

$$\tfrac{1}{2}\varphi^{\cdot 2} = \frac{g}{l}\cos\varphi + c. \tag{4}$$

$\varphi = \varphi_0$ and $s^{\cdot} = 0$ for $t = 0$; therefore by (1) $\varphi^{\cdot} = 0$. From equation (4) for $t = 0$ we get $0 = g\cos\varphi_0\,/\,l + c$, whence $c = -g\cos\varphi_0\,/\,l$, and hence $\tfrac{1}{2}\varphi^{\cdot 2} = g(\cos\varphi - \cos\varphi_0)\,/\,l$; therefore

$$\varphi^{\cdot} = \pm\sqrt{2g\,/\,l}\sqrt{\cos\varphi - \cos\varphi_0}. \tag{5}$$

Let us suppose that we are investigating the motion of the point from the time $t = 0$ to the time when the point reaches the same elevation on the opposite side of the line OS. Therefore $\varphi^{\cdot} \leqq 0$, and (5) will be

$$\varphi^{\cdot} = -\sqrt{\frac{2g}{l}}\sqrt{\cos\varphi - \cos\varphi_0}, \text{ whence } -\sqrt{\frac{l}{2g}}\frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = dt.$$

Denoting the period of oscillation by T, we obtain

$$-\sqrt{\frac{l}{2g}}\int_{\varphi_0}^{-\varphi_0}\frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = \tfrac{1}{2}T;$$

therefore

$$T = \sqrt{\frac{2l}{g}}\int_{-\varphi_0}^{\varphi_0}\frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = 2\sqrt{\frac{2l}{g}}\int_{0}^{\varphi_0}\frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}}. \tag{6}$$

Let us introduce a new variable u by means of the equation $\sin\tfrac{1}{2}\varphi = \sin u \sin\tfrac{1}{2}\varphi_0$. Since $\cos\varphi - \cos\varphi_0 = 2(\sin^2\tfrac{1}{2}\varphi_0 - \sin^2\tfrac{1}{2}\varphi)$, we obtain

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\frac{1}{2}\pi}\frac{du}{\sqrt{1 - \sin^2 u \sin^2\frac{1}{2}\varphi_0}}. \tag{7}$$

Evaluating the integral by means of a series expansion, we obtain:

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + (\tfrac{1}{2})^2\sin^2(\tfrac{1}{2}\varphi_0) + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\sin^4(\tfrac{1}{2}\varphi_0) + \right.$$
$$\left. + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2\sin^6(\tfrac{1}{2}\varphi_0) + \cdots\right].$$

For small φ_0 we obtain formula (3) by omitting the terms of the series beginning with the second term.

§ 21. Equilibrium of a constrained point. If a constrained point is in equilibrium it means that the acting force **P** balances the reaction **R**. Therefore

$$\mathbf{P} + \mathbf{R} = 0. \tag{I}$$

The above equation represents the *necessary condition for equilibrium.*

If there is no friction and the point is constrained to remain on the surface, then — as we know — the reaction is perpendicular to the surface. In the case of equilibrium, therefore, the acting force **P** must also be perpendicular to the surface.

Conversely, if at a certain time t the force **P** is perpendicular to the

surface S, and the point has a velocity $\mathbf{v} = 0$, then $\mathbf{P} + \mathbf{R} = 0$, so that the point will remain at rest. For, suppose that $\mathbf{P} + \mathbf{R} \neq 0$; then the point would move along a certain curve C lying on a surface S. Let us note that at the time t the normal acceleration is $p_n = v^2 / \varrho = 0$. From the equation $m\mathbf{p} = \mathbf{P} + \mathbf{R}$, after forming the projections on the tangent to C, we obtain $mp_t = 0$, because $\mathbf{P}$ and $\mathbf{R}$ are perpendicular to the tangent. Since $p_n = 0$ and $p_t = 0$, it follows that $\mathbf{p} = 0$. Therefore we would have $\mathbf{P} + \mathbf{R} = m\mathbf{p} = 0$, which is contrary to hypothesis.

Hence: *the necessary and sufficient condition for equilibrium of a constrained point having to remain (without friction) on a certain surface is that the acting force be perpendicular to the surface.*

A similar theorem holds for a curve.

Stable equilibrium. We define the stable equilibrium of a constrained point in a manner similar to that for an unconstrained point (p. 119), with this difference, that the displacement from the position of equilibrium has to be consistent with the constraints. A point will therefore be in *stable equilibrium* if after a small displacement (consistent with the constraints) from the position of equilibrium, and after receiving initially a small amount of kinetic energy, it will move constantly in the vicinity of the position of equilibrium and possess constantly a small amount of kinetic energy.

Equilibrium in a potential field. Let a material point in a potential field be constrained to remain on a certain surface whose equation is $F(x, y, z) = 0$. Let us assume that there is no friction.

If at a certain point $A(x, y, z)$ of the surface S the potential V attains an extremum with respect to the points on that surface, then the point A is the position of equilibrium.

For, by hypothesis, the point A is an extremum of the function V with the subsidiary condition $F(x, y, z) = 0$. Therefore by a theorem from the theory of maxima and minima there exists a constant λ such that:

$$\frac{\partial V}{\partial x} + \lambda \frac{\partial F}{\partial x} = 0, \qquad \frac{\partial V}{\partial y} + \lambda \frac{\partial F}{\partial y} = 0, \qquad \frac{\partial V}{\partial z} + \lambda \frac{\partial F}{\partial z} = 0.$$

Therefore

$$P_x + \lambda \frac{\partial F}{\partial x} = 0, \quad P_y + \lambda \frac{\partial F}{\partial y} = 0, \quad P_z + \lambda \frac{\partial F}{\partial z} = 0.$$

Since $\partial F / \partial x$, $\partial F / \partial y$, $\partial F / \partial z$ are proportional to the direction

cosines of the normal at A, the force $\mathbf{P}$ has the direction of the normal, i. e. the point A is actually the position of equilibrium.

If a point A is a proper maximum of a potential with respect to the points of a surface S, then the point A is the position of stable equilibrium.

The proof is similar to that on p. 119.

The above remarks apply equally to the case when the material point is constrained to remain on a curve.

Let a point in a gravitational field be constrained to remain on a surface S whose equation is $z = f(x, y)$ (the z-axis being directed vertically upwards). The positions of equilibrium are those points at which the force of gravity is perpendicular to the surface, i. e. at which the tangent plane is horizontal. These points can be the highest or lowest points (relative to the surrounding ones) or so-called saddle points. The proper maximum of the potential $V = -mgz$ occurs at those points for which the function $z = f(x, y)$ attains a proper minimum. Stable equilibrium therefore occurs at the lowest points. The points A, B are then positions of stable equilibrium; whereas C is a position of unstable equilibrium (see Fig. 93.).

If we displace the point from the position A, e. g. to A' and impart to it a small velocity, then it will move in the depression around the point A with a small velocity. If, on the other hand, we displace the point (even ever so slightly) from the position C to the position C', then obviously it will move away from C under the influence of its weight.

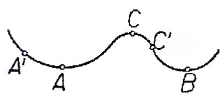

Fig. 93.

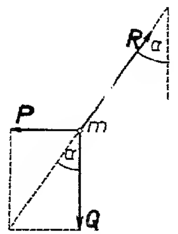

Fig. 94.

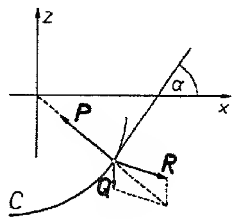

Fig. 95.

Example I. A heavy material point hanging on a string making an angle α with the vertical is in equilibrium under the influence of a horizontal force $\mathbf{P}$ (Fig. 94). The point is acted upon by the reaction $\mathbf{R}$ of the string directed along the string (towards the point of suspension), the weight $\mathbf{Q}$, and the force $\mathbf{P}$. Therefore

$$\mathbf{R} + \mathbf{Q} + \mathbf{P} = 0.$$

Putting $|\mathbf{R}| = R$, $|\mathbf{P}| = P$ and $|\mathbf{Q}| = mg$, we obtain from the triangle formed by the forces $\mathbf{R}$, $\mathbf{Q}$ and $\mathbf{P}$

$$P = mg \tan \alpha, \qquad R = mg / \cos \alpha.$$

***Example* 2.** The equation of a curve C lying in the xz-plane is $z = f(x)$. A heavy point on the curve C is attracted towards the origin O of the coordinate system by a force $\mathbf{P}$ whose magnitude is proportional to the distance of the point from O. In what position will the point be in equilibrium if we assume that there is no friction?

In a position of equilibrium the force $\mathbf{P}$, the weight $\mathbf{Q}$, and the reaction $\mathbf{R}$ balance each other (Fig. 95, p. 133); hence

$$\mathbf{P} + \mathbf{Q} + \mathbf{R} = 0. \tag{1}$$

The projections of the force $\mathbf{P}$ on the axes of the coordinate system are:

$$P_x = -\lambda^2 x, \qquad P_z = -\lambda^2 z, \tag{2}$$

where λ is a constant of proportionality. Let α denote the angle which the tangent at the position of equilibrium makes with the x-axis. Projecting on the tangent, we obtain from (1) and (2) $-\lambda^2 x \cos\alpha - \lambda^2 z \sin\alpha - mg \cdot \sin\alpha = 0$. Dividing by $\cos\alpha$ and noting that $\tan\alpha = z'$, we get

$$\lambda^2 x + \lambda^2 z z' + mgz' = 0. \tag{3}$$

Knowing the function $z = f(x)$, we can determine the x coordinate of the position of equilibrium from equation (3).

For example, if the curve C is the parabola $z = x^2 - a$, then by (3) we have $\lambda^2 x + 2\lambda^2(x^2 - a)\, x + 2mgx = 0$, whence

$$x_1 = 0, \text{ and } x_{2,3} = \pm \sqrt{\frac{\lambda^2(2a-1) - 2mg}{2\lambda^2}}.$$

The solutions $x_{2,3}$ exist provided that the expression under the radical is positive.

Let us ask now: *what is the curve on which a point is everywhere in equilibrium?*

For such a curve equation (3) must be satisfied identically. Integrating it, we obtain $\frac{1}{2}\lambda^2 x^2 + \frac{1}{2}\lambda^2 z^2 + mgz = \text{const.}$, whence

$$x^2 + \left(z + \frac{mg}{\lambda^2}\right) = \text{const.}$$

Such a curve is therefore an arbitrary circle with centre at the point $(0, -mg / \lambda^2)$.

III. DYNAMICS OF RELATIVE MOTION

§ 22. Laws of motion. Let us suppose that we are investigating the motion of a material point in a frame (x, y, z) moving relative to the inertial frame. Considering the inertial frame (*vide* p. 69) as fixed and the frame (x, y, z) as moving, we obtain (p. 60):

$$\mathbf{p}_a = \mathbf{p}_r + \mathbf{p}_t + \mathbf{p}_C \quad \text{or} \quad \mathbf{p}_r = \mathbf{p}_a - \mathbf{p}_t - \mathbf{p}_C, \tag{1}$$

where $\mathbf{p}_a, \mathbf{p}_r, \mathbf{p}_t, \mathbf{p}_C$ denote the accelerations: absolute, relative, transport and Coriolis. Multiplying (1) on both sides by the mass m of the given point, we get

$$m\mathbf{p}_r = m\mathbf{p}_a - m\mathbf{p}_t - m\mathbf{p}_C. \tag{2}$$

Let us put:

$$\mathbf{P}_a = m\mathbf{p}_a, \quad \mathbf{P}_t = -m\mathbf{p}_t, \quad \mathbf{P}_C = -m\mathbf{p}_C. \tag{I}$$

Since $\mathbf{p}_a$ is the acceleration of a point relative to the inertial frame, $\mathbf{P}_a$ is according to Newton's law the force acting on the given material point; it is called the *absolute force*. The vector $\mathbf{P}_t$ is called the *force of transport* or the *centrifugal force*, and the vector $\mathbf{P}_C$ the *force of Coriolis* or the *compound centrifugal force*.

It should be noted that the vectors $-m\mathbf{p}_t$ and $-m\mathbf{p}_C$ do not represent any forces; we have called them forces of transport and of Coriolis only for practical reasons.

By (2) and (I)

$$m\mathbf{p}_r = \mathbf{P}_a + \mathbf{P}_t + \mathbf{P}_C. \tag{II}$$

According to Newton's law we have $m\mathbf{p} = \mathbf{P}$ in an inertial frame; we see that equation (II) has a similar form.

Hence: *the laws of motion in a moving frame of reference are such as if the frame were an inertial frame, subject to the condition, however, that to the acting forces we add the force of transport and the force of Coriolis.*

The sum of the forces: absolute, transport, and Coriolis, is called the *relative force* and we denote it by $\mathbf{P}_r$.

Therefore

$$\mathbf{P}_r = \mathbf{P}_a + \mathbf{P}_t + \mathbf{P}_C. \tag{3}$$

Equation (II) can therefore be written in the form

$$m\mathbf{p}_r = \mathbf{P}_r. \tag{III}$$

An observer, being at rest relative to a moving frame and taking it as the inertial frame, will judge that the force acting on the material point is just the relative force $\mathbf{P}_r$. If the frame began its motion at a certain time t_0, then it will seem to

the observer that in addition to the force P_a acting previously, a new force $P_t + P_C$ began to act from the time t_0. For instance, a person riding on a merry-go-round judges that in addition to the force of gravity, there acts on him still another force directed from the centre of motion and trying to throw him off the merry-go-round (the centrifugal force). However, to an observer at rest relative to the inertial frame, the forces of transport and Coriolis obviously do not exist.

If a moving frame moves with an advancing motion with a constant velocity relative to a certain inertial frame, then $\mathbf{p}_t = 0$ and $\mathbf{p}_C = 0$ (*vide* p. 61); because of this $\mathbf{P}_t = 0$ and $\mathbf{P}_C = 0$, and by (II)

$$m\mathbf{p}_r = \mathbf{P}_a.$$

Therefore for such a moving frame hold the Newton's laws.

Hence: *every coordinate system which moves with an advancing motion with a constant velocity relative to an inertial frame is also an inertial frame.*

We see from this that the laws of mechanics will never enable us to decide whether a given inertial frame is at rest or not.

If we are investigating the motion of a material point in a certain frame of reference (x, y, z), then we can obtain the relative force $\mathbf{P}_r$ from equation (III). If we know in addition the absolute force $\mathbf{P}_a$ from another source and if we observe that $\mathbf{P}_a \neq \mathbf{P}_r$, then we shall be able to establish that the frame (x, y, z) is not an inertial frame and hence that it moves relative to every inertial frame.

§ 23. Examples of motion. Advancing motion of a frame. If a frame moves with an advancing motion, the acceleration of Coriolis $\mathbf{p}_C = 0$ (p. 61), and hence the force of Coriolis $\mathbf{P}_C = 0$. The acceleration of transport is constant for all points and is equal to the acceleration of the origin of the frame (relative to the inertial frame). Therefore the force of transport is constant. It follows from this that the force of transport forms a potential field (p. 100). By (II), p. 135, we then have

$$m\mathbf{p}_r = \mathbf{P}_a + \mathbf{P}_t. \tag{I}$$

Example 1. An inclined plane moves with a constant horizontal acceleration $\mathbf{a}$. A heavy point of mass m is situated on the inclined plane. Friction is not considered. What acceleration will the point m have with respect to the inclined plane?

The absolute forces are: the weight $\mathbf{Q}$ and the reaction $\mathbf{R}$ perpendicular to the inclined plane. The force of transport is $-m\mathbf{a}$. Let us select as the x-axis the intersection of the inclined plane with the vertical plane passing through m and give to it a downward sense (Fig. 96). Denoting by α the angle which the inclined plane makes with the horizontal and forming the projections on the x-axis, we obtain from (I)

$$p = g \sin\alpha - a \cos\alpha, \tag{1}$$

where $p = p_{r_x}$, and $a = |\mathbf{a}|$. We see from this that $p > 0$ or $p < 0$, depending on whether $a < g \tan\alpha$ or $a > g \tan\alpha$.

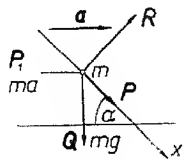

Fig. 96.

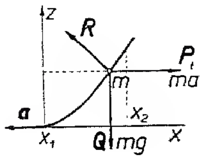

Fig. 97.

Example 2. A frame (x, y, z) moves with an advancing motion with a constant horizontal acceleration $\mathbf{a}$ in a gravitational force field. Let us assume that the z-axis is directed vertically upwards and that the x-axis has the direction of the acceleration $\mathbf{a}$, but an opposite sense.

The force of transport is $\mathbf{P}_t = -m\mathbf{a}$; putting $a = |\mathbf{a}|$ we obtain $P_{t_x} = ma$, $P_{t_y} = 0$, and $P_{t_z} = 0$. It is easy to see that the force of transport forms a potential field having the potential $V_t = max$; the potential of the force of gravity is $V_g = -mgz$. The relative force therefore forms a field having the potential

$$V = max - mgz. \tag{2}$$

If only the force of gravity $\mathbf{Q}$ acts on the material point, then applying the theorem on the conservation of total energy and setting $v = |\mathbf{v}_r|$, we obtain by (2) $\frac{1}{2}mv^2 - V = \text{const}$, whence

$$v^2 - 2ax + 2gz = h, \tag{3}$$

where h is a certain constant.

Let us suppose now that we are investigating the motion of a constrained point which is to remain on a curve $z = x^2$ lying in the xz-plane (Fig. 97).

Let us assume that at $t = 0$, $x = 0$ and $v = 0$. If friction is neglected, then the reaction is perpendicular to the path and does no work. Hence equation (3) applies to the motion. From the initial conditions it follows that $h = 0$; hence $v^2 - 2ax + 2gx^2 = 0$, whence

$$v^2 = 2x(a - gx). \tag{4}$$

Since $v^2 \geq 0$, $2x(a - gx) \geq 0$; it follows from this that $0 \leq x \leq a / g$. The motion will therefore take place along the arc closed between the abscissae $x_1 = 0$ and $x_2 = a / g$. Since $v = \mathrm{d}s / \mathrm{d}t = (\mathrm{d}s / \mathrm{d}x) \cdot (\mathrm{d}x / \mathrm{d}t) = x^{\cdot}\sqrt{1 + (\mathrm{d}z / \mathrm{d}x)^2} = x^{\cdot}\sqrt{1 + 4x^2}$, it follows in virtue of (4) that $x^{\cdot 2}(1 + 4x^2) = 2x(a - gx)$, whence

$$\sqrt{\frac{1 + 4x^2}{2x(a - gx)}}\,\mathrm{d}x = \mathrm{d}t,$$

and hence

$$\int_0^x \sqrt{\frac{1 + 4x^2}{2x(a - gx)}}\,\mathrm{d}x = t.$$

The above formula is valid from the momet $t = 0$ until the time when the point reaches the abscissa $x_2 = a / g$. For $x_2 = a / g$ in virtue of (4) we have $v = 0$. After that the return motion will take place until the time when the point reaches the abscissa $x = 0$, etc.

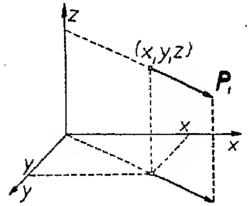

Fig. 98.

Rotary motion of a frame. Let a frame (x, y, z) rotate about the z-axis with a constant angular velocity ω (Fig. 98). The acceleration of transport has the projections: $p_{t_x} = - x\omega^2$, $p_{t_y} = - y\omega^2$ and $p_{t_z} = 0$. Therefore for the force of transport we have:

$$P_{t_x} = mx\omega^2, \quad P_{t_y} = my\omega^2, \quad P_{t_z} = 0.$$

It is easy to see that the force of transport forms a field having the potential

$$V = \tfrac{1}{2}m\omega^2(x^2 + y^2). \tag{5}$$

The acceleration of Coriolis will be $\mathbf{p}_C = 2\mathbf{v}_r \times \omega$ (p. 62). The projections of the relative velocity on the x, y, z axes are $x^{\cdot}, y^{\cdot}, z^{\cdot}$, whereas $\omega_x = 0$, $\omega_y = 0$, and $\omega_z = \omega$. Therefore $p_{C_x} = 2y^{\cdot}\omega$, $p_{C_y} = - 2x^{\cdot}\omega$ and $p_{C_z} = 0$, whence

$$P_{C_x} = - 2my^{\cdot}\omega, \quad P_{C_y} = 2mx^{\cdot}\omega, \quad P_{C_z} = 0.$$

The equations of motion will therefore have the form (p. 135, formula (II)):

$$mx^{\cdot\cdot} = P_{a_x} + mx\omega^2 - 2my^{\cdot}\omega, \quad my^{\cdot\cdot} = P_{a_y} + my\omega^2 + 2mx^{\cdot}\omega, \quad mz^{\cdot\cdot} = P_{a_z}. \tag{6}$$

The work of a force in relative motion is called *relative work*.

Since $\mathbf{p}_C$ is perpendicular to $\mathbf{v}_t$, $\mathbf{P}_C$ is perpendicular to $\mathbf{v}_r$; therefore the force of Coriolis does no relative work. The relative work of a relative force is hence reduced to the work of the absolute force and the force of transport.

If the absolute force is the force of gravity, then taking the z-axis as directed vertically upwards, we obtain $V_g = - mgz$ as the potential of the force of gravity. The force of gravity together with the force of transport forms a potential field having the potential

$$V = - mgz + \tfrac{1}{2}m\omega^2(x^2 + y^2).$$

Therefore, if we set $v = |\mathbf{v}_r|$, then by the principle of equivalence of work and kinetic energy (p. 105) we get $\tfrac{1}{2}mv^2 - V = \text{const}$; hence

$$\tfrac{1}{2}mv^2 + mgz - \tfrac{1}{2}m\omega^2(x^2 + y^2) = \text{const}.$$

Therefore

$$v^2 + 2gz - \omega^2(x^2 + y^2) = h, \tag{7}$$

where h is a constant.

If we are investigating the motion of a constrained point along a curve (or surface) motionless relative to a frame (x, y, z), then under the assumption that there is no friction, the reaction does no relative work; hence formula (7) also applies in this case.

***Example* 3.** A plane curve C revolves with a constant angular velocity ω about a vertical axis lying in its plane. Determine the motion of a constrained point moving along a curve C under the influence of the force of gravity.

Let us choose the z-axis directed vertically upwards as the axis of revolution, and the xz-plane as the plane of the curve C. Let the equation of the curve C be $z = f(x)$. Because $y = 0$, we get by (7): $v^2 + 2gz - \omega^2x^2 = h$. Assuming that at $t = 0$, $x = x_0$, $z = z_0 = f(x_0)$, and $v = 0$, we obtain $h = 2gz_0 - \omega^2x_0^2$; $\mathrm{d}s = \mathrm{d}x\sqrt{1 + f'^2(x)}$, whence $v = s^{\cdot} = x^{\cdot}\sqrt{1 + f'^2(x)}$; therefore

$$x^{\cdot 2}(1 + f'^2(x)) + 2g\,f(x) - \omega^2x^2 = h. \tag{8}$$

From this equation we can determine x as a function of the time t.

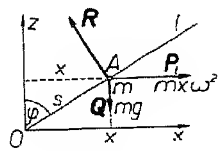

Fig. 99.

***Example* 4.** In particular, let the curve C (in above example) be the straight line l passing through the origin O of the frame and inclined at an angle φ with the z-axis (Fig. 99).

In order to determine the motion of a point along the line l, one could apply formula (8) by substituting $z = f(x) = x \cot \varphi$. However, we shall derive the equations of motion directly.

The force of transport is perpendicular to the z-axis and equal to $mx\omega^2$. Let s denote the length of the segment OA. Since the force of Coriolis and the reaction are perpendicular to the line l, the projection of the relative force on l is equal to $-mg \cos \varphi + mx\omega^2 \sin \varphi$. As $x = s \sin \varphi$, we obtain

$$ms^{\cdot\cdot} = -mg \cos \varphi + ms\omega^2 \sin^2\varphi,$$

whence

$$s^{\cdot\cdot} - s\omega^2 \sin^2\varphi = -g \cos \varphi. \tag{9}$$

The homogeneous equation $s^{\cdot\cdot} - s\omega^2 \sin^2\varphi = 0$ has a general solution of the form $s = ae^{\omega t \sin \varphi} + be^{-\omega t \sin \varphi}$. Since a particular solution of equation (9) is

$$s = \frac{g \cos \varphi}{\omega^2 \sin^2\varphi},$$

the general solution of this equation will be

$$s = ae^{\omega t \sin \varphi} + be^{-\omega t \sin \varphi} + \frac{g \cos \varphi}{\omega^2 \sin^2\varphi}. \tag{10}$$

The constants a and b are determined from initial conditions. In particular, if $\varphi = \frac{1}{2}\pi$, i. e. the line l is the x-axis, then

$$s = ae^{\omega t} + be^{-\omega t}. \tag{11}$$

§ 24. Relative equilibrium. If a material point is in equilibrium (i. e. at rest) relative to a moving frame, then the relative acceleration $\mathbf{p}_r = 0$, and the relative velocity $\mathbf{v}_r = 0$. It follows from this that the acceleration of Coriolis $\mathbf{p}_C$ is also equal to zero, and hence the force of Coriolis $\mathbf{P}_C = 0$. From equation (II), p. 135, we therefore obtain

$$\mathbf{P}_a + \mathbf{P}_t = 0. \tag{I}$$

Hence: *when a point is in relative equilibrium, the absolute force is in equilibrium with the force of transport.*

Relative equilibrium in a frame moving with an advancing motion. If a frame moves with an advancing motion, the acceleration of transport has a constant value for all points; hence the force of transport must also be the same at every point.

If, in particular, the moving frame moves with an advancing motion with a constant velocity, then $\mathbf{p}_t = 0$, whence $\mathbf{P}_t = 0$, and equation (I) expressing the condition for equilibrium reduces to the form $\mathbf{P}_a = 0$.

Example 1. A heavy point of mass m is hanging on an inextensible string in an elevator moving with an acceleration $\mathbf{p}$. Let the string have a vertical direction and let the point be in equilibrium relative to the elevator (i. e. to the frame attached to the elevator). The acting forces, namely the weight $\mathbf{Q}$ and the tension in the string $\mathbf{T}$, are therefore balanced by the force of transport (Fig. 100). Let us put $p = |\mathbf{p}|$ and $T = |\mathbf{T}|$.

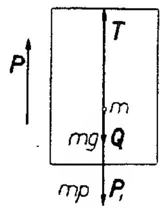

Fig. 100.

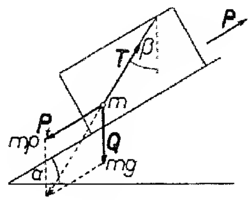

Fig. 101.

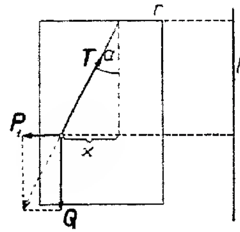

Fig. 102.

The acceleration of transport is $\mathbf{p}$. Let us assume that it is directed upwards. Therefore the force of transport is directed downwards and is in magnitude equal to mp. Forming the projections of the forces on the axis directed vertically upwards, we obtain $T - mg - mp = 0$, whence $T = mg + mp$. The tension in the string is therefore greater than the weight. If one held the body in one's hand, one would feel an increase in its weight.

Conversely, if the acceleration is directed downwards, then $T = mg - mp$; the tension in the string is smaller than the weight and the body seems lighter in this case.

Finally, if $p = 0$, then $T = mg$. Hence the tension in the string is equal to the weight during the uniform motion of the elevator.

Example 2. A carriage of a cog-wheel railway moves with an acceleration $\mathbf{p}$ along a path inclined at an angle α with the horizontal. A material point hanging on an inextensible string is in equilibrium relative to the carriage. Let β denote the angular deviation of the string from the vertical.

The weight $\mathbf{Q}$ of the point and the tension $\mathbf{T}$ of the string are in equilibrium with the force of transport $\mathbf{P}_t$ (Fig. 101); therefore

$$\mathbf{T} + \mathbf{Q} + \mathbf{P}_t = 0. \tag{1}$$

The acceleration of transport is $\mathbf{p}$. Let us assume that it is directed upwards. $\mathbf{P}_t$ is hence directed downwards, and $|\mathbf{P}_t| = m|\mathbf{p}|$. Forming the projections on the horizontal and vertical axes, we obtain from (1):

$$T \sin \beta - mp \cos \alpha = 0, \quad T \cos \beta - mp \sin \alpha - mg = 0, \tag{2}$$

where $T = |\boldsymbol{T}|$, and $p = |\boldsymbol{p}|$. From equations (2) we obtain:

$$T = m\sqrt{p^2 + g^2 + 2pg\sin\alpha}, \quad \tan\beta = \frac{p\cos\alpha}{g + p\sin\alpha}.$$

In particular, when $\alpha = 0$, i. e. when the path is horizontal, $T = m\sqrt{p^2 + g^2}$, and $\tan\beta = p\,/\,g$. Hence in the railway carriage we can determine the acceleration of the carriage from the angular deviation of the string from the vertical: because we have $p = g\tan\beta$.

***Example* 3.** A railway carriage moves along a horizontal curved path with a constant velocity v. We may suppose that the carriage turns about a certain vertical line l. Let a heavy point of mass m hanging on an inextensible string be in equilibrium relative to the carriage (Fig. 102).

Let us denote by α the angle made by the string with the vertical, and by r the distance from the point of suspension of the string to the line l. The distance of the point m from the l-axis is therefore $r' = r + x = r + d\sin\alpha$ (where d is the length of the string). The acceleration of transport $\boldsymbol{p}_t$ of the point m is perpendicular to l and directed towards l, while $|\boldsymbol{p}_t| = v^2\,/\,(r + x)$. The force of transport, having an opposite sense, is $|\boldsymbol{P}_t| = mv^2\,/\,(r + x)$. Since the weight $\boldsymbol{Q}$ and the tension $\boldsymbol{T}$ in the string are in equilibrium with the force of transport, we obtain from the triangle of forces

$$\tan\alpha = |\boldsymbol{P}_t|\,/\,|\boldsymbol{Q}| = v^2\,/\,g(r + x).$$

When x is small in comparison with r, then $\tan\alpha = v^2\,/\,gr$.

***Example* 4.** A heavy point of mass m is constrained to remain on a curve C revolving about a fixed vertical line l with an angular velocity ω. Friction is not considered. In what position will the point be in equilibrium relative to the curve C?

Let us choose a moving frame (x, y, z) revolving together with the curve C about the l-axis with an angular velocity ω, taking l as the z-axis directed upwards. Let the curve C which is at rest relative to the frame (x, y, z) be given parametrically by means of the functions:

$$x = f(\sigma), \quad y = \varphi(\sigma), \quad z = \psi(\sigma). \tag{3}$$

In a position of relative equilibrium the weight $\boldsymbol{Q}$, the reaction $\boldsymbol{R}$, and the force of transport $\boldsymbol{P}_t$ balance each other (Fig. 103). Therefore the sum $\boldsymbol{Q} + \boldsymbol{P}_t$ is perpendicular to the curve C. Denoting the coordinates of the point in relative equilibrium by x, y, z, we obtain $p_{t_x} = -x\omega^2$, $p_{t_y} = -y\omega^2$, and $p_{t_z} = 0$, from which $P_{t_x} = mx\omega^2$, $P_{t_y} = my\omega^2$, and $P_{t_z} = 0$. Therefore the sum $\boldsymbol{P}_t + \boldsymbol{Q}$ has the projections: $mx\omega^2$, $my\omega^2$, and $-mg$.

The direction numbers of the tangent are proportional to the derivatives $x' = f'(\sigma)$, $y' = \varphi'(\sigma)$, $z' = \psi'(\sigma)$. From the condition that $\boldsymbol{P}_t + \boldsymbol{Q}$ is perpendicular to the tangent it therefore follows (after dividing by m) that

$$xx'\omega^2 + yy'\omega^2 - gz' = 0. \tag{4}$$

From this equation we can determine the value of the parameter σ corresponding to the position of relative equilibrium.

In particular, if the curve C having the equation $z = \psi(x)$ is a plane curve lying in the xz-plane, then for the position of equilibrium we obtain from equation (4) (putting $x = \sigma$, $y = 0$, and $z = \psi(\sigma)$) or directly from the figure:

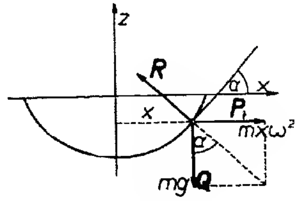

Fig. 103.

$$\tan\alpha = \psi'(x) = x\omega^2\,/\,g. \tag{5}$$

For example, if the equation of the curve C is $z = -\sqrt{r^2 - x^2}$ (i. e. the lower portion of the circle $x^2 + z^2 = r^2$), then from (5) we get $x\,/\sqrt{r^2 - x^2} = x\omega^2\,/\,g$, whence $x_1 = 0$, and $x_{2,3} = \pm\sqrt{r^2 - g^2\,/\,\omega^4}$. The solutions $x_{2,3}$ exist only when $r^2 - g^2\,/\,\omega^4 \geqq 0$, i. e. when $\omega \geqq \sqrt{g\,/\,r}$.

We ask now: *what are the curves on which a point is at every position in relative equilibrium?*

For such curves equation (4) must be satisfied identically, i. e. for every value of the parameter σ. Therefore we obtain from (4)

$$\tfrac{1}{2}\frac{d(x^2 + y^2)}{d\sigma}\omega^2 - g\frac{dz}{d\sigma} = 0.$$

Integrating, we get $\frac{1}{2}(x^2 + y^2)\,\omega^2 - gz = \text{const}$, whence

$$z = \frac{\omega^2}{2g}(x^2 + y^2) + c, \tag{6}$$

where $c = \text{const}$.

Equation (6) represents a system of paraboloids of revolution generated by revolving the parabola $z = \dfrac{\omega^2}{2g}x^2 + c$ about the z-axis. By (6) the curve lying on any one of these paraboloids satisfies equation (4) identically. The plane curves satisfying equation (4) are obtained by forming section of the paraboloid with an arbitrary plane. We get ellipses and parabolas as sections. In particular, the section with the vertical plane $y = 0$ will be the parabola $z = \dfrac{\omega^2}{2g}x^2 + c$.

§ 25. Motion relative to the earth. Force of gravity. Let us take as a frame of reference an inertial frame whose origin is in the sun and whose axes are directed towards the fixed stars. The earth is not at rest relative to such a frame. When investigating the motion of a point during a short interval of time, we can confine ourselves to the rotary motion of the earth only about a certain axis.

Let a point hung on a string in a certain place on the earth's surface be at rest relative to the earth. The absolute forces are: the attraction of earth $\mathbf{A}$ and the tension $\mathbf{T}$ in the string (equal in magnitude and direction to the weight, but opposite in sense).

The earth's force of attraction is not equal to the weight because, in the contrary case, it would be in equilibrium with the tension in the string, and the point would be at rest or in uniform motion along a straight line. However, this is not the case because the point rotates together with the earth about its axis.

Applying the conditions of relative equilibrium (§ 24, p. 140) to the frame attached to the earth, we can say that the attraction $\mathbf{A}$ of the earth and the tension $\mathbf{T}$ in the string are in equilibrium with the force of transport $\mathbf{P}_t$. Hence

$$\mathbf{A} + \mathbf{T} + \mathbf{P}_t = 0.$$

Since the weight of the body $\mathbf{Q} = -\mathbf{T}$, it follows that $\mathbf{A} - \mathbf{Q} + \mathbf{P}_t = 0$, whence

$$\mathbf{Q} = \mathbf{A} + \mathbf{P}_t. \tag{I}$$

Hence: *the weight of a body is the resultant of the centrifugal force* (force of transport) *and the earth's force of attraction.*

Magnitude and direction of the earth's attraction. Let us suppose that the earth has the form of a solid of revolution whose axis is the earth's axis of revolution. In addition, let us suppose that the density of the earth is distributed symmetrically with respect to the centre of mass. Then it can be proved that the force of attraction is directed constantly towards the earth's centre of mass.

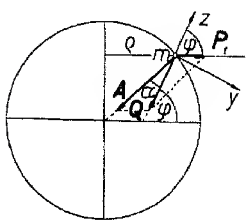

Fig. 104.

Let α be the angle which the force $\mathbf{A}$ makes with the vertical, i. e. with the weight $\mathbf{Q}$ (Fig. 104). Let us denote the radius of the parallel of latitude on which the material point lies by ϱ, the latitude of this point by φ (i. e. the angle made by the vertical passing through the point and the equatorial plane), and finally the angular velocity of the earth rotating by ω.

The force of transport lies in the plane of the parallel of latitude and is directed away from the axis of revolution, while $|\mathbf{P}_t| = m\varrho\omega^2$. Let us form the projections of the force $\mathbf{A}$, $\mathbf{Q}$, and $\mathbf{P}_t$ on the z-axis directed vertically as well as on the horizontal y-axis (i. e. perpendicular to the vertical) lying in the plane of the meridian (i. e. in the plane of the forces) and directed southwards. Setting $A = |\mathbf{A}|$, and $Q = |\mathbf{Q}|$, we obtain by (I)

$$-\mathbf{Q} = -A\cos\alpha + m\varrho\omega^2\cos\varphi, \quad -A\sin\alpha + m\varrho\omega^2\sin\varphi = 0.$$

Therefore:

$$A\cos\alpha = Q + m\varrho\omega^2\cos\varphi, \quad A\sin\alpha = m\varrho\omega^2\sin\varphi. \tag{1}$$

Hence, knowing Q, ϱ, ω, and φ, we can calculate A and α. On the equator $\varphi = 0$; therefore by (1) we get $\alpha = 0$. Denoting by A_0, Q_0, and ϱ_0 the corresponding values on the equator, we obtain

$$A_0 = Q_0 + m\varrho_0\omega^2. \tag{2}$$

Knowing Q_0 and ϱ_0, we can calculate A_0. Knowing A_0, we obtain

$$m\varrho_0\omega^2 / A_0 = \tfrac{1}{289} = (\tfrac{1}{17})^2. \tag{3}$$

If the velocity of the earth were $\omega_1 = 17\omega$, then $m\varrho_0\omega_1^2 / A_0 = 1$, whence $A_0 = m\varrho_0\omega_1^2$. From this and (2) we obtain $Q_0 = A_0 - m\varrho_0\omega_1^2 = 0$; hence if the earth were to turn 17 times faster, then bodies on the equator would be deprived of their weight.

Let us now assume that the earth is a sphere composed of concentric layers of constant density. Then, as can be demonstrated, A must be constant on the earth's surface. Therefore $A = A_0$. Denoting the radius of the earth by R, we obtain

$$\varrho = R\cos(\varphi - \alpha). \tag{4}$$

By (1)

$$\sin\alpha = \frac{mR\omega^2}{A_0}\cos(\varphi - \alpha)\sin\varphi,$$

and since $\varrho_0 = R$, by (3)

$$\sin\alpha = \tfrac{1}{289}\cos(\varphi - \alpha)\sin\varphi.$$

Angle α is very small; hence taking as an approximation $\sin\alpha = \alpha$, and $\cos(\varphi - \alpha) = \cos\varphi$, we get

$$\alpha = \frac{1}{2\cdot 289}\sin 2\varphi.$$

We see from this that α has the greatest value for $\varphi = 45°$. Putting $\alpha = 0$, we get by (1) and (4)

$$Q = A_0 - mR\omega^2 \cos^2\varphi = A_0 \left(1 - \frac{mR\omega^2}{A_0} \cos^2\varphi\right),$$

whence by (3), as $\varrho_0 = R$, we obtain

$$Q = A_0(1 - \cos^2\varphi \,/\, 289).$$

Force of Coriolis. When investigating the motion of a point relative to the earth, it is necessary to add the forces of transport and Coriolis to the absolute forces. Let us assume that in addition to the force of attraction $\boldsymbol{A}$, a force $\boldsymbol{P}$ acts on a material point. Denoting the acceleration relative to the earth by $\boldsymbol{p}$, we obtain $m\boldsymbol{p} = \boldsymbol{P} + \boldsymbol{A} + \boldsymbol{P}_t + \boldsymbol{P}_C$, and since $\boldsymbol{A} + \boldsymbol{P}_t$ is equal to the weight $\mathbf{Q}$,

$$m\boldsymbol{p} = \boldsymbol{P} + \mathbf{Q} + \boldsymbol{P}_C. \tag{5}$$

Therefore: *when inquiring into the motion of a point relative to the earth it is necessary to add the force of Coriolis* $\boldsymbol{P}_C$ *to the force* $\boldsymbol{P}$ *and the weight* $\mathbf{Q}$.

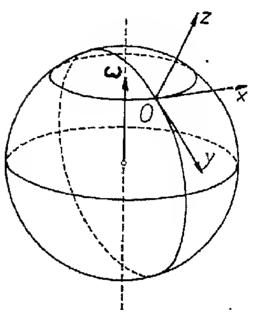

Fig. 105.

At a given place on the earth let us select a z-axis directed vertically upwards, a horizontal x-axis directed towards the east and a horizontal y-axis directed towards the south (Fig. 105). The axis of revolution will lie in the yz-plane and make an angle of $90° - \varphi$ with the z-axis (cf. Fig. 104). Since the earth revolves from west to east, the vector of angular velocity $\boldsymbol{\omega}$ lying on the axis of revolution is directed from the south pole to the north pole. Putting $|\boldsymbol{\omega}| = \omega$, we obtain

$$\omega_x = 0, \quad \omega_y = -\omega\cos\varphi, \quad \omega_z = \omega\sin\varphi \tag{N}$$

on the northern hemisphere, and

$$\omega_x = 0, \quad \omega_y = -\omega\cos\varphi, \quad \omega_z = -\omega\sin\varphi \tag{S}$$

on the southern hemisphere.

Denoting by $\boldsymbol{v}$ the velocity of the point relative to the earth, we obtain $\boldsymbol{p}_C = 2\boldsymbol{v} \times \boldsymbol{\omega}$; therefore $\boldsymbol{P}_C = -m\boldsymbol{p}_C = -2m\boldsymbol{v} \times \boldsymbol{\omega}$, whence

$$\boldsymbol{P}_C = 2m\boldsymbol{\omega} \times \boldsymbol{v}, \tag{6}$$

whence by (N), for the northern hemisphere:

$$P_{C_x} = -2m\omega(v_y \sin\varphi + v_z\cos\varphi), \quad P_{C_y} = 2m\omega v_x \sin\varphi, \qquad P_{C_z} = 2m\omega v_x \cos\varphi. \tag{II}$$

If the point has only a vertical velocity, i. e. if $v_x = 0$ and $v_y = 0$, we obtain $P_{C_x} = -2m\omega v_z\cos\varphi$, $P_{C_y} = 0$, and $P_{C_z} = 0$. When the point rises, $v_z > 0$; hence $P_{C_x} < 0$ and $\boldsymbol{P}_C$ is directed horizontally towards the west; whereas when the point falls, $v_z < 0$, $P_{C_x} > 0$, and therefore $\boldsymbol{P}_C$ is directed horizontally towards the east.

It follows from this that *a falling body is deflected towards the east under the influence of the force of Coriolis.*

If a point moves constantly in a horizontal plane, i. e. when $v_z = 0$, then $P_{C_x} = -2m\omega v_y \sin\varphi$, and $P_{C_y} = 2m\omega v_x \sin\varphi$. The horizontal component of the force of Coriolis is therefore perpendicular to the velocity and has with respect to it a sense to the right.

Therefore: *a point moving in a horizontal plane in the northern hemisphere tends to be deflected* (under the influence of the force of Coriolis) *to the right of the direction of the velocity.*

It is for this reason, for example, that the right rail is pressed down more than the left rail by moving trains.

The effects of the force of Coriolis are small because the force is small. For in virtue of (6) we have $|\boldsymbol{P}_C| = 2m\omega|\boldsymbol{v}|\sin\varepsilon$, where ε denotes the angle between $\boldsymbol{v}$ and the axis of revolution. Since

$$\omega = \frac{2\pi}{T}\sec^{-1} = \frac{2\pi}{24 \cdot 60 \cdot 60}\sec^{-1} = 0.00007 \sec^{-1},$$

where T denotes the period of one revolution of the earth about its axis, thus $\boldsymbol{P}_C$ is small.

Let us form the projections of (5) on the x, y and z axes. By (II) we obtain:

$$mx^{\cdot\cdot} = P_x - 2m\omega(y^{\cdot}\sin\varphi + z^{\cdot}\cos\varphi), \quad my^{\cdot\cdot} = P_y + 2m\omega x^{\cdot}\sin\varphi, \qquad mz^{\cdot\cdot} = P_z - mg + 2m\omega x^{\cdot}\cos\varphi. \tag{III}$$

Deviation to the east of a falling body. We shall concern ourselves with the determination of the deviation from the vertical of a freely falling material point.

Let us assume that for $t = 0$ we have

$$x = 0, \quad y = 0, \quad z = 0, \quad \boldsymbol{v} = 0. \tag{7}$$

By (III), under the assumption that $\boldsymbol{P} = 0$, we obtain:

$$x^{\cdot\cdot} = -2\omega(y^{\cdot}\sin\varphi + z^{\cdot}\cos\varphi), \quad y^{\cdot\cdot} = 2\omega x^{\cdot}\sin\varphi, \qquad z^{\cdot\cdot} = -g + 2\omega x^{\cdot}\cos\varphi. \tag{8}$$

Integrating and making use of the initial conditions (7), we get

$$x^{\cdot} = -2\omega(y \sin\varphi + z \cos\varphi), \quad y^{\cdot} = 2\omega x \sin\varphi,$$
$$z^{\cdot} = -gt + 2\omega x \cos\varphi. \tag{9}$$

Substituting the values $y^{\cdot}$ and $z^{\cdot}$ in (8), we obtain

$$x^{\cdot\cdot} = -4\omega^2 x + 2\omega gt \cos\varphi.$$

The above equation could be integrated and the result substituted in (9) from which y and z could be determined. We shall obtain an approximate solution by neglecting the term $-4\omega^2 x$, which is very small in comparison with $2\omega gt \cos\varphi$. We get $x^{\cdot\cdot} = 2\omega gt \cos\varphi$, whence

$$x = \tfrac{1}{3}\omega g t^3 \cos\varphi. \tag{10}$$

Omitting the term $2\omega x \cos\varphi$ in the third of the equations (9), as being small compared with $-gt$, we obtain $z^{\cdot} = -gt$, whence $z = -\frac{1}{2}gt^2$. When the point reaches the level $z = -h$, then $-h = -\frac{1}{2}gt^2$; therefore $t = \sqrt{2h/g}$. Hence by (10)

$$x = \tfrac{2}{3}\omega h \cos\varphi \sqrt{2h/g}. \tag{11}$$

This formula represents the deviation to the east (because $x > 0$) of a body falling from a height h.

At Harvard University experiments were performed with $h = 23$ m and $\varphi = 42°$. From about a thousand experiments deviation between 1.3 mm and 1.7 mm was obtained. From the approximation formula (11) one gets instead 1.8 mm. The difference is therefore not great.

Foucault's pendulum. Let us investigate the influence of the force of Coriolis on the motion of a pendulum. Let us place the origin of the coordinate system (x, y, z) at the point of suspension of an inextensible string at whose end a heavy point of mass m is fastened. Let l be the length of the pendulum (i. e. of the string). Since the reaction $\boldsymbol{P}$ of the string acts on the point along the string, denoting the coordinates of the point m by x, y, z, we obtain

$$P_x = \lambda m x, \quad P_y = \lambda m y, \quad P_z = \lambda m z,$$

where λ is a factor of proportionality depending on time.

By (III), p. 147, after dividing by m, we obtain

$$x^{\cdot\cdot} = \lambda x - 2\omega(y^{\cdot} \sin\varphi + z^{\cdot} \cos\varphi), \quad y^{\cdot\cdot} = \lambda y + 2\omega x^{\cdot} \sin\varphi, \tag{12}$$
$$z^{\cdot\cdot} = \lambda z - g + 2\omega x^{\cdot} \cos\varphi. \tag{13}$$

We shall concern ourselves only with an approximate solution of equations (12) and (13). Let us assume that the angle of oscillation is sufficiently small so that we can take as an approximation

$$z = -l, \; z^{\cdot} = 0, \; z^{\cdot\cdot} = 0. \tag{14}$$

From equation (13) we then obtain $0 = -\lambda l - g + 2\omega x^{\cdot} \cos\varphi$, whence $\lambda = (-g + 2\omega x^{\cdot} \cos\varphi)/l$. Omitting the second term in the numerator as being small compared with the first one, we get

$$\lambda = -g/l. \tag{15}$$

The factor λ can therefore be considered approximately as a constant. By (14) and (15) the equations (12) take the form

$$x^{\cdot\cdot} = -\frac{g}{l}x - 2\omega y^{\cdot} \sin\varphi, \quad y^{\cdot\cdot} = -\frac{g}{l}y + 2\omega x^{\cdot} \sin\varphi. \tag{16}$$

Multiplying the first of the equations (16) by $x^{\cdot}$, the second by $y^{\cdot}$, and adding, we obtain

$$x^{\cdot\cdot}x^{\cdot} + y^{\cdot\cdot}y^{\cdot} = -\frac{g}{l}(xx^{\cdot} + yy^{\cdot}),$$

whence after integrating

$$x^{\cdot 2} + y^{\cdot 2} = -\frac{g}{l}(x^2 + y^2) + a, \tag{17}$$

where a is a certain constant. Multiplying the first of the equations (16) by y, the second by x, and substracting, we get

$$yx^{\cdot\cdot} - xy^{\cdot\cdot} = -2\omega(yy^{\cdot} + xx^{\cdot}) \sin\varphi,$$

whence after integrating

$$yx^{\cdot} - xy^{\cdot} = -\omega(x^2 + y^2)\sin\varphi + b, \tag{18}$$

where b is a certain constant. Let us introduce the polar coordinates:

$$x = r\cos\psi, \quad y = r\sin\psi.$$

From (17) and (18) we obtain

$$r^{\cdot 2} + r^2\psi^{\cdot 2} = -\frac{g}{l}r^2 + a, \tag{19}$$

$$r^2\psi^{\cdot} = r^2\omega\sin\varphi - b. \tag{20}$$

Let us introduce a new coordinate system (x_1, y_1, z_1) having with the preceding system (x, y, z) a common origin as well as a common z-axis, and revolving about the z-axis with an angular velocity $\omega \sin\psi$ in the direction from east to south, i. e. from x to y. For the polar coordinates r_1, ψ_1 we obtain in the new system the formulae:

$$r = r_1, \quad \psi = \psi_1 + \omega t \sin\varphi. \tag{21}$$

By substituting (21) in (20) we obtain in terms of the new coordinates r_1, ψ_1 the equation $r_1^{\cdot 2}(\psi_1^{\cdot} + \omega \sin \varphi) = r_1^2 \omega \sin \varphi - b$, whence

$$r_1^2 \psi_1^{\cdot} = -b, \tag{22}$$

and by substituting (21) in equation (19) we get

$$r_1^{\cdot 2} + r_1^2 \psi_1^{\cdot 2} + 2 r_1^2 \psi_1^{\cdot} \omega \sin \varphi + r_1^2 \omega^2 \sin^2 \varphi = -g r_1^2 / l + a,$$

from which after neglecting the term $r_1^2 \omega^2 \sin^2 \varphi$ as being very small and applying equation (22), we obtain

$$r_1^{\cdot 2} + r_1^2 \psi_1^{\cdot 2} = -g r_1^2 / l + a_1, \tag{23}$$

where $a_1 = a + 2b\omega \sin \varphi = \text{const.}$

It is easy to verify that (22) and (23) are the equations of the motion whose equations in terms of the coordinates x_1, y_1, z_1 are:

$$x_1^{\cdot\cdot} = -g x_1 / l, \quad y_1^{\cdot\cdot} = -g y_1 / l. \tag{24}$$

Indeed, this is the form which equations (16) assume for $\omega = 0$. Hence, introducing polar coordinates, we obtain, as is seen from equations (19) and (20) for $\omega = 0$, equations (22) and (23).

Equations (24) represent the motion of a point under the influence of a force $\boldsymbol{P}$ whose projections are:

$$P_{x_1} = -g m x_1 / l, \quad P_{y_1} = -g m y_1 / l. \tag{25}$$

This is an elastic force, i. e. one directed constantly towards the origin of the coordinate system and directly proportional to the distance of the point from the origin of the system. On p. 112 we showed that motion under the influence of an elastic force takes place along an ellipse. Hence a material point will execute a motion in the system (x_1, y_1, z_1) along an ellipse. Because this system also revolves about the z-axis with an angular velocity $\omega \sin \varphi$, the axis of this ellipse will revolve with an angular velocity $\omega \sin \varphi$ from east to south. The period of revolution is

$$T = 2\pi / \omega \sin \varphi.$$

Since one revolution of the earth lasts 24 hours it follows that, $2\pi / \omega = 24$ h, whence $T = 24 / \sin \varphi$ h. For $\varphi = 45°$ we get $T = 34$ h.

This phenomenon was first confirmed experimentally by L. Foucault; it constitutes a proof of the earth's rotation about its axis.

CHAPTER IV

GEOMETRY OF MASSES

I. SYSTEMS OF POINTS

§ 1. Statical moments. Statical moment of a point. Let us consider an arbitrary plane Π. It divides space into two parts; we can consider one of these parts as positive, and the other as negative. Let A denote a certain material point and d its distance from the plane Π. We shall write $\sigma = +d$ or $\sigma = -d$, depending on whether A lies in the positive or negative part of space.

Denoting the mass of the point A by m, we shall call the expression

$$M_\Pi = m\sigma$$

the *statical moment* of the material point A with respect to the plane Π.

The statical moment of a point can therefore be a positive or negative number or zero (it is zero for every point A lying in the plane Π).

If we choose one of the coordinate planes xy, yz, zx, as the plane Π, then we shall consider as the positive part of space that part in which is found the positive part of the axis perpendicular to the chosen coordinate plane. If the point A of mass m has the coordinates x, y, z, then by the preceding convention we have:

$$M_{xy} = mz, \quad M_{yz} = mx, \quad M_{zx} = my,$$

where M_{xy}, M_{yz}, M_{zx} denote the corresponding statical moments of the point A with respect to the xy, yz and zx planes.

Statical moment of a system of points. A collection of material points is called a *system of points*, and the sum of the statical moments of its separate points is called the *(total) statical moment of the system of points.*

If the statical moments with respect to the plane Π of the material

By substituting (21) in (20) we obtain in terms of the new coordinates r_1, ψ_1 the equation $r_1^2(\dot{\psi}_1 + \omega \sin\varphi) = r_1^2\omega \sin\varphi - b$, whence

$$r_1^2\dot{\psi}_1 = -b, \tag{22}$$

and by substituting (21) in equation (19) we get

$$\dot{r}_1^2 + r_1^2\dot{\psi}_1^2 + 2r_1^2\dot{\psi}_1\omega \sin\varphi + r_1^2\omega^2 \sin^2\varphi = -gr_1^2 / l + a,$$

from which after neglecting the term $r_1^2\omega^2 \sin^2\varphi$ as being very small and applying equation (22), we obtain

$$\dot{r}_1^2 + r_1^2\dot{\psi}_1^2 = -gr_1^2 / l + a_1, \tag{23}$$

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It is easy to verify that (22) and (23) are the equations of the motion whose equations in terms of the coordinates x_1, y_1, z_1 are:

$$\ddot{x}_1 = -gx_1 / l, \quad \ddot{y}_1 = -gy_1 / l. \tag{24}$$

Indeed, this is the form which equations (16) assume for $\omega = 0$. Hence, introducing polar coordinates, we obtain, as is seen from equations (19) and (20) for $\omega = 0$, equations (22) and (23).

Equations (24) represent the motion of a point under the influence of a force **P** whose projections are:

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$$M_{xy} = mz, \quad M_{yz} = mx, \quad M_{zx} = my,$$

where M_{xy}, M_{yz}, M_{zx} denote the corresponding statical moments of the point A with respect to the xy, yz and zx planes.

Statical moment of a system of points. A collection of material points is called a *system of points*, and the sum of the statical moments of its separate points is called the *(total) statical moment of the system of points*.

If the statical moments with respect to the plane Π of the material

points of masses $m_1, m_2, \ldots, m_n$ are $m_1\sigma_1, m_2\sigma_2, \ldots, m_n\sigma_n$, respectively, then the total statical moment of the system of these points will be

$$M_\Pi = m_1\sigma_1 + m_2\sigma_2 + \ldots + m_n\sigma_n = \sum_{i=1}^{n} m_i\sigma_i.$$

If the material points of a given system of points have the coordinates $x_1, y_1, z_1,\ x_2, y_2, z_2, \ldots, x_n, y_n, z_n$ respectively, then the total statical moments of this system of points with respect to the corresponding coordinate planes are expressed by the formulae:

$$M_{xy} = m_1z_1 + m_2z_2 + \ldots + m_nz_n = \sum_{i=1}^{n} m_iz_i,$$

$$M_{yz} = m_1x_1 + m_2x_2 + \ldots + m_nx_n = \sum_{i=1}^{n} m_ix_i,$$

$$M_{zx} = m_1y_1 + m_2y_2 + \ldots + m_ny_n = \sum_{i=1}^{n} m_iy_i.$$

Statical moments are also called *moments of first order.*

§ 2. Centre of mass. Let there be given a system U of material points $m_1(x_1, y_1, z_1)$, $m_2(x_2, y_2, z_2)$, $\ldots$, $m_n(x_n, y_n, z_n)$. Let us consider a point S whose coordinates are:

$$x_0 = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n},\quad y_0 = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{m_1 + m_2 + \ldots + m_n},$$

$$z_0 = \frac{m_1z_1 + m_2z_2 + \ldots + m_nz_n}{m_1 + m_2 + \ldots + m_n}. \tag{I}$$

The point S is called the *centre of mass* or the *centre of gravity* of the given system of points U.

The sum of the masses of the individual points (appearing in the denominators of the fractions (I)) will be called the *total mass* of the system of points.

Although we have defined the centre of mass with the aid of a coordinate system, we shall show that the position of the centre of mass does not depend on the coordinate system, but only on the masses of the points and their mutual distribution. This follows easily from the following theorem:

Theorem 1. *The statical moment of a system of points with respect to an arbitrary plane is equal to the statical moment of the total mass placed at the centre of gravity.*

Proof. Let Π be an arbitrary plane whose normal equation has the form

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0.$$

Let us take as positive that one of the two parts of space for which $x\cos\alpha + y\cos\beta + z\cos\gamma - p > 0$, when the coordinates of an arbitrary point of this part are substituted for x, y, and z. Hence, if $A(x, y, z)$ is an arbitrary point of space, then, since the distance of the point A from the plane Π is expressed by the formula

$$d = |x\cos\alpha + y\cos\beta + z\cos\gamma - p|,$$

according to our convention we can put

$$\sigma = x\cos\alpha + y\cos\beta + z\cos\gamma - p.$$

The statical moment of the point A of mass m with respect to the plane Π is therefore

$$M_\Pi = m\sigma = mx\cos\alpha + my\cos\beta + mz\cos\gamma - mp. \tag{1}$$

Let there be given a system of material points $m_1(x_1, y_1, z_1), \ldots, m_n(x_n, y_n, z_n)$. By (1) the statical moment of this system of points with respect to the plane Π will be

$$M_\Pi = (m_1x_1\cos\alpha + m_1y_1\cos\beta + m_1z_1\cos\gamma - m_1p) + \ldots$$
$$\ldots + (m_nx_n\cos\alpha + m_ny_n\cos\beta + m_nz_n\cos\gamma - m_np);$$

thus

$$M_\Pi = (m_1x_1 + m_2x_2 + \ldots + m_nx_n)\cos\alpha + (m_1y_1 + \ldots + m_ny_n)\cos\beta +$$
$$+ (m_1z_1 + \ldots + m_nz_n)\cos\gamma - (m_1 + \ldots + m_n)\,p. \tag{2}$$

Putting $m_1 + m_2 + \ldots + m_n = m$, we have by (I):

$$mx_0 = \sum m_ix_i,\quad my_0 = \sum m_iy_i,\quad mz_0 = \sum m_iz_i,\quad i = 1, 2, \ldots, n, \tag{II}$$

whence by (2)

$$M_\Pi = mx_0\cos\alpha + my_0\cos\beta + mz_0\cos\gamma - mp. \tag{3}$$

Since the right hand side of equation (3) represents by (1) the statical moment of the mass m placed at the centre of gravity S having coordinates x_0, y_0, z_0, the theorem has been proved.

In order to show now that the centre of mass of the system of points does not depend on the choice of the coordinate system, let us suppose that another point S' possesses, in addition to point S, the property of the centre of gravity S described in the theorem. We shall prove that this is impossible.

With this end in view, let us pass through the point S an arbitrary plane Π not passing through S'. Therefore

$$M_\Pi = m\sigma \quad \text{and} \quad M_\Pi = m\sigma', \tag{4}$$

where σ and σ' denote the corresponding distances (positive or negative) of the points S and S' from the plane Π. From (4) it follows that $\sigma = \sigma'$. But $\sigma = 0$, since Π passes through S. Therefore σ' would also have to be equal to zero, which is impossible, because S' does not lie in the plane Π.

We see, therefore, that the *position of the centre of mass of a system of points is independent of the coordinate system.*

Centre of mass of two systems of points. Let a system U be composed of the material points

$$m_1'(x_1', y_1', z_1'), \quad m_2'(x_2', y_2', z_2'), \; \ldots, \; m_1''(x_1'', y_1'', z_1''), \quad m_2''(x_2'', y_2'', z_2''), \; \ldots$$

The centre of mass S' of the system of points $m_1', m_2', \ldots$ has by definition the coordinates:

$$x_0' = (m_1'x_1' + m_2'x_2' + \ldots) / m', \quad y_0' = (m_1'y_1' + m_2'y_2' + \ldots) / m',$$
$$z_0' = (m_1'z_1' + m_2'z_2' + \ldots) / m', \tag{5}$$

where $m' = m_1' + m_2' + \ldots$ On the other hand, for the centre of mass S of the entire system U, there will be

$$x_0 = \frac{(m_1'x_1' + m_2'x_2' + \ldots) + (m_1''x_1'' + m_2''x_2'' + \ldots)}{(m_1' + m_2' + \ldots) + (m_1'' + m_2'' + \ldots)},$$

whence by (5)

$$x_0 = \frac{m'x_0' + (m_1''x_1'' + m_2''x_2'' + \ldots)}{m' + (m_1'' + m_2'' + \ldots)}. \tag{6}$$

Similar formulae are obtained for y_0 and z_0. Formula (6) represents the x-coordinate of the centre of mass of the system that is obtained from the given system U, if a part of it, namely the points of masses $m_1', m_2', \ldots$, is replaced by a single material point of mass $m' = m_1' + m_2' + \ldots$ placed at the centre of mass of this part. Therefore, we have obtained

Theorem 2. *The centre of mass of a system of points is not altered if a part of it is replaced by a material point having a mass equal to the mass of this part and placed at centre of its mass.*

Hence if we have in particular two systems of points U' and U'' of total masses m' and m'' and with centres of gravity S' and S'', then we obtain the centre of mass of the system $U' + U''$ by determining the centre of mass of the system of two material points having masses m' and m'' placed at the points S' and S'', respectively. This is so because the systems U' and U'' can be considered as parts of the system $U' + U''$.

Plane system of points. A system of material points is said to be a *plane* system if all points of the system lie in one plane. Selecting this plane as the xy-plane (which is possible for us to do since the centre of mass is independent of the choice of the coordinate system), we shall have for the points of the system $z_1 = 0, z_2 = 0, \ldots, z_n = 0$, whence by formulae (I), p. 152, we get $z_0 = 0$.

The centre of gravity of a plane system therefore lies in the plane of the system.

The statical moment of a plane system with respect to an arbitrary line l lying in the plane of the system is defined by the expression

$$M_l = \sum_{i=1}^{n} m_i\sigma_i, \tag{7}$$

where $|\sigma_i|$ is the distance of the point of mass m_i from l, and the sign of σ_i depends on whether m_i is situated in the positive or negative parts of the plane into which this plane is divided by the line l. We see from this that the statical moment of a plane system with respect to the line l is the statical moment of this system with respect to a plane perpendicular to the plane of the system and intersecting it along the line l. Hence, in particular, the moments with respect to the x and y axes are expressed by the formulae:

$$M_x = \Sigma m_i y_i, \quad M_y = \Sigma m_i x_i. \tag{8}$$

Linear system of points. If the points of a system lie on one line l, then the centre of mass of the system also lies on this line, because choosing the line l as the x-axis, we have $y_1 = 0, y_2 = 0, \ldots$ and $z_1 = 0, z_2 = 0, \ldots$, whence by formulae (I), p. 152, we get $y_0 = 0, z_0 = 0$. The centre of mass will therefore also lie on the x-axis.

Centre of mass of two points. Let the material points of masses m_1 and m_2 be at a distance d from each other. The centre of mass obviously lies on a line joining these points. Let us place at m_1 the origin of the x-axis, and pass its positive part through m_2. The points m_1 and m_2 will therefore have the coordinates $x_1 = 0$ and $x_2 = d$, respectively, and the centre of mass the coordinate

$$x_0 = m_2 d / (m_1 + m_2). \tag{9}$$

Since $0 < x_0 < d$, the centre of mass lies between the points. Denoting the distances of the centre of mass from the points m_1 and m_2 by d_1 and d_2, respectively, we obtain $d_1 = x_0 = m_2 d / (m_1 + m_2)$ and $d_2 = d - d_1 = m_1 d / (m_1 + m_2)$, whence

$$d_1 : d_2 = m_2 : m_1. \tag{10}$$

Hence we have the following

Theorem 3. *The centre of mass of a two point system lies between the points of the system and divides the line segment joining these points in the inverse ratio to their masses.*

Making use of theorem 2, p. 154, we can determine the centre of mass of a finite system of points $A_1, A_2, A_3, \ldots$ of masses $m_1, m_2, m_3, \ldots$ (Fig. 106) in the following manner: we determine at first the centre of mass S_1 of the system of points A_1 and A_2; next we determine the centre of mass S_2 of the system of two points consisting of the point of mass $m_1 + m_2$, situated at S_1 and of the point A_3 of mass m_3; continuing in this manner, we obtain the centre of mass of the entire given system.

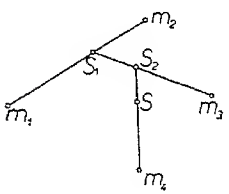

Fig. 106.

Symmetric systems of points. A point O (line l, plane Π) is called a *centre (line, plane) of symmetry of the system of material points*, if to each point m_i there exists in the system a point having the same mass m_i, placed symmetrically with respect to point O (line l, plane Π).

If the centre of symmetry is the origin of the coordinate system (Fig. 107), then along with each point $m_i(x_i, y_i, z_i)$ the system of points includes the point $m_i(-x_i, -y_i, -z_i)$. If the plane of symmetry is the xy-plane (Fig. 108), then along with each point $m_i(x_i, y_i, z_i)$ the system

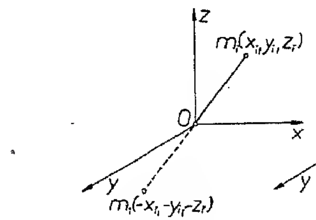

Fig. 107.

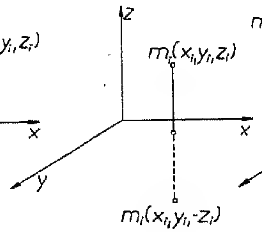

Fig. 108.

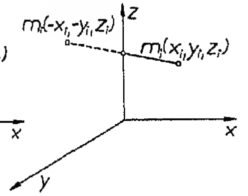

Fig. 109.

includes the point $m_i(x_i, y_i, -z_i)$. If the axis of symmetry is the z-axis (Fig. 109), then along with each point $m_i(x_i, y_i, z_i)$ the system includes the point $m_i(-x_i, -y_i, z_i)$.

It is easy to show that the *centre of symmetry is always the centre of mass.*

For by theorem 3, the centre of mass of a pair of symmetric points lies at the centre of symmetry. Hence by theorem 2, p. 154, we can replace such a pair by a material point situated at the centre of symmetry. Doing this with every pair, we come to the conclusion that the centre of symmetry is the centre of mass of the entire system.

Similarly, the *centre of mass lies on a line of symmetry (and on a plane of symmetry).*

Because for these same reasons the centre of a pair of symmetric points lies on a line (and on a plane) of symmetry.

§ 3. Moments of second order. Moment of inertia. Let there be given a material point A of mass m and a certain plane Π. Let r denote the distance of the point A from the plane Π. The expression

$$I = mr^2 \tag{1}$$

is called the *moment of inertia* of the point A with respect to the plane Π. If we denote by r the distance of the material point A from a certain line l (or from a certain point O), then (1) will be the moment of inertia of the point A with respect to the line l (or with respect to the point O).

The total moment of inertia of a system of points is defined as the sum of the moments of inertia of the separate points of this system.

Product of inertia. Let there be given two mutually perpendicular planes Π_1 and Π_2. Put $\sigma_1 = \pm d_1$, where d_1 denotes the distance of the material point A from Π_1, and the sign depends on whether the point is in the positive or negative of the two parts into which the plane Π_1 divides space. We define σ_2 with respect to the plane Π_2 similarly. The expression

$$D = m\sigma_1\sigma_2 \tag{2}$$

is called the *product of inertia* of the material point A with respect to the planes Π_1 and Π_2.

The *total product of inertia* of a system of points $A_1, A_2, \ldots$ with respect to the planes Π_1 and Π_2 is defined as the sum of the products of inertia of the separate points. Hence

$$D = \Sigma m_i \sigma_1^{(i)} \sigma_2^{(i)}, \tag{3}$$

where m_i, $\sigma_1^{(i)}$, $\sigma_2^{(i)}$ denote respectively the mass of the point A_i and its distances from the planes Π_1 and Π_2 (preceded by proper signs).

Moments of inertia and products of inertia are called *moments of second order.*

Radius of gyration. Let I denote the total moment of inertia of a system of points U with respect to a plane Π (line l, point O). The number

$$k = \sqrt{I / m}, \quad \text{where } m = m_1 + m_2 + \ldots \tag{4}$$

is called the *radius of gyration* of the system of points U with respect to the plane Π (line l, point O). In virtue of (4)

$$I = mk^2. \tag{5}$$

Therefore: *the radius of gyration k is the distance at which the total mass of a system has a moment of inertia equal to the total moment of inertia of the system.*

Concentrated mass. Let r be an arbitrary positive number. The mass of a system *concentrated at a distance* r with respect to a plane (line or point) is defined by the number

$$m_r = I / r^2. \tag{6}$$

Therefore $I = m_r r^2$. Hence: *the moment of inertia of a system with respect to a plane (line, point) is equal to the moment of inertia of its concentrated mass m_r situated at a distance r from this plane (line, point).*

The moments of second order of the system of points

$$m_1(x_1, y_1, z_1), \quad m_2(x_2, y_2, z_2), \quad \ldots, \quad m_n(x_n, y_n, z_n)$$

with respect to a plane, axis and the origin of a coordinate system are expressed by means of the following formulae:

The moments of inertia with respect to the planes xy, yz and xz:

$$I_{xy} = \sum_{i=1}^{n} m_i z_i^2, \quad I_{yz} = \sum_{i=1}^{n} m_i x_i^2, \quad I_{zx} = \sum_{i=1}^{n} m_i y_i^2. \tag{7}$$

The moments of inertia with respect to the x, y, and z axes:

$$I_x = \sum_{i=1}^{n} m_i(y_i^2 + z_i^2), \; I_y = \sum_{i=1}^{n} m_i(x_i^2 + z_i^2), \; I_z = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2). \tag{8}$$

The moments of inertia with respect to tho origin O of a coordinate system:

$$I_o = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2 + z_i^2). \tag{9}$$

The products of inertia with respect to the pairs of planes xy and zx, xy and yz, as well as xz and zy:

$$D_x = \sum_{i=1}^{n} m_i y_i z_i, \; D_y = \sum_{i=1}^{n} m_i x_i z_i, \; D_z = \sum_{i=1}^{n} m_i x_i y_i. \tag{10}$$

From formulae (7)—(10) the following relations can be easily derived:

$$I_x = I_{xy} + I_{xz}, \quad I_y = I_{xy} + I_{yz}, \quad I_z = I_{xz} + I_{yz};$$
$$I_{xy} = \tfrac{1}{2}[I_x + I_y - I_z], \quad I_{yz} = \tfrac{1}{2}[I_y + I_z - I_x], \quad I_{xz} = \tfrac{1}{2}[I_x + I_z - I_y];$$
$$I_o = \tfrac{1}{2}[I_x + I_y + I_z] = I_{xy} + I_{yz} + I_{zx},$$
$$I_o = I_x + I_{yz} = I_y + I_{xz} = I_z + I_{xy}.$$

Moments of inertia with respect to parallel lines. Knowing the moments of inertia of a system of points of total mass m with respect to lines passing through one point e. g. through the origin of the coordinate system, we can easily determine the moment of inertia of this system of points with respect to an arbitrary line in space by making use of the following theorem:

If a line l passing through the centre of mass of a system of points is parallel to the line l', then

$$I_{l'} = I_l + md^2, \tag{I}$$

where d denotes the distance between l and l', whereas I_l and $I_{l'}$ denote the moments of inertia with respect to these lines.

Proof. Let us choose the centre S of the mass of the system of points through which the line l passes as the origin of the coordinate system, the line l as the x-axis, and the plane passed through the parallel lines l and l' as the xy-plane (Fig. 110). Denoting by r and r' respectively the distances of an arbitrary point $A(x, y, z)$ from the straight lines l and l', we have $r'^2 = z^2 + (d - y)^2$ and $r^2 = z^2 + y^2$, whence $r'^2 = r^2 + d^2 - 2dy$, and hence

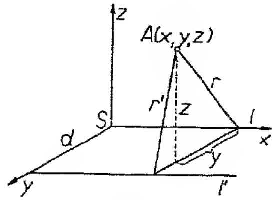

Fig. 110.

$$I_{l'} = \sum_{i=1}^{n} m_i r_i'^2 = \sum_{i=1}^{n} m_i[r_i^2 + d^2 - 2dy_i] =$$
$$= \sum_{i=1}^{n} m_i r_i^2 + d^2 \sum_{i=1}^{n} m_i - 2d \sum_{i=1}^{n} m_i y_i = I_l + md^2 - 2d \sum_{i=1}^{n} m_i y_i.$$

But $\sum_{i=1}^{n} m_i y_i = m y_0 = 0$, since the centre S of the mass of the system of points lies at the origin of the system of coordinates. Therefore $I_{l'} = I_l + md^2$, q. e. d.

From formula (I) it follows that if all lines parallel to a line having a certain given direction are taken into consideration, then the moment of inertia will be the least with respect to that line which passes through the centre of mass. It is equally obvious that if lines l' and l'' are parallel, then denoting by d_1 and d_2 the distances of the centre of mass from these lines, we shall have

$$I_{l'} - md_1^2 = I_{l''} - md_2^2, \tag{11}$$

because by (I) both sides of the equality are equal to I_l, where l is a line parallel to l' and l'' and passing through the centre of mass.

From formula (11) we have

$$I_{l''} = I_{l'} + m(d_2^2 - d_1^2). \tag{II}$$

This formula enables one to compute the moment of inertia of a system of points with respect to an arbitrary line in space, if the moments of inertia of this system with respect to every line passing through one point and the position of the centre of mass of the system are known.

Products of inertia with respect to parallel planes. For products of inertia we can prove a theorem similar to the theorem on moments of inertia (p. 159).

Let the planes Π_1, Π_2 be perpendicular to each other and pass through the centre of mass m of a given system of material points. Let us select arbitrary planes Π_1', Π_2' parallel to the planes Π_1, Π_2, respectively. Let σ_1 denote the distance between the planes Π_1, Π_1', preceded by a $+$ or $-$ sign depending on whether the plane Π_1' lies in the positive or negative part of space into which the plane Π_1 divides space. Let us define σ_2 for the pair of planes Π_2, Π_2' analogously. Finally, let us denote the products of inertia of the given system with respect to the pairs of planes Π_1, Π_2 and Π_1', Π_2' by D and D'. We then have a formula which is analogous to (I), namely:

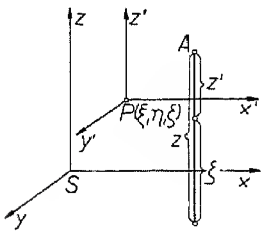

Fig. 111.

$$D' = D + m\sigma_1\sigma_2. \tag{III}$$

Remark. Let us note that the product $m\sigma_1\sigma_2$ denotes the product of inertia with respect to the pair of planes Π_1, Π_2 of the total mass m of the system placed anywhere on the intersection of the pair of planes Π_1' and Π_2'.

Proof of formula (III). Let us choose the origin of the coordinate system (x, y, z) at the centre S of the total mass m of the given system of points (Fig. 111). As the x-axis we shall take the intersection of the planes Π_1 and Π_2, and we select these planes as the xy and xz planes, respectively.

Analogously, we select a second system of coordinates (x', y', z') for the pair of planes Π_1', Π_2', taking as the origin an arbitrary point P lying on the line of intersection of the planes Π_1' and Π_2'.

Let us denote the coordinates of the point P with respect to the coordinate system (x, y, z) by ξ, η, ζ. Obviously $\eta = \sigma_2$, and $\zeta = \sigma_1$. Let x, y, z be the coordinates of an arbitrary point A with respect to the coordinate system (x, y, z), and x', y', z' the coordinates of this point A with respect to the coordinate system (x', y', z'). Then:

$$x' = x - \xi,\ y' = y - \eta,\ z' = z - \zeta. \tag{12}$$

Since

$$D' = \Sigma m_i z_i' y_i',\ D = \Sigma m_i z_i y_i, \tag{13}$$

it follows by (11) that

$$\begin{aligned} D' &= \Sigma m_i(z_i - \zeta)(y_i - \eta) = \Sigma m_i[z_i y_i + \zeta\eta - \zeta y_i - \eta z_i] = \\ &= \Sigma m_i z_i y_i + \zeta\eta\, \Sigma m_i - \zeta\, \Sigma m_i y_i - \eta\, \Sigma m_i z_i. \end{aligned} \tag{14}$$

But $\Sigma m_i y_i = m y_0 = 0$, and $\Sigma m_i z_i = m z_0 = 0$, because by hypothesis the centre S of mass m of the given system of points lies at the origin of the system (x, y, z). Therefore, by (13) and (14), $D' = D + m\zeta\eta$, and since $\zeta = \sigma_1$, and $\eta = \sigma_2$, we obtain finally $D' = D + m\sigma_1\sigma_2$, q. e. d.

§ 4. Ellipsoid of inertia. Principal axes of inertia. Let $O(x, y, z)$ be an arbitrary rectangular coordinate system with origin at O. We shall prove that it is possible to determine the moments of inertia with respect to an arbitrary line l passing through O, if the moments of inertia with respect to the axes and the products of inertia with respect to the planes of this coordinate system are known.

Let the line l form the angles α, β, γ with the axes of the coordinate system $O(x, y, z)$ (Fig. 112). Let us select an arbitrary point $A(x, y, z)$ and let P be the projection of the point A on the line l. Therefore $AP = r$ is the distance of the point A from the line l. Let us put

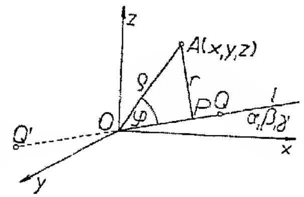

Fig 112.

$$OA = \varrho = \sqrt{x^2 + y^2 + z^2}. \tag{1}$$

Denoting by φ the angle between the line OA and the line l, we obtain

$$AP = r = \varrho \sin\varphi. \tag{2}$$

Since

$$OP = x\cos\alpha + y\cos\beta + z\cos\gamma,$$

as is known from analytic geometry, and because $OP = \varrho\cos\varphi$, we obtain

$$\cos\varphi = (x\cos\alpha + y\cos\beta + z\cos\gamma)\,/\,\varrho.$$

By (2) $r^2 = \varrho^2\sin^2\varphi = \varrho^2(1 - \cos^2\varphi)$; hence

$$r^2 = \varrho^2\left[1 - \frac{(x\cos\alpha + y\cos\beta + z\cos\gamma)^2}{\varrho^2}\right] =$$
$$= \varrho^2 - [x\cos\alpha + y\cos\beta + z\cos\gamma]^2,$$

whence by (1),

$$r^2 = x^2[1 - \cos^2\alpha] + y^2[1 - \cos^2\beta] + z^2[1 - \cos^2\gamma] -$$
$$- 2xy\cos\alpha\cos\beta - 2xz\cos\alpha\cos\gamma - 2yz\cos\beta\cos\gamma.$$

Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, substituting $1 - \cos^2\alpha = \cos^2\beta + \cos^2\gamma$, etc. we obtain

$$r^2 = (y^2 + z^2)\cos^2\alpha + (x^2 + z^2)\cos^2\beta + (x^2 + y^2)\cos^2\gamma -$$
$$- 2xy\cos\alpha\cos\beta - 2xz\cos\alpha\cos\gamma - 2yz\cos\beta\cos\gamma.$$

Denoting by m_i the mass and by r_i the distance of the point A_i of a given system of points $A_1, A_2, \ldots$ from the line l, we obtain for the moment of inertia I_l of this system of points with respect to the line l the formula

$$I_l = \Sigma m_i r_i^2 =$$
$$= \cos^2\alpha\,\Sigma m_i(y_i^2 + z_i^2) + \cos^2\beta\,\Sigma m_i(x_i^2 + z_i^2) + \cos^2\gamma\,\Sigma m_i(x_i^2 + y_i^2) -$$
$$- 2\cos\alpha\cos\beta\,\Sigma m_i x_i y_i - 2\cos\alpha\cos\gamma\,\Sigma m_i x_i z_i - 2\cos\beta\cos\gamma\cdot\Sigma m_i y_i z_i,$$

whence

$$I_l = I_x\cos^2\alpha + I_y\cos^2\beta + I_z\cos^2\gamma - \qquad \text{(I)}$$
$$- 2D_z\cos\alpha\cos\beta - 2D_y\cos\alpha\cos\gamma - 2D_x\cos\beta\cos\gamma.$$

From formula (I) we can determine the moment of inertia of a system of material points with respect to the line l, if we know $I_x, I_y, I_z, D_x, D_y, D_z$ as well as the angles which the line l forms with the axes of the coordinate system.

Retaining the previous notation, let us denote the length of the radius of gyration of a given system of points with respect to l by k_l. Therefore (p. 157)

$$k_l = \sqrt{I_l / m}. \qquad (3)$$

Let us assume that $I_l \neq 0$ for every line l passing through O. This assumption is equivalent to the assumption that the material points of the given system do not lie on one and the same line passing through O. Since $I_l \neq 0$, it follows by (3) that also $k_l \neq 0$.

On each line l, let us cut off segments OQ and OQ' (Fig. 113) whose lengths are inversely proportional to the radius of gyration k_l, i. e.

$$OQ = OQ' = a / k_l = a\sqrt{m / I_l}, \qquad (4)$$

where a is an arbitrary positive constant independent of the line l. Denoting the coordinates of the point Q by x, y, z, we have:

$$x = \pm a\sqrt{m / I_l}\cos\alpha, \quad y = \pm a\sqrt{m / I_l}\cos\beta,$$
$$z = \pm a\sqrt{m / I_l}\cos\gamma. \qquad (5)$$

The point Q' has the coordinates $-x, -y, -z$. The collection of all points Q and Q' will form a certain surface Σ. In order to obtain its equation we shall determine $\cos\alpha$, $\cos\beta$, $\cos\gamma$ from (5) and substitute the values obtained into equation (I). We get

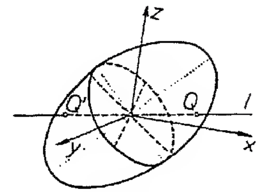

Fig. 113.

$$I_l = \frac{I_l}{ma^2}[I_x x^2 + I_y y^2 + I_z z^2 - 2D_x yz - 2D_y xz - 2D_z xy],$$

whence

$$I_x x^2 + I_y y^2 + I_z z^2 - 2D_x yz - 2D_y xz - 2D_z xy = c^2, \qquad (6)$$

where $c^2 = ma^2$. We see from this that the surface Σ is of the second degree.

We shall show that Σ is an ellipsoid. For this purpose it is sufficient to prove that Σ is a bounded surface (i. e. that the distances of its points from the origin of the coordinate system do not exceed a certain number). Indeed, we have assumed that $I_l \neq 0$, and hence we have $I_l > 0$ constantly. Since by formula (I) I_l is a continuous function of the angles α, β, γ, the minimum of I_l is also positive. Denoting this minimum by h, we have by (3) $k_l \geqq \sqrt{h / m}$, whence by (4), $OQ' = OQ \leqq a\sqrt{m / h}$. Therefore the surface Σ is a bounded surface. It follows from this that the surface Σ is an ellipsoid, because the only bounded surface of the second degree is an ellipsoid.

The ellipsoid Σ is called the *ellipsoid of inertia* of the given system of points with respect to the point O.

Since equation (6) lacks terms of the first degree, i. e. x, y, z, the point O is the centre of the ellipsoid of inertia.

Therefore: *the ellipsoid of inertia of a system of material points with respect to a point O has this property, that the distance of each of its points from O is inversely proportional to the radius of gyration of the system with respect to the diameter passing through this point.*

The axes of the ellipsoid of inertia with respect to the point O are called the *principal axes of inertia* with respect to the point O.

The ellipsoid of inertia with respect to the centre of gravity is called

the *central* ellipsoid of inertia, its axes, *central* axes of inertia, and the planes passing through two axes, *central planes.*

Obviously, there exist infinitely many ellipsoids of inertia of a given system with respect to one and the same point O. They depend on the choice of the constant of proportionality. However, all these ellipsoids nevertheless have a common centre and common directions of the principal axes. In addition, the ratio of the principal axes is the same for all ellipsoids. All the ellipsoids of inertia with respect to one and the same point O are therefore similar to each other.

The radius of gyration has the greatest value with respect to the minor axis; hence the moment of inertia has the smallest value with respect to the minor axis. The converse is true with respect to the major axis.

In particular, if the ellipsoid is a sphere, the point O is termed a *spherical* point.

The moments of inertia with respect to every line passing through a spherical point are the same (and conversely).

Determination of principal axes of inertia. If we take the principal axes of inertia as the coordinate axes x, y, z, then the equation of the ellipsoid of inertia will have the form

$$Ax^2 + By^2 + Cz^2 = F. \tag{7}$$

Comparing equations (6) and (7), we see that in this case $D_x = 0$, $D_y = 0$ and $D_z = 0$; therefore the ellipsoid of inertia will be

$$I_x x^2 + I_y y^2 + I_z z^2 = c^2. \tag{8}$$

Hence: *the necessary and sufficient condition that the coordinate axes be axes of inertia is that the products of inertia D_x, D_y, D_z be equal to zero.*

If the coordinate axes x, y, z are selected so that only one of them, e. g. the z-axis, coincides with one of the principal axes of inertia, then the equation of the ellipsoid will have the form

$$Ax^2 + By^2 + Cz^2 + Exy = F. \tag{9}$$

Comparing equations (6) and (9), we see that $D_x = 0$ and $D_y = 0$. The equation of the ellipsoid of inertia in this case will therefore be

$$I_x x^2 + I_y y^2 + I_z z^2 - 2D_z xy = c^2. \tag{10}$$

Hence: *the necessary and sufficient condition that the z-axis be a principal axis of inertia is that the products of inertia D_x and D_y be equal to zero.*

It is easy to formulate analogous conditions that the principal axes of inertia be the x-axis or the y-axis.

Let us now take the centre S of the total mass m of a given system of material points as the origin of the coordinate system (x, y, z), and its central axes as the coordinate axes x, y, z. We obviously have

$$D_x = 0, \quad D_y = 0, \quad D_z = 0. \tag{11}$$

Let us next take an arbitrary point $O(\xi, \eta, \zeta)$ as the origin of a new coordinate system (x', y', z') whose axes are parallel to the axes x, y and z, respectively. By (III), p. 160,

$$D_{x'} = D_x + m\zeta\eta, \quad D_{y'} = D_y + m\xi\zeta, \quad D_{z'} = D_z + m\xi\eta,$$

whence by (11):

$$D_{x'} = m\zeta\eta, \quad D_{y'} = m\xi\zeta, \quad D_{z'} = m\xi\eta. \tag{12}$$

Let us assume that the point $O(\xi, \eta, \zeta)$ lies on one of its central planes, e. g. on the xy-plane. Therefore $\zeta = 0$. Hence by (12) we obtain $D_{x'} = 0$ and $D_{y'} = 0$. It follows from this that the z'-axis is a principal axis of inertia with respect to the point O. Hence we have the theorem:

One of the principal axes of inertia with respect to a point lying in a central plane is perpendicular to this plane.

In particular, if the point O lies on a central axis, e. g. on the x-axis, then $\eta = 0$ and $\zeta = 0$, whence by (12) $D_{x'} = 0$, $D_{y'} = 0$ and $D_{z'} = 0$. It follows from this that the x', y', z' axes are principal axes of inertia with respect to the point O. We therefore obtain the corollaries:

1° *The principal axes of inertia with respect to a point lying on a central axis are parallel to the central axes.*

2° *The central axis is the principal axis of inertia with respect to each of its points.*

If the given system of material points possesses an axis or a plane of symmetry, then we can prove the following theorem:

An axis of symmetry of a system of material points is a central axis; similarly, *a plane of symmetry is a central plane.*

Proof. In order to prove the first part of the theorem, let us note that the centre S of mass m of the system lies on the axis of symmetry. Let us take S as the origin of the coordinate system (x, y, z) and the axis of symmetry as the z-axis. Because of this the given system of material points includes in addition to each point A_i of mass m_i and coordinates x_i, y_i, z_i, another point A_i' of mass equal to m_i and having coordinates $-x_i, -y_i, z_i$. We therefore have

$$D_x = \Sigma[m_i y_i z_i + m_i(-y_i) z_i] = 0, \quad D_y = \Sigma[m_i x_i z_i + m_i(-x_i) z_i] = 0.$$

It follows from this that the axis of symmetry z is at the same time the principal axis of inertia with respect to the centre of mass S, and therefore it is a central axis.

In order to prove the second part of the theorem, let us note that the centre of mass S lies in the plane of symmetry. Let us take the origin of the coordinate system at S and the x and y axes in the plane of symmetry. Since the xy-plane is a plane of symmetry, to each point A_i of mass m_i and coordinates x_i, y_i, z_i, there exists in our system of material points a point A_i' of mass equal to m_i and having coordinates $x_i, y_i, -z_i$. We therefore have

$$D_x = \Sigma[m_i y_i z_i + m_i y_i(-z_i)] = 0, \quad D_y = \Sigma[m_i x_i z_i + m_i x_i(-z_i)] = 0.$$

It follows from this that the z-axis is a central axis, and hence the plane of symmetry xy, being perpendicular to the central axis z, is a central plane.

§ 5. Second moments of a plane system. Let a system of material points lying in a plane Π be given. Since the plane Π is a plane of symmetry, by the preceding theorem it is a central plane. Therefore, at every point of the plane Π, one of the principal axes of inertia is perpendicular to the plane Π, while the two remaining principal axes of inertia lie in the plane Π.

If only the moments of inertia with respect to lines lying in the plane Π are taken into account, then the considerations of §§ 3 and 4 can be simplified.

Let us choose the given point O as the origin of the coordinate system (x, y) of the plane Π. From the point O let us draw an arbitrary line l lying in this plane and forming an angle α with the x-axis (Fig. 114). The moment of inertia with respect to l is obtained from formula (I), p. 162, by putting $\beta = 90° - \alpha$, and $\gamma = 90°$. Consequently

$$I_l = I_x \cos^2\alpha + I_y \sin^2\alpha - D_z \sin 2\alpha. \tag{1}$$

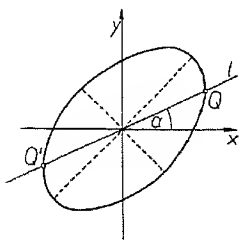

Fig. 114.

We shall call the product of inertia $D_z = \Sigma m_i x_i y_i$ the *product of inertia with respect to the x and y axes*.

On the line l let us mark off the points Q and Q' whose distances from O are inversely proportional to the radius of gyration. The collection of such pairs of points marked off on all lines l which pass through O will form a curve which will be the intersection of the plane Π with the ellipsoid of inertia with respect to the point O. This curve will therefore be an ellipse; we shall call it the *ellipse of inertia with respect to O*.

Its equation is obtained from equation (6), p. 163, by putting $z = 0$. Hence the equation of the ellipse of inertia will be

$$I_x x^2 + I_y y^2 - 2D_z xy = c^2. \tag{2}$$

If the x and y axes are chosen as the principal axes of inertia, then the equation of the ellipse will assume the form

$$I_x x^2 + I_y y^2 = c^2. \tag{3}$$

II. SOLIDS, SURFACES AND MATERIAL LINES

§ 6. Density. If a body is not so small that it can be considered as a material point, then, in addition to the mass of the body, we also give the distribution of the mass in the body; for in many problems of mechanics, not only a knowledge of the mass of the entire body is of great importance, but also a knowledge of the mass of its separate parts.

Frequently it happens that the mass of a part of a body is proportional to the volume. Then, denoting the mass by m, and the volume of the body by v, we obtain as the mass per unit volume

$$\varrho = m / v. \tag{I}$$

The number ϱ is called the *density* of the body.

In this case we say that the mass of the body is distributed uniformly, or that the body is *homogeneous*, or finally, that the *density is constant*.

The mass of a part of a body of volume v' is then $m' = v'\varrho$. By (I) the dimension of density is

$$[\text{density}] = L^{-3}M. \tag{1}$$

Let us pass on now to the general case, i. e. we do not assume that the mass in a given body is distributed uniformly. Let A be any point of the given body. Let us select in the body an arbitrary cube of mass Δm and volume Δv, whose centre is the point A. The limit

$$\lim_{\Delta v \to 0} \frac{\Delta m}{\Delta v} = \varrho \tag{II}$$

is called the *density of the body at the point A*.

In general, the density ϱ depends on the point A. If A has the co-

ordinates x, y, z, then ϱ is a function of the variables x, y, z. We can therefore write $\varrho = \varrho(x, y, z)$. If $\varrho = \text{const}$, then the mass of the body is distributed uniformly, and we have the case of the homogeneous body already considered.

We shall always assume that ϱ is a continuous function.

Calculation of mass. Knowing the density at every point of a body, we can calculate its mass as well as the mass of an arbitrary part of it.

By means of planes parallel to the xy, yz and zx planes, let us divide the given body into small rectangular parallelepipeds (the so-called *elements of volume*) and possibly into boundary pieces of irregular form. Let us denote the volumes of successive rectangular parallelepipeds by $\Delta v_1, \Delta v_2, \ldots$ and let us select one point in each one of them whose coordinates are $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ respectively. The masses of the separate rectangular parallelepipeds are approximately $\varrho(x_1, y_1, z_1)\,\Delta v_1$, $\varrho(x_2, y_2, z_2)\,\Delta v_2, \ldots$ Therefore the sum

$$\varrho(x_1, y_1, z_1)\,\Delta v_1 + \varrho(x_2, y_2, z_2)\,\Delta v_2 + \ldots$$

represents approximately the mass of the body. Forming subdivisions into smaller and smaller rectangular parallelepipeds whose dimension tend to zero and passing to the limit, we obtain for the mass m of a given body the formula

$$m = \iiint_D \varrho(x, y, z)\,\mathrm{d}v, \tag{III}$$

where the region of integration D extends over the entire body.

In particular, if $\varrho = \text{const}$, we obtain from formula (III) $m = \varrho v$ which agrees with formula (I).

Formula (III) also gives the mass of an arbitrary part of the given body if we assume that D denotes the region occupied by this part.

Material surface, material line. Sometimes one or two dimensions of a body are small in comparison with the remaining ones. Examples of such bodies are plates, rods, wires etc. In these cases we represent the body as a surface or a material line and say that its *mass* is *distributed along a surface* or *along a line*.

Let a mass be distributed along a surface S, and let A denote an arbitrary point on this surface. Let us denote by $\Delta\sigma$ the area of a small part of the surface S containing the point A (the so-called *element of area*), and by Δm the mass of this part. If the dimensions of the element tend to zero, then

$$\lim_{\Delta\sigma \to 0} \frac{\Delta m}{\Delta\sigma} = \varrho \tag{IV}$$

is called the *surface density at the point A*.

In particular, if $\varrho = \text{const}$, then ϱ represents the mass of an element 1 cm² in area cut out from the surface S.

It can be shown (in a manner similar to that used for solids) that the mass of the surface S is expressed by the formula

$$m = \iint_S \varrho\,\mathrm{d}\sigma, \tag{V}$$

where the region of integration S extends over the entire surface.

If $\varrho = \text{const}$, and P denotes the area of the surface S, we have $m = \varrho P$, whence

$$\varrho = m / P. \tag{2}$$

From the above formula we obtain as the dimension of surface density

$$[\text{surface density}] = L^{-2}M. \tag{3}$$

We proceed similarly in the case of a mass distributed linearly along a certain curve C. If A is a point of the curve C, then we choose an arbitrary arc of the curve C containing the point A. If Δs denotes the length of this arc (the so-called *element of length*), and Δm its mass, then

$$\lim_{\Delta s \to 0} \frac{\Delta m}{\Delta s} = \varrho \tag{VI}$$

is called the *linear density at the point A*.

In particular, if $\varrho = \text{const}$, then ϱ represents the mass of an arc (of the curve C) 1 cm in length.

The mass of the entire curve C is

$$m = \int_C \varrho\,\mathrm{d}s, \tag{VII}$$

where the region of integration extends over the entire curve.

If $\varrho = \text{const}$, and s denotes the length of the curve C, we have by (VII):

$$m = \varrho s, \quad \text{whence} \quad \varrho = m / s. \tag{4}$$

Hence, as the dimension of linear density, we obtain the formula

$$[\text{linear density}] = L^{-1}M \tag{5}$$

§ 7. Statical moments and moments of inertia. Centre of mass. The *statical moment* of a body with respect to a certain plane, e. g. the xy-plane,

is defined in the following manner. We divide the body into small rectangular parallelepipeds of volumes $\Delta v_1, \Delta v_2, \ldots$ and possibly into certain irregular boundary pieces. In each rectangular parallelepiped we select arbitrarily one point whose coordinates are $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ respectively. Let us denote the density of the body at the point x, y, z, by $\varrho(x, y, z)$. The mass of the rectangular parallelepiped Δv_1 is approximately $\varrho(x_1, y_1, z_1) \cdot \Delta v_1$; if its entire mass were situated at the point x_1, y_1, z_1, then the statical moment of this mass with respect to the xy-plane would be equal to $z_1 \cdot \varrho(x_1, y_1, z_1) \cdot \Delta v_1$. We can therefore consider the sum

$$z_1 \cdot \varrho(x_1, y_1, z_1) \cdot \Delta v_1 + z_2 \cdot \varrho(x_2, y_2, z_2) \cdot \Delta v_2 + \ldots$$

as representing approximately the statical moment of the body with respect to the xy-plane. It is for this reason that the limit of the above sum, when the dimensions of the rectangular parallelepipeds tend to zero, is called the *statical moment of the body with respect to the xy-plane.*

Since the limit of the above sum is the triple integral

$$\iiint_D z\varrho \, dv,$$

where the region of integration D extends over the entire body, we have

$$M_{xy} = \iiint_D z\varrho \, dv, \text{ and similarly } M_{xz} = \iiint_D y\varrho \, dv, \quad M_{yz} = \iiint_D x\varrho \, dv. \qquad \text{(I)}$$

We define the statical moment of surfaces and curves analogously. Instead of triple integrals there occur double integrals over surfaces and single integrals over curves.

In the case of a mass distributed over a surface S we obtain

$$M_{xy} = \iint_S z\varrho \, d\sigma, \quad M_{xz} = \iint_S y\varrho \, d\sigma, \quad M_{yz} = \iint_S x\varrho \, d\sigma, \qquad \text{(II)}$$

where $d\sigma$ is an element of area, and for a mass distributed linearly along a curve C:

$$M_{xy} = \int_C z\varrho \, ds, \quad M_{xz} = \int_C y\varrho \, ds, \quad M_{yz} = \int_C x\varrho \, ds, \qquad \text{(III)}$$

where ds is an element of arc length.

The statical moments of plane figures with respect to the x and y axes are expressed by the formulae

$$M_x = \iint_D y\varrho \, dx \, dy, \quad M_y = \iint_D x\varrho \, dx \, dy. \qquad (1)$$

For plane curves we have:

$$M_x = \int_C y\varrho \, ds, \quad M_y = \int_S x\varrho \, ds. \qquad (2)$$

The *centre of mass* of a body, surface or a material curve is defined as the point having coordinates:

$$x_0 = M_{zy} / m, \quad y_0 = M_{xz} / m, \quad z_0 = M_{xy} / m, \qquad \text{(IV)}$$

where M_{zy}, M_{xz}, M_{xy} denote the statical moments with respect to the zy, xz, xy planes, and m is the mass of the body.

For plane figures and curves we get:

$$x_0 = M_y / m, \quad y_0 = M_x / m. \qquad \text{(V)}$$

If the density of the body $\varrho = \text{const}$, then

$$M_{zy} = \varrho \iiint_D x \, dv, \quad \text{and} \quad m = \varrho \iiint_D dv = \varrho v,$$

whence

$$x_0 = \frac{\iiint_D x \, dv}{v}, \quad y_0 = \frac{\iiint_D y \, dv}{v}, \quad z_0 = \frac{\iiint_D z \, dv}{v}. \qquad \text{(VI)}$$

We see from this that x_0, y_0, z_0 do not depend on the density.

Hence: *if the density is constant, then the position of the centre of gravity does not depend on the density.*

The same relates to surfaces and curves.

It can be shown that the theorems concerning the centre of mass for material systems of points, proved in § 2, hold also in the case of material bodies, surfaces and curves.

Geometric solids, surfaces and curves. The *statical moment of a geometric solid* is defined as the statical moment of a material body having the form of the given solid and a density $\varrho = \text{const}$; usually we suppose that $\varrho = 1$.

The centre of mass of this body (which does not depend on ϱ) is called the *centre of gravity of the geometric solid.*

In the same manner we define the statical moment and the centre of gravity for geometric surfaces and curves.

Statical moments and centres of mass of geometric configurations are obtained, therefore, by putting $\varrho = 1$ in the given formulae (I)—(V), (1) and (2), and assuming because of this that m denotes the volume, area, or length, depending on whether the geometric configuration is a solid, surface, or curve.

We shall now become acquainted with a theorem which in many cases facilitates the finding of the centre of mass. Let us cut the given solid D by planes parallel to a certain plane Π. Let us assume that the centres of gravity of these sections lie in a certain plane σ.

Under these assumptions it can be proved that the centre of gravity of the solid D also lies in the plane σ.

This is intuitively evident. For let us cut the solid D into thin layers by means of planes parallel to the plane Π. We can assume, as an approximation, that the centre of mass of each layer lies in the plane σ. Therefore the statical moment of each layer with respect to the plane σ is equal to zero. It follows from this that the statical moment of the entire solid D with respect to the plane σ is zero (because it is equal to the sum of the moments of the separate layers). Hence the centre of mass of the solid D will also lie in the plane σ.

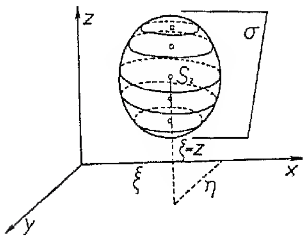

Fig. 115.

A rigorous proof can be carried out in the following manner. Let us suppose that Π is a horizontal plane, and σ is a plane having the equation

$$Ax + By + Cz + E = 0. \tag{3}$$

Let us denote the section of the solid D made by a horizontal plane at the height z by D_z (Fig. 115). Let ξ, η, and $\zeta = z$ be the coordinates of the centre of gravity S_z of the section D_z. Obviously ξ, η, and ζ are functions of z, and by (3)

$$A\xi + B\eta + C\zeta + E = 0. \tag{4}$$

Denoting by x_0, y_0, z_0 the coordinates of the centre of mass of the solid D, we obtain from (VI)

$$Ax_0 + By_0 + Cz_0 + E =$$
$$= \frac{1}{v}\Big(A\iiint\limits_D x\,dv + B\iiint\limits_D y\,dv + C\iiint\limits_D z\,dv + Ev\Big). \tag{5}$$

Let P_z be the area of the section D_z, and z' and z'' (where $z' < z''$) the limits between which z varies. Then

$$P_z = \iint\limits_{D_z} dx\,dy. \tag{6}$$

Resolving the triple integral into an iterated integral, we obtain:

$$v = \iiint\limits_D dv = \int_{z'}^{z''} dz \iint\limits_{D_z} dx\,dy = \int_{z'}^{z''} P_z\,dz, \tag{7}$$

$$\iiint\limits_D z\,dv = \int_{z'}^{z''} z\,dz \iint\limits_{D_z} dx\,dy = \int_{z'}^{z''} P_z z\,dz = \int_{z'}^{z''} P_z \zeta\,dz, \tag{8}$$

$$\iiint\limits_D x\,dv = \int_{z'}^{z''} dz \iint\limits_{D_z} x\,dx\,dy. \tag{9}$$

Since

$$\iint\limits_{D_z} x\,dx\,dy$$

represents the statical moment of the section D_z with respect to the yz-plane,

$$\iint\limits_{D_z} x\,dx\,dy = P_z\xi.$$

Therefore by (9)

$$\iiint\limits_D x\,dv = \int_{z'}^{z''} P_z\,\xi\,dz, \quad \text{and similarly} \quad \iiint\limits_D y\,dv = \int_{z'}^{z''} P_z\eta\,dz. \tag{10}$$

Hence by (6)—(10) we get

$$Ax_0 + By_0 + Cz_0 + E = \frac{1}{v}\int_{z'}^{z''}(A\xi + B\eta + C\zeta + E)\,P_z\,dz.$$

From formula (4) we obtain then

$$Ax_0 + By_0 + Cz_0 + E = 0.$$

Therefore the centre of gravity of the solid D lies in the plane σ. We have thus proved the

Theorem. *If the centres of gravity of parallel sections of a given solid lie in one plane, then the centre of gravity of this solid lies in this same plane.*

In particular, it follows that if the centres of gravity of the sections lie on one line, then the centre of gravity of this solid lies on this line. For if an arbitrary plane is passed through this line, then by the theorem just proved, the centre of gravity of the solid will lie in this plane.

Similar theorems hold for surfaces and plane figures.

Guldin's rules. Let a given curve C whose equation is $y = f(x)$, $f(x) \geq 0$ for $a \leq x \leq b$, lie in the xy-plane. Denote the length of the curve by l. By (V) the centre of gravity is expressed by the formulae:

$$x_0 = M_y / m = \int_a^b x\,ds / l, \quad y_0 = M_x / m = \int_a^b y\,ds / l. \tag{11}$$

The area of the surface generated by revolving the given curve about the x-axis is

$$P = 2\pi \int_a^b y\,ds.$$

Hence by (11)

$$P = 2\pi l y_0. \qquad \text{I.}$$

A similar formula is obtained for an arbitrary curve lying above the x-axis.

Since the centre of mass describes a circle of radius y_0 as the curve revolves, $2\pi y_0$ denotes the circumference of this circle.

Hence: *the area of a surface generated by revolving a plane curve about an axis lying in the plane of this curve and not cutting it, is equal to the product of the length of the curve and the length of the path described by the centre of gravity.*

This is the so-called *Guldin's first rule.*

Let us take under consideration for the same curve the region D bounded by the curve, the x-axis, and the ordinates $x = a$ and $x = b$. Let us denote the area of the region D by F. By (V), p. 171, the centre of gravity of the region D has the coordinates:

$$x_0 = M_y / F = (\iint_D x \,\mathrm{d}x \,\mathrm{d}y) / F, \quad y_0 = M_x / F = (\iint_D y \,\mathrm{d}x \,\mathrm{d}y) / F. \quad (12)$$

But

$$\iint_D y \,\mathrm{d}x \,\mathrm{d}y = \int_a^b \mathrm{d}x (\int_0^y y \,\mathrm{d}y) = \tfrac{1}{2} \int_a^b y^2 \,\mathrm{d}x.$$

Therefore by (12)

$$F y_0 = \tfrac{1}{2} \int_a^b y^2 \,\mathrm{d}x. \quad (13)$$

If the curve revolves about the x-axis, then the volume of the solid generated will be

$$V = \pi \int_a^b y^2 \,\mathrm{d}x,$$

whence by (13)

$$V = 2\pi y_0 F. \quad \text{II.}$$

A similar formula would be obtained for an arbitrary region lying above the x-axis.

Hence: *the volume of a solid generated by revolving a plane region about an axis lying in the plane of the region and not intersecting it, is equal to the product of the area of the region and the length of the path traversed by the centre of gravity of the region.*

This is the so-called *Guldin's second rule.*

Moments of inertia and products of inertia. Proceeding as we did in connection with statical moments, we come to the definitions of moments of inertia for solids, surfaces, and curves.

If $\varrho(x, y, z)$ denotes the density of the solid, then the moments of inertia with respect to the xy, yz and zx planes are defined by the formulae:

$$I_{xy} = \iiint_D \varrho z^2 \,\mathrm{d}v, \quad I_{yz} = \iiint_D \varrho x^2 \,\mathrm{d}v, \quad I_{zx} = \iiint_D \varrho y^2 \,\mathrm{d}v, \quad \text{(VII)}$$

the moments of inertia with respect to the coordinate axes:

$$\begin{aligned} I_x &= \iiint_D \varrho (y^2 + z^2) \,\mathrm{d}v, \quad I_y = \iiint_D \varrho (x^2 + z^2) \,\mathrm{d}v, \\ I_z &= \iiint_D \varrho (x^2 + y^2) \,\mathrm{d}v, \end{aligned} \quad \text{(VIII)}$$

and the products of inertia with respect to the coordinate planes:

$$D_x = \iiint_D \varrho yz \,\mathrm{d}v, \quad D_y = \iiint_D \varrho zx \,\mathrm{d}v, \quad D_z = \iiint_D \varrho xy \,\mathrm{d}v. \quad \text{(IX)}$$

In order to obtain the moments of inertia of surfaces (curves), it is necessary to replace the triple integral by a double (single) integral over a surface (over a curve) and to substitute $\mathrm{d}\sigma$ ($\mathrm{d}s$) for $\mathrm{d}v$ in the given formulae just as in the case of statical moments. The definitions of the radius of gyration as well as those of a concentrated mass remain unchanged. The theorems proved for systems of material points obtain here also.

§ 8. Centres of gravity of some curves, surfaces and solids.

If a line, surface, or solid has a centre of symmetry, then it is at the same time its centre of gravity. Therefore the centre of gravity of a segment, parallelogram, circle, parallelepiped, sphere and cylinder is the centre of symmetry of these configurations.

Broken line. The centre of gravity of a broken line, e. g. $ABCD$, is obtained by replacing a line segment by a material point situated at the centre of the segment, and having a mass equal to the length of the segment. The centre of gravity of this system of points will be the centre of gravity of the broken line $ABCD$ (Fig. 116).

Let d_1, d_2, d_3 denote the lengths of the segments AB, BC, CD, and $S_1(x_1, y_1)$, $S_2(x_2, y_2)$, $S_3(x_3, y_3)$ the centres of these segments. The centre of gravity of the broken line $ABCD$ will therefore have the coordinates

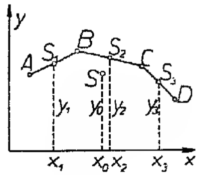

Fig. 116.

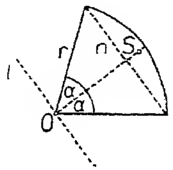

Fig. 117.

$$x_0 = \frac{d_1x_1 + d_2x_2 + d_3x_3}{d_1 + d_2 + d_3}, \qquad y_0 = \frac{d_1y_1 + d_2y_2 + d_3y_3}{d_1 + d_2 + d_3}. \tag{1}$$

The arc of a circle of radius r, subtending a central angle of 2α, has the bisector of this angle as an axis of symmetry. Therefore the centre of gravity of the arc lies on this bisector (Fig. 117).

In order to determine the distance of the centre of gravity S from the centre of the circle O, we make use of Guldin's first rule. As the arc rotates about the diameter l, perpendicular to the bisector of the angle 2α, it describes a zone of area $2r\pi h$ (where h denotes the length of the chord subtended by the arc). The length of the arc is $s = 2r\alpha$, and that of the path of the centre of gravity is $2\pi \cdot OS$. Therefore $2r\pi h = 4r\pi\alpha \cdot OS$, whence $OS = h \,/\, 2\alpha$. Since $h = 2r \sin\alpha$,

$$OS = r\frac{\sin\alpha}{\alpha}. \tag{2}$$

In particular, for the semicircle $2\alpha = \pi$; consequently $OS = 2r \,/\, \pi = 0.64r$.

Triangle. Let us cut a triangle by lines parallel to one of its sides. The centres of the segments lie on the median, and hence so does the centre of gravity of the triangle.

It follows from this that the centre of gravity of the triangle lies at the point of intersection of the three medians, and hence at a distance of one third of the corresponding altitude from each side.

Trapezoid. The centres of the segments parallel to the base of a trapezoid lie on the median, and hence so does the centre of gravity S of the trapezoid.

In order to determine the distance y_0 of the centre of gravity S from the base, let us calculate the statical moment of the trapezoid with respect to the base. Let a denote the base, b the parallel side, h the altitude and P the area of the trapezoid. The statical moment with respect to the base is

$$M = Py_0 = \tfrac{1}{2}(a + b)\,hy_0. \tag{3}$$

Dividing the trapezoid into a parallelogram and a triangle, we get

$$M = bh \cdot \tfrac{1}{2}h + \tfrac{1}{2}(a - b)\,h \cdot \tfrac{1}{3}h = \tfrac{1}{6}h^2(a + 2b). \tag{4}$$

By comparing (3) and (4) we get

$$y_0 = \tfrac{1}{3} \cdot \frac{a + 2b}{a + b}\,h. \tag{5}$$

From this follows the geometric construction of the centre of gravity shown in Fig. 118. From the similarity of triangles BCS and ADS we get $(h - y_0) \,/\, y_0 = (\tfrac{1}{2}b + a) \,/\, (\tfrac{1}{2}a + b)$, from which we get y_0 in agreement with formula (5).

Polygon. In order to determine the centre of gravity of a polygon we break it up into triangles (trapezoids, rectangles), and then we compute the statical moments of the separate parts with respect to the axes of the system.

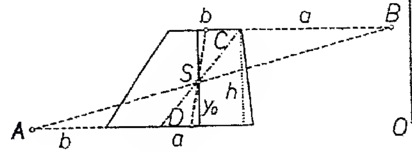

Fig. 118.

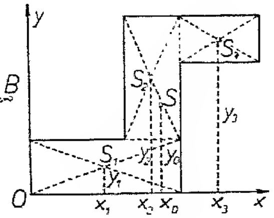

Fig. 119.

Let us denote the area of the configuration given in Fig. 119 by p. We break it up into three rectangles having areas p_1, p_2, and p_3. Let x_1, y_1, x_2, y_2 and x_3, y_3 be the coordinates of the centres of gravity with respect to the x and y axes. We have $M_x = p_1y_1 + p_2y_2 + p_3y_3$, and $M_y = p_1x_1 + p_2x_2 + p_3x_3$; hence the centre of gravity S has the coordinates

$$x_0 = M_y \,/\, p, \qquad y_0 = M_x \,/\, p. \tag{6}$$

Sector of a circle. Let us consider the sector of the circle OAB (Fig. 120). Because of symmetry the centre of gravity S of the sector lies on the bisector of the central angle 2α. The distance OS of the centre of mass from the centre of the circle O is obtained by using Guldin's second rule. The sector OAB generates a spherical sector by revolving about the radius $OA = r$. The altitude of the segment of the spherical sector will be $CA = OA - OC = r - r\cos 2\alpha = 2r\sin^2\alpha$, from which the volume of the spherical sector $v = \tfrac{2}{3}r^2\pi \cdot 2r\sin^2\alpha = \tfrac{4}{3}r^3\pi\sin^2\alpha$. The centre of gravity will describe a circle of radius $y_0 = OS \cdot \sin\alpha$. The area of the sector is $r^2\alpha$. Hence by Guldin's second rule $\tfrac{4}{3}r^3\pi\sin^2\alpha = 2\pi y_0 \cdot r^2\alpha = 2\pi OS \sin\alpha \cdot r^2\alpha$, whence

$$OS = \frac{2\sin\alpha}{3\alpha}\,r. \tag{7}$$

For a semicircle we have in particular $2\alpha = \pi$, whence

$$OS = 4r \,/\, 3\pi = 0.42r. \tag{8}$$

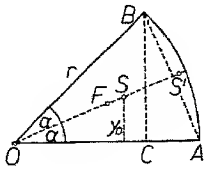

Fig. 120.

Segment of a circle. The centre of gravity S' of a segment of a circle is situated on the bisector of the central angle subtended by this segment. We obtain the distance OS' of the centre of gravity of the segment from the centre of the circle from the formula representing the statical moment of the sector OAB with respect to OA as the sum of the moments of the triangle OAB and the segment of the circle. Denoting by p the area of the sector, by p' the area of the triangle OAB, by p'' the area of the segment, and by F the centre of mass of the triangle OAB, we obtain $p \cdot OS \cdot \sin\alpha = p' \cdot OF \sin\alpha + p'' \cdot OS' \sin\alpha$. Since $p = r^2\alpha$, $p' = \frac{1}{2}r^2 \sin 2\alpha$, $p'' = p - p'$ and $OF = \frac{2}{3}r\cos\alpha$,

$$OS' = \frac{4\sin^3\alpha}{3(2\alpha - \sin 2\alpha)}\,r. \tag{9}$$

Prism. Cylinder. The centres of gravity of sections of a prism made by planes parallel to a base lie on a line joining the centres of gravity of both bases. The sections made by planes parallel to one of the lateral faces are parallelograms (or consist of several parallelograms); the centres of gravity of these sections lie in a plane parallel to the base and passing half way up the altitude. The discussion for the cylinder is similar.

It follows from this that *the centre of gravity of a prism or a cylinder lies halves the straight line joining the centres of gravity of both its bases.*

Pyramid. Cone. The centres of gravity of sections parallel to the base of a pyramid lie on a line joining the vertex with the centre of gravity of the base. Hence the centre of gravity S of the pyramid also lies on this line. In order to determine the height at which this centre of gravity lies we shall calculate the statical moment of the pyramid with respect to the plane of the base. Selecting the plane of the base as the horizontal plane, we obtain $M_{xy} = \iiint z \,dx\,dy\,dz$. The region of integration extends over the entire pyramid. Let us denote the altitude of the pyramid by h (Fig. 121), the section made by a horizontal plane at a height z by D_z, and the area of the section D_z by P_z. Resolving the triple integral into an iterated integral we get

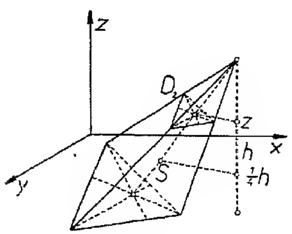

Fig. 121.

$$M_{xy} = \int_0^h z\,dz \iint_{D_z} dx\,dy = \int_0^h z P_z\,dz.$$

Let P denote the area of the base. As is known $P_z / P = (h - z)^2 / h^2$, or $P_z = [(1 - z) / h]^2 P$. Therefore

$$M_{xy} = \int_0^h z[(1-z)\,/\,h]^2 P\,dz = \tfrac{1}{12}h^2P.$$

On the other hand, denoting the volume of the pyramid by v and its centre of gravity by z_0, we have

$$M_{xy} = z_0 v = z_0 \cdot \tfrac{1}{3}hP.$$

By equating both formulae for M_{xy} we obtain

$$z_0 = \tfrac{1}{4}h. \tag{10}$$

The discussion for the cone is similar.

Hence: *the centre of gravity of a pyramid (cone) lies one fourth of the way up the straight line joining the vertex with the centre of gravity of the base.*

§ 9. Moments of inertia of some curves, surfaces and solids. In this § we shall assume that the curves, surfaces, and solids considered have a constant density ϱ.

Segment. Let us calculate the moment of inertia of the line segment AB of length a with respect to the line l passing through the centre O of this segment and inclined at an angle α to it (Fig. 122).

Let us suppose that AB lies on the x-axis and that O is the origin of the coordinate system. Let us subdivide the segment AB into small segments by means of the points $x_1, x_2, \ldots$ Set $\Delta x_1 = x_2 - x_1$, $\Delta x_2 = x_3 - x_2$, etc. The moment of inertia of the segment Δx_i with respect to the line l is approximately $r_i^2\,\Delta m_i$, where Δm_i denotes the mass of the i-th segment, and r_i the distance of its left end point from l. Since $r_i = x_i \sin\alpha$, $\Delta m_i = \varrho\,\Delta x_i$, it follows that $r_i^2\,\Delta m_i = x_i^2\,\varrho\,\Delta x_i \sin^2\alpha$. We can say, therefore, that the moment of inertia I_l with respect to l is approximately $\Sigma x_i^2 \varrho\,\Delta x_i \sin^2\alpha$. Passing to the limit, we obtain

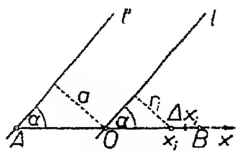

Fig. 122.

$$I_l = \int_{-\frac{1}{2}a}^{+\frac{1}{2}a} x^2\varrho \sin^2\alpha\,dx = \tfrac{1}{12}a^3\varrho\sin^2\alpha.$$

The mass m of the segment AB is $m = a\varrho$. Therefore

$$I_l = \tfrac{1}{12}ma^2\sin^2\alpha. \tag{1}$$

O is the centre of gravity of the segment AB; therefore the moment of inertia with respect to the line l' parallel to l and lying at a distance d from O is according to formula (I), p. 159, $I_{l'} = I_l + md^2$, i. e.

$$I_{l'} = \tfrac{1}{12}m(a^2\sin^2\alpha + 12d^2). \tag{2}$$

In particular, if l' passes through the end point A, then $d = \frac{1}{2}a \sin\alpha$, whence

$$I_{l'} = \tfrac{1}{3}ma^2 \sin^2\alpha. \tag{3}$$

If the lines l and l' are perpendicular to AB, then $\alpha = \frac{1}{2}\pi$, and the moments I_l and $I_{l'}$ are reduced to the moments of inertia with respect to the points O and A. From (1) and (3) we obtain for $\alpha = \frac{1}{2}\pi$:

$$I_o = \tfrac{1}{12}ma^2, \qquad I_A = \tfrac{1}{3}ma^2. \tag{4}$$

Rectangle. Let us pass the x and y axes of a coordinate system through the centre of a rectangle of sides a and b. Since these axes are axes of symmetry, they are at the same time central axes and therefore

$$I_y = \iint_D x^2\varrho \,\mathrm{d}x\,\mathrm{d}y = \varrho \int_{-\frac{1}{2}b}^{+\frac{1}{2}b} \mathrm{d}y \int_{-\frac{1}{2}a}^{+\frac{1}{2}a} x^2 \,\mathrm{d}x = \tfrac{1}{12}a^3b\varrho.$$

The mass of the rectangle is $m = ab\varrho$; hence

$$I_y = \tfrac{1}{12}ma^2; \quad \text{similarly} \quad I_x = \tfrac{1}{12}mb^2. \tag{5}$$

The product of inertia D_z is zero, hence the central ellipse of inertia has the equation (p. 167, (3)) $I_x x^2 + I_y y^2 = c^2$, or $\frac{1}{12}mb^2x^2 + \frac{1}{12}ma^2y^2 = c^2$. The constant c is arbitrary; putting $c^2 = \frac{1}{12}ma^2b^2\lambda^2$, where λ is a new arbitrary constant, we obtain

$$(x / \lambda a)^2 + (y / \lambda b)^2 = 1. \tag{6}$$

Hence: *central ellipses of inertia have axes proportional to the sides of the rectangle.*

The moment of inertia with respect to the line l (Fig. 123) passing through O and making an angle α with the x-axis is (p. 166, formula (1)) $I_l = I_x \cos^2\alpha + I_y \sin^2\alpha$ or

$$I_l = \tfrac{1}{12}m(b^2 \cos^2\alpha + a^2 \sin^2\alpha). \tag{7}$$

The moments of inertia I_a and I_b with respect to the sides a and b of the rectangle are $I_a = I_x + m(\frac{1}{2}b)^2$ and $I_b = I_x + m(\frac{1}{2}a)^2$, or

$$I_a = \tfrac{1}{3}mb^2, \qquad I_b = \tfrac{1}{3}ma^2. \tag{8}$$

Square. Retaining the notation used for the rectangle, we have $a = b$, whence $I_x = I_y$. It follows from this that the central ellipse of inertia is a circle. The centre of the square is therefore a circular point.

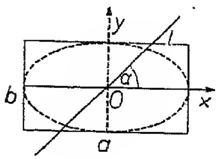

Fig. 123.

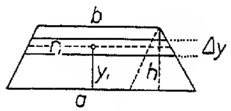

Fig. 124.

Trapezoid. In order to determine the moment of inertia of a trapezoid with respect to the base a (Fig. 124), let us divide the trapezoid into narrow strips parallel to the base. Let us denote the widths of these strips by $\Delta y_1, \Delta y_2, \ldots$, the distances of their centres from the base by $y_1, y_2, \ldots$, and the lengths of the segments passing through the centres of the strips and parallel to the base by $r_1, r_2, \ldots$ We can assume that the moment of inertia of the i-th strip with respect to the side a is approximately $\Delta m_i y_i^2$, where Δm_i denotes the mass of the i-th strip. The moment of inertia I_a with respect to the side a is approximately equal to $\Sigma \Delta m_i y_i^2$. But $\Delta m_i = \Delta y_i r_i \varrho$. From Fig. 124 we see that $(r_i - b) / (a - b) = (h - y_i) / h$, whence $r_i = a - (a - h)\, y_i / h$. Therefore I_a is approximately

$$\sum \left[a - (a - b)\frac{y_i}{h}\right] \varrho y_i^2 \,\Delta y_i.$$

Passing to the limit, we get

$$I_a = \int_0^h \left[a - (a - b)\frac{y}{h}\right] \varrho y^2 \,\mathrm{d}y = \tfrac{1}{12}\varrho(a + 3b)\, h^3.$$

Since $m = \frac{1}{2}(a + b)\, h\varrho$,

$$I_a = \tfrac{1}{6} \cdot \frac{a + 3b}{a + b} mh^2. \tag{9}$$

Triangle. From the last formula we obtain the moment of inertia of a triangle with respect to the base by putting $b = 0$. We get

$$I_a = \tfrac{1}{6}mh^2. \tag{10}$$

Parallelogram. Putting $b = a$ in formula (9), we obtain the moment of inertia of a parallelogram with respect to one of its sides:

$$I_a = \tfrac{1}{3}mh^2. \tag{11}$$

Rectangular parallelepiped. Let us place the origin of the coordinate system at the centre of a rectangular parallelepiped, so that the x, y and z axes be parallel to the edges, whose lengths we denote by a, b, and c. The moment of inertia with respect to the x-axis is

$$I_x = \iiint \varrho(y^2 + z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \varrho\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\mathrm{d}x\int_{-\frac{1}{2}b}^{\frac{1}{2}b}\mathrm{d}y\int_{-\frac{1}{2}c}^{\frac{1}{2}c}(y^2 + z^2)\,\mathrm{d}z = \tfrac{1}{12}abc\varrho(b^2 + c^2).$$

Setting $m = abc\varrho$, we obtain

$$I_x = \tfrac{1}{12}m(b^2 + c^2), \tag{12}$$

and similarly $I_y = \frac{1}{12}m(a^2 + c^2)$, $I_z = \frac{1}{12}m(a^2 + b^2)$.

The moment of inertia I_a with respect to the edge a is $I_a = I_x + md^2$, where $d = \frac{1}{2}\sqrt{b^2 + c^2}$; hence

$$I_a = \tfrac{1}{3}m(b^2 + c^2), \tag{13}$$

and similarly $I_b = \frac{1}{3}m(a^2 + c^2)$, $I_c = \frac{1}{3}m(a^2 + b^2)$.

Circumference of a circle. The moment of inertia of the circumference of a circle of radius r with respect to the centre O is obviously

$$I_o = mr^2. \tag{14}$$

In order to determine the moment of inertia with respect to a diameter, let us choose O as the origin of the coordinate system (x, y). We obviously have $I_x = I_y$. Since $I_o = I_x + I_y$, then $I_o = 2I_x$ or $I_x = \frac{1}{2}I_o$. From this the moment of inertia with respect to a diameter is

$$I_x = \tfrac{1}{2}mr^2. \tag{15}$$

Circle. Because of symmetry the moments of inertia of a circle with respect to the diameters are equal. Let us select the centre of the circle O as the centre of the coordinate system (x, y) (Fig. 125).

Therefore $I_x = I_y$, and since the moment of inertia with respect to the centre $I_o = I_x + I_y$, then $I_o = 2I_x$ and $I_x = \frac{1}{2}I_o$. In order to calculate I_o, let us divide the circle into rings by means of concentric circles of radii $x_1, x_2, \ldots$ Let us put $\Delta x_1 = x_2 - x_1$, $\Delta x_2 = x_3 - x_2, \ldots$ We can assume that the moment of inertia of the i-th ring with respect to O is approximately $\Delta m_i x_i^2$, where Δm_i denotes the mass of this ring. The area of a ring is approximately $2\pi x_i \, \Delta x_i$; hence $\Delta m_i = 2\pi x_i \varrho \, \Delta x_i$. Therefore aproximately $I_o = \Sigma 2\pi x_i^3 \varrho \Delta x_i$. Passing to the limit, we obtain

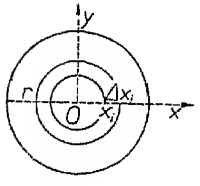

Fig. 125.

$$I_o = \int_0^r 2\pi x^3 \varrho \, dx = \tfrac{1}{2}\pi\varrho r^4. \tag{16}$$

Since the mass of a circle $m = r^2\pi\varrho$,

$$I_o = \tfrac{1}{2}mr^2 \quad \text{and} \quad I_x = \tfrac{1}{4}mr^2. \tag{17}$$

Surface of a sphere. The moment of inertia of a surface of a sphere with respect to the centre is obviously

$$I_o = mr^2. \tag{18}$$

In order to determine the moment of inertia of a sphere with respect to a diameter, let us place the origin of the coordinate system at the centre

of the sphere. Because of symmetry we have $I_x = I_y = I_z$. Since $2I_o = I_x + I_y + I_z$, $I_o = \frac{3}{2}I_x$. Therefore $I_x = \frac{2}{3}I_o$, whence

$$I_x = \tfrac{2}{3}mr^2. \tag{19}$$

Sphere. Taking the centre of the sphere as the origin of the coordinate system, we have because of symmetry as before $I_x = \frac{2}{3}I_o$. Let us calculate the moment I_o by proceeding as in the case of the circle, i. e. dividing the sphere into layers by means of concentric spheres. We obtain

$$I_o = \tfrac{3}{5}mr^2 \text{ and } I_x = \tfrac{2}{5}mr^2. \tag{20}$$

Cylinder of revolution. Let us denote the radius of the base by r and the altitude of the cylinder by h (Fig. 126). Let us take the centre of the axis of the cylinder as the origin of the coordinate system, and the axis of the cylinder as the z-axis.

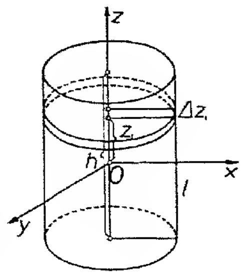

Fig. 126.

In order to calculate I_z, let us proceed as in the case of the circle, i. e. let us divide the cylinder into layers by means of cylinders whose bases are concentric with the base of the cylinder. We obtain

$$I_z = \tfrac{1}{2}mr^2. \tag{21}$$

In order to calculate I_x and I_y, let us cut the cylinder into slices by means of planes parallel to the base. Let us denote the thicknesses of the slices by $\Delta z_1, \Delta z_2, \ldots$, the coordinates of the centres of their bases by $z_1, z_2, \ldots$, and the masses of the slices by $\Delta m_1, \Delta m_2, \ldots$ The moment of inertia of the i-th slice with respect to a line parallel to the x-axis and passing through the centre of gravity of this slice is approximately equal to $\frac{1}{4}\Delta m_i r^2$ (like the moment of inertia of a circle with respect to a diameter). Hence the moment of inertia of a slice with respect to the x-axis is approximately $\frac{1}{4}\Delta m_i r^2 + \Delta m_i z_i^2$. Since $\Delta m_i = r^2\pi \, \Delta z_i \varrho$, approximately $I_x = \Sigma(\frac{1}{4}r^2 + z_i^2)\, r^2\pi\varrho \, \Delta z_i$, whence, passing to the limit, we obtain

$$I_x = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} (\tfrac{1}{4}r^2 + z^2)\, r^2\pi\varrho \, dz = \tfrac{1}{12}r^2\pi\varrho h(3r^2 + h^2).$$

Since the mass of the cylinder is $m = r^2\pi\varrho h$,

$$I_x = \tfrac{1}{12}m(3r^2 + h^2). \tag{22}$$

On account of symmetry we obviously have $I_x = I_y$.

The moment of inertia of the cylinder with respect to the generatrix l is $I_l = I_z + mr^2$; hence

$$I_l = \tfrac{3}{2}mr^2. \tag{23}$$

The z-axis is an axis of symmetry, and hence a central axis as well. Because of symmetry the x and y axes are also central axes. Hence the ellipsoid of inertia has the equation $I_x x^2 + I_y y^2 + I_z z^2 = c^2$, whence because of (22) $\frac{1}{12}m(3r^2 + h^2)(x^2 + y^2) + \frac{1}{2}mr^2z^2 = c^2$ and hence

$$(x / r)^2 + (y / r)^2 + (z / \sqrt{\tfrac{1}{6}(3r^2 + h^2)}) = \lambda^2, \tag{24}$$

where λ^2 is an arbitrary constant.

The ellipsoid of inertia is therefore an ellipsoid of revolution. When $r\sqrt{3} = h$, the ellipsoid is a sphere.

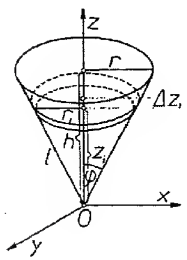

Fig. 127.

Cone of revolution. Let us denote the radius of the base by r and the altitude of the cone by h. Let us place the origin O of the coordinate system at the vertex of the cone, and let us take the axis of the cone as the z-axis (Fig. 127).

As an axis of symmetry it is also a central axis of inertia, and hence by theorem 2°, p. 165, the principal axis of inertia at the point O. Because of symmetry the x and y axes are also principal axes of inertia at the point O.

Let us cut the cone into slices of thickness Δz_i be means of planes parallel to the base. The moment of inertia of the i-th slice with respect to the z-axis is approximately $\Delta m_i r_i^2 / 2$ (like the moment of inertia of a cylinder with respect to the axis), where r_i denotes the radius of the lower base of the i-th slice. Let z_i denote the coordinate of the centre of the lower base of the i-th slice; then $r_i / r = z_i / h$, whence

$$r_i = rz_i / h. \tag{25}$$

Since $\Delta m_i = r_i^2\pi\, \Delta z_i \varrho$ approximately, by (25) we have

$$\tfrac{1}{2}\Delta m_i r_i^2 = \tfrac{1}{2}(r / h)^4 \pi\varrho z_i^4\, \Delta z_i, \tag{26}$$

whence $I_z = \Sigma\frac{1}{2}(r / h)^4 \pi\varrho z_i^4\, \Delta z_i$ approximately. Passing to the limit, we obtain

$$I_z = \int_0^h \tfrac{1}{2}(r / h)^4 \pi\varrho z^4\, \mathrm{d}z = \tfrac{1}{10}r^4 h\pi\varrho.$$

The mass of the cone is $m = \frac{1}{3}r^2\pi h\varrho$; hence

$$I_z = \tfrac{3}{10}mr^2. \tag{27}$$

In order to calculate I_x, let us note that the moment of inertia of the i-th slice with respect to a line parallel to the x-axis and passing

through the centre of gravity of this slice is approximately $\frac{1}{4}\Delta m_i r_i^2$. Therefore with respect to the x-axis it is $\frac{1}{4}\Delta m_i r_i^2 + \Delta m_i z_i^2$. By (25) and (26) this sum is equal to $\frac{1}{4}(r / h)^2 \pi\varrho z_i^4(4 + (r / h)^2)\, \Delta z_i = \Delta w_i$, or I_x is approximately equal to $\Sigma\Delta w_i$. Passing to the limit, we obtain

$$I_x = \int_0^h [\tfrac{1}{4}(r / h)^2 \pi\varrho z^4(4 + (r / h)^2)]\, \mathrm{d}z = \tfrac{1}{20}r^2\pi\varrho(4h^2 + r^2)\, h,$$

whence

$$I_x = \tfrac{3}{20}m(r^2 + 4h^2). \tag{28}$$

Obviously $I_x = I_y$.

Let φ denote the angle between the z-axis and the generatrix l (lying in the xz-plane). Since the x, y, z axes are principal axes of inertia at O, by formula (I), p. 162, $I_l = I_x \cos^2\alpha + I_y \cos^2\beta + I_z \cos^2\gamma$, where α, β, γ denote the angles between the generatrix l and the axes of the coordinate system. We have $\alpha = \frac{1}{2}\pi - \varphi$, $\beta = \frac{1}{2}\pi$, and $\gamma = \varphi$, whence

$$I_l = I_x \sin^2\varphi + I_z \cos^2\varphi,$$

and hence by (27) and (28)

$$I_l = \tfrac{3}{20}m[(r^2 + 4h^2)\sin^2\varphi + 2r^2\cos^2\varphi]. \tag{29}$$

As $\tan\varphi = r / h$, we get

$$I_l = \frac{3m}{20}\cdot\frac{r^2 + 6h^2}{r^2 + h^2}r^2. \tag{30}$$

CHAPTER V

SYSTEMS OF MATERIAL POINTS

§ 1. Equations of motion. Let there be given a system of material points of masses $m_1, m_2, \ldots, m_n$. Let us denote the sums of the forces acting on the individual points by $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, and the accelerations of these points with respect to an inertial system of coordinates by $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$. Then according to Newton's law:

$$m_1\boldsymbol{p}_1 = \boldsymbol{P}_1, \quad m_2\boldsymbol{p}_2 = \boldsymbol{P}_2, \quad \ldots, \quad m_n\boldsymbol{p}_n = \boldsymbol{P}_n.$$

We write these equations compactly as

$$m_i\boldsymbol{p}_i = \boldsymbol{P}_i \qquad (i = 1, 2, \ldots, n). \quad \text{(I)}$$

Let the point m_i have the coordinates x_i, y_i, z_i. Equations (I) can be written in the form:

$$m_i\ddot{x}_i = P_{i_x}, \quad m_i\ddot{y}_i = P_{i_y}, \quad m_i\ddot{z}_i = P_{i_z} \quad (i = 1, 2, \ldots, n). \quad \text{(II)}$$

Unconstrained systems. We assume that the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, in the most general case, depend on the time, position, and acceleration, of the system of points. We shall therefore suppose that the forces are functions of: the time t, the variables $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$ defining the positions of the points, as well as the variables $\dot{x}_1, \dot{y}_1, \dot{z}_1, \ldots, \dot{x}_n, \dot{y}_n, \dot{z}_n$ defining the velocities of the points. Hence we can write:

$$P_{i_x} = F_i(t, x_1, y_1, z_1, \ldots, x_n, y_n, z_n, \dot{x}_1, \dot{y}_1, \dot{z}_1, \ldots, \dot{x}_n, \dot{y}_n, \dot{z}_n),$$

and similarly $P_{i_y} = \Phi_i$, $P_{i_z} = \Psi_i$.

We shall assume that the functions P_{i_x}, P_{i_y} and P_{i_z} are continuous and that they have continuous first partial derivatives with respect to each variable.

Equations (II) are called *Newton's equations of motion.*

They constitute a system of differential equations of the second order. From the theory of differential equations it follows that equations (II)

determine the motion of the system of points if there are given at the initial moment $t = t_0$, the initial positions of the points (i. e. coordinates $x_1^0, y_1^0, z_1^0, \ldots, z_n^0$), and the initial velocities (i. e. $\dot{x}_1^0, \dot{y}_1^0, \dot{z}_1^0, \ldots, \dot{z}_n^0$).

Internal and external forces. The forces acting on the points of a system are divided into two groups.

In the first group are included those forces which arise from the mutual interactions of the points of the system. These forces are called *internal* forces.

The remaining forces are called *external* forces.

The internal forces are assumed to conform to the law of action and reaction (p. 72).

Let us consider the pair of forces which the two points m' and m'' of a system exert on each other. The sum of these forces is zero, and because they act along the line joining the points m' and m'', their moment with respect to an arbitrary point is zero. Since the internal forces can be grouped in such pairs, *the sum and the total moment of the internal forces are zero.*

Equilibrium of a system of points. A system of points is in equilibrium if each point is in equilibrium.

Therefore, if a system of points is in equilibrium, then the sum of the forces acting on each point is equal to zero. A system of forces having this property is said to be in *equilibrium* or that the *forces* of this systems *balance each other*.

Let a given system of forces be in equilibrium. Let us consider the forces of this system acting on an arbitrary material point. Since the sum of these forces as well as the total moment with respect to an arbitrary point of space are equal to zero, the given system is *equipollent to zero* (p. 22).

Hence: *if the forces acting on a material system of points balance each other, then the sum of these forces and the total moment are zero.*

This condition is a necessary condition for equilibrium of forces, but it is not a sufficient condition. Since the internal forces have a sum and total moment equal to zero, from the given condition it follows that *if a system of material points is in equilibrium, then the sum and the total moment of the external forces are zero.*

This condition, as the preceding one, is only a necessary condition for the equilibrium of a system.

D'Alembert's principle. We can write equations (I), p. 186, in the form

$$\boldsymbol{P}_i + (-m_i\ddot{\boldsymbol{p}}_i) = 0 \qquad (i = 1, 2, \ldots, n).$$

We have called the vectors $-m_i\ddot{\boldsymbol{p}}_i$ *forces of inertia* of the points m_i (p. 73). We can therefore say that *the forces of inertia balance the forces acting on a system of points.*

The above theorem is called *d'Alembert's principle.*

The significance of this principle appears chiefly in systems of constrained points which we shall consider further on. It is necessary to remember what we have said on p. 73, that forces of inertia are not forces, but vectors, which we have called forces only for the sake of convenience.

Example I. Two points A_1, A_2 of masses m_1, m_2, attracting each other with a force $\boldsymbol{P}$ according to Newton's law, move along the x-axis. Hence $|\boldsymbol{P}| = Km_1m_2 / r^2$, where $r = A_1A_2$. Denoting the coordinates of the points by x_1 and x_2 $(x_1 < x_2)$ (Fig. 128), we obtain the equations of motion in the form:

$$m_1\ddot{x}_1 = Km_1m_2 / (x_2 - x_1)^2, \quad m_2\ddot{x}_2 = -Km_1m_2 / (x_2 - x_1)^2. \qquad (1)$$

Fig. 128.

Let us suppose that at the time $t = 0$, $x_1 = x_1^0$, $x_2 = x_2^0$, $\dot{x}_1 = 0$, $\dot{x}_2 = 0$. Adding equations (1), we get $m_1\ddot{x}_1 + m_2\ddot{x}_2 = 0$, whence after integrating

$$m_1\dot{x}_1 + m_2\dot{x}_2 = a, \quad m_1x_1 + m_2x_2 = at + b.$$

In view of the initial conditions we obtain $a = 0$, and $b = m_1x_1^0 + m_2x_2^0$. Therefore:

$$m_1\dot{x}_1 + m_2\dot{x}_2 = 0, \quad m_1x_1 + m_2x_2 = m_1x_1^0 + m_2x_2^0. \qquad (2)$$

From equations (1) we obtain in addition

$$\ddot{x}_2 - \ddot{x}_1 = -K(m_1 + m_2) / (x_2 - x_1)^2. \qquad (3)$$

Let us set $x_2 - x_1 = r$, and $K(m_1 + m_2) = h$. We get $\ddot{r} = -h / r^2$. Multiplying both sides by $\dot{r}$ and integrating, we obtain $\frac{1}{2}\dot{r}^2 = h / r + c$. Since at $t = 0$, $\dot{r} = \dot{x}_2^0 - \dot{x}_1^0 = 0$ as well as $r = x_2^0 - x_1^0 = r_0$, it follows that $c = -h / r_0$. Therefore $\frac{1}{2}\dot{r}^2 = h / r - h / r_0$, whence

$$\dot{r} = -\sqrt{2h(1 / r - 1 / r_0)}. \qquad (4)$$

We have taken the minus sign because the points will come closer to each other, and hence r will become smaller, whence $\dot{r} < 0$. From (4) we get

$$-\frac{dr}{\sqrt{2h(1 / r - 1 / r_0)}} = dt; \text{ hence } -\int \frac{dr}{\sqrt{2h(1 / r - 1 / r_0)}} + c' = t.$$

After integrating we obtain

$$\frac{1}{2}\frac{\sqrt{r_0}}{\sqrt{2h}}\left[2\sqrt{r_0r - r^2} + r_0 \arcsin\frac{r_0 - 2r}{r_0}\right] + c' = t. \qquad (5)$$

Since $r = r_0$ at $t = 0$,

$$c' = \frac{\pi r_0\sqrt{r_0}}{4\sqrt{2h}}. \qquad (6)$$

The time T at which the points meet is obtained from (5) by setting $r = 0$. Therefore

$$T = \frac{1}{4}\pi\frac{r_0\sqrt{r_0}}{\sqrt{2h}} = \frac{r_0^{\frac{3}{2}}\pi}{2\sqrt{2K(m_1 + m_2)}}.$$

We obtain $T = 3055$ sec for $m_1 = m_2 = 1$ g, $r_0 = 1$ cm and $K = 6.6 \cdot 10^{-8}\ \mathrm{cm^3\,g^{-1}\,sec^{-2}}$.

Formulae (2), (5) and (6) (where $r = x_2 - x_1$) determine the motion of the points.

Constrained systems. If conditions exist which limit the possible motion of a system of material points, then it is called a *constrained* system and the limiting conditions are called *constraints*. As in the case of one constrained point, we assume that the constraints are the result of certain forces termed *reactions*, which cause the system to maintain the constraints.

If the forces of reaction are added to the forces acting on the points of a system, then the system can be considered as unconstrained. In this way the investigation of motions of constrained systems is reduced to the investigation of motions of unconstrained systems.

Therefore, if forces $\{\boldsymbol{P}_i\}$ act on a system of points of masses $\{m_i\}$, then, denoting the reactions by $\{\boldsymbol{R}_i\}$, we have

$$m_i\ddot{\boldsymbol{p}}_i = \boldsymbol{P}_i + \boldsymbol{R}_i \qquad (i = 1, 2, \ldots, n). \quad \text{(III)}$$

In particular, a constrained system is in equilibrium if the acting forces and the reactions balance each other.

The constraints of a system can be such that some points must constantly remain on certain curves or surfaces. In addition to this kind of constraints, already considered (cf. p. 121), we also meet with others.

For instance, two material points can be joined by an inextensible string of length l; consequently the distance between the points must be constantly $\leq l$. The string acts on the points only when it is in tension (Fig. 129). If we assume that the mass of the string is so small that it can be neglected, then the forces that the string exerts on the points will be of equal magnitude, even if the string is wound around some body (Fig. 129) — provided that there is no friction. These forces are obviously reactions. The reactions are tangent to the string and have a sense in the direction of the string.

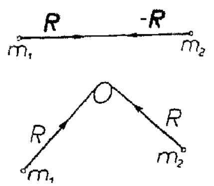

Fig. 129.

Rigid system. An important example of a constrained system of points is the so-called *rigid system*. It is a constrained system whose constraints are such that the mutual distances of the points of the system remain unchanged. Let us suppose that in this case there appear certain internal forces (i. e. forces acting between the points of the system) which cause the points to maintain constant distances in spite of the actions of external forces.

A solid physical body will in general be deformed, i. e. it will change its form under the influence of forces acting on it. It can happen, however, that when the forces do not exceed a certain limit, the deformations are so small that, practically, we can disregard them. In this case, as a model of such a body acted upon by small forces, we can choose a system of points which we have called a rigid system. The results that we shall obtain will then be approximately valid for a physical body. Thus we can apply to solid physical bodies the theorems that we shall obtain for rigid systems. Because of the important role that the theory of rigid body plays, we shall concern ourselves with this theory in all detail in chapter VI. In this chapter we shall limit ourselves to giving only a few examples based on the general theory of a system of points.

The simplest example of a rigid system is a system composed of two material points whose distance r is constant.

We can realize such a system by joining two material points with a rigid rod of a small mass, which in comparison with the masses of the points themselves, can be neglected. We then say that the points are joined by a rigid massless rod. In this manner the internal forces between the points are replaced by forces with which the rod reacts on these points. These forces are therefore equal in magnitude, have the direction of the rod, but opposite senses.

Example 2. Two heavy material points of masses m_1 and m_2 are connected by a (massless) inextensible string passing over a pulley. Point m_2 must remain on a straight vertical line l. What angle φ does the string make with the line l in the position of equilibrium if there is no friction?

The forces acting on the point m_1 are: the tension T of the string directed vertically upwards, and the weight $m_1 g$ directed vertically downwards (Fig. 130). Therefore $T - m_1 g = 0$, and hence

$$T = m_1 g. \tag{7}$$

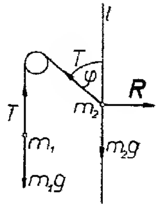

Fig. 130.

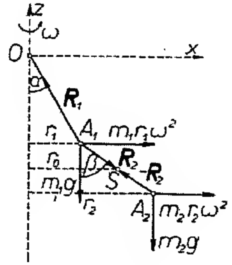

Fig. 131.

The forces acting on the point m_2 are: the tension T of the string directed along the string, the weight $m_2 g$ directed vertically downwards, and the reaction **R** perpendicular to the line l. Forming the projections of the forces on the line l, we get $T\cos\varphi - m_2 g = 0$, and hence

$$T\cos\varphi = m_2 g. \tag{8}$$

From equations (7) and (8) we obtain

$$\cos\varphi = m_2 / m_1.$$

Equilibrium is therefore possible only when $m_2 < m_1$.

Example 3. Heavy material points A_1 and A_2 of masses m_1, m_2 are connected by a massless inextensible string. Point A_1 is suspended from the string OA_1 which is also massless and inextensible. The entire system rotates about a vertical axis with a constant angular velocity ω, while the angles α and β, which the strings OA_1 and A_1A_2 form with the vertical, do not undergo any change during this rotation. To determine the angles α and β.

Let us choose the point O as the origin of the moving frame (x, y, z) rotating about the vertical z-axis with an angular velocity ω (Fig. 131). We

can suppose that the point A_1 is always situated in the xz-plane. Since the angle α is constant, the point A_1 is in relative equilibrium with respect to the frame (x, y, z).

We shall first show that the point A_2 is also situated in the xz-plane.

Since the point A_1 is in relative equilibrium, its force of transport $\mathbf{P}_t^{(1)}$ is in equilibrium with the acting forces, i. e. with the weight $\mathbf{Q}_1$, the reaction $\mathbf{R}_1$ of the string OA_1, and the reaction $\mathbf{R}_2$ of the string A_1A_2. Therefore

$$\mathbf{P}_t^{(1)} + \mathbf{Q}_1 + \mathbf{R}_1 + \mathbf{R}_2 = 0. \tag{9}$$

From this equation it follows that $\mathbf{R}_2 = -\mathbf{P}_t^{(1)} - \mathbf{Q}_1 - \mathbf{R}_1$. As the forces $-\mathbf{P}_t^{(1)}, -\mathbf{Q}_1$, and $-\mathbf{R}_1$, lie in the xz-plane, $\mathbf{R}_2$ also lies in the xz-plane. In addition, since $\mathbf{R}_2$ has the direction of the string A_1A_2, the string A_1A_2 also lies in the xz-plane; therefore the point A_2 likewise lies in the xz-plane.

Let us proceed to determine the angles α and β. In view of the fact that the angles are constant, and that the lengths OA_1 and A_1A_2 also remain unchanged, it follows that the point A_2 is likewise in relative equilibrium with respect to the frame (x, y, z). Denoting its force of transport by $\mathbf{P}_t^{(2)}$ and its weight by $\mathbf{Q}_2$, we obtain

$$\mathbf{P}_t^{(2)} + \mathbf{Q}_2 - \mathbf{R}_2 = 0. \tag{10}$$

Let us denote the distances of the points A_1 and A_2 from the axis of revolution by r_1 and r_2. We have:

$$r_1 = OA_1 \sin\alpha, \qquad r_2 = OA_1 \sin\alpha + A_1A_2 \sin\beta, \tag{11}$$

$$|\mathbf{P}_t^{(1)}| = m_1 r_1 \omega^2, \qquad |\mathbf{P}_t^{(2)}| = m_2 r_2 \omega^2. \tag{12}$$

Let us form the projections of (9) and (10) on the x and z axes. Putting $R_1 = |\mathbf{R}_1|$ and $R_2 = |\mathbf{R}_2|$, we get:

$$m_1 r_1 \omega^2 - R_1 \sin\alpha + R_2 \sin\beta = 0, \quad -m_1 g + R_1 \cos\alpha - R_2 \cos\beta = 0, \tag{13}$$

$$m_2 r_2 \omega^2 - R_2 \sin\beta = 0, \qquad -m_2 g + R_2 \cos\beta = 0. \tag{14}$$

From equations (13) and (14) we obtain

$$\tan\alpha = (m_1 r + m_2 r_2)\, \omega^2 \,/\, (m_1 + m_2)\, g, \qquad \tan\beta = r_2\omega^2 \,/\, g. \tag{15}$$

If we denote the distance of the centre of mass S of system of points A_1, A_2 from the axis of rotation z by r_0, then we obtain $(m_1 + m_2)\, r_0 = m_1 r_1 + m_2 r_2$, whence by (15) $\tan\alpha = r_0 \omega^2 / g$, and as $r_1 < r_0 < r_2$, we have $\tan\alpha < \tan\beta$ or $\alpha < \beta$.

Knowing OA_1 and A_1A_2, we find α and β from equations (11) and (15).

Example 4. Atwood's machine. At the ends of a string (inextensible, massless), parsing over a pulley (massless), are suspended two heavy material points of masses m_1 and m_2 (Fig. 132). Let us assume that both points move vertically. Since the string is inextensible, the paths traversed by both points are equal. Therefore the accelerations and velocities of both points are equal in magnitude, but they have opposite senses. Let us denote by p the projection of the acceleration of m_1 on the z-axis directed vertically downwards. Let R denote the absolute value of the force with which the string acts on the points m_1 and m_2. The weight and the reaction of the string acts on the point m_1. Therefore

$$m_1 p = m_1 g - R. \tag{16}$$

Similarly, for the point m_2 we obtain

$$-m_2 p = m_2 g - R. \tag{17}$$

From equations (16) and (17) we get

$$p = \frac{m_1 - m_2}{m_1 + m_2} g, \quad R = \frac{2m_1 m_2}{m_1 + m_2} g. \tag{18}$$

Hence the points will move with a constant acceleration.

From equation (18) we get $(m_1 + m_2)\, p = m_1 g - m_2 g$.

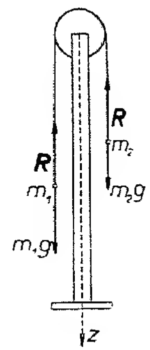

Fig. 132.

Therefore the acceleration is such as if a force $m_1 g - m_2 g$, i. e. a force equal to the difference of the weights, were acting on a material point of mass $m_1 + m_2$.

If $m_1 > m_2$, then $p > 0$, which means that the acceleration of the point m_1 is directed downwards and that of the point m_2 upwards.

If $m_1 = m_2$, then $p = 0$, which means that the points move with uniform motion.

Example 5. Two heavy points of masses m_1 and m_2, connected by an inextensible and massless string, move in a vertical plane along two lines l_1 and l_2. The tension of the string $\mathbf{T}_1$, the weight $m_1\mathbf{g}$, and the reaction of the line $\mathbf{R}_1$ act on the point m_1. Similarly, the forces $\mathbf{T}_2, m_2\mathbf{g}$, and $\mathbf{R}_2$ act on the point m_2. Let us assume that there is no friction and therefore that $\mathbf{R}_1$ and $\mathbf{R}_2$ are perpendicular to the lines l_1 and l_2, respectively (Fig. 133).

Since the string is inextensible, the paths traversed by both points will be equal, and the acceleration will then be equal in magnitude.

Let l_1 and l_2 be given downward senses. Let p denote the component (with respect to l_1) of the acceleration of the point m_1; therefore the component (with respect to l_2) of the acceleration of the point m_2 is $-p$. The forces $\mathbf{T}_1$ and $\mathbf{T}_2$ are equal in magnitude; set $T = |\mathbf{T}_1| = |\mathbf{T}_2|$. Let us denote the angles made by l_1 and l_2 with the horizontal by α_1 and α_2. Forming the projection on the lines l_1 and l_2, we obtain $m_1 p = -T + m_1 g \sin\alpha_1$, and $-m_2 p = -T + m_2 g \sin\alpha_2$, whence

$$p = \frac{m_1 \sin\alpha_1 - m_2 \sin\alpha_2}{m_1 + m_2} g.$$

Hence the points will move along the lines with uniformly accelerated motion.

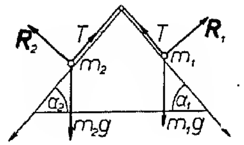

Fig. 133.

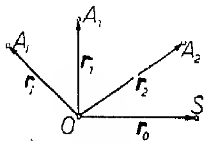

Fig. 134.

§ 2. Motion of the centre of mass. Kinematic properties of the centre of mass. Let there be given a system of material points $A_1, A_2, \ldots$ of masses $m_1, m_2, \ldots$, whose centre of mass is the point S (Fig. 134). Let us select an arbitrary point O. Put $m = \Sigma m_i$ as well as:

$$\mathbf{r}_0 = \overline{OS}, \quad \mathbf{r}_i = \overline{OA_i} \quad (i = 1, 2, \ldots).$$

In terms of the above notation the following equation holds

$$m\mathbf{r}_0 = \Sigma m_i \mathbf{r}_i. \tag{I}$$

Proof. Let us choose arbitrarily a system of coordinates with its origin at O. If x_i, y_i, z_i denote the coordinates of the point A_i, and x_0, y_0, z_0 the coordinates of the centre S, then by (II), p. 153,

$$mx_0 = \Sigma m_i x_i, \quad my_0 = \Sigma m_i y_i, \quad mz_0 = \Sigma m_i z_i. \tag{1}$$

Since the vector $\mathbf{r}_i$ has the projections x_i, y_i, z_i, and $\mathbf{r}_0$ has the projections x_0, y_0, z_0, equation (I) is only the vector form of the equalities (1).

Let us take the derivative of both sides of the equality (I) with respect to time; we obtain $m\mathbf{r}_0^{\cdot} = \Sigma m_i \mathbf{r}_i^{\cdot}$. Since $\mathbf{r}_i$ denotes the velocity $\mathbf{v}_i$ of the point A_i, and $\mathbf{r}_0^{\cdot}$ the velocity $\mathbf{v}_0$ of the centre of mass S with respect to the system $O(x, y, z)$,

$$m\mathbf{v}_0 = \Sigma m_i \mathbf{v}_i. \tag{II}$$

The vector $m\mathbf{v}_i$ is the momentum (p. 72) of the point A_i. The right side of the equality (II) therefore denotes the sum of the momenta of the separate points of the system. This sum is called the *(total) momentum of the system.*

The vector $m\mathbf{v}_0$ can be considered as the momentum of a material point, having a mass equal to the total mass of the system, situated at the centre of mass (and moving together with the centre of mass).

Therefore: *the (total) momentum of a system is equal to the momentum of the total mass situated at the centre of mass.*

Let us differentiate both sides of equation (II) with respect to the time t. We get $m\mathbf{v}_0^{\cdot} = \Sigma m_i \mathbf{v}_i^{\cdot}$. But $\mathbf{v}_i^{\cdot}$ denotes the acceleration $\mathbf{p}_i$ of the point A_i, and $\mathbf{v}_0^{\cdot}$ the acceleration $\mathbf{p}_0$ of the centre of mass S. Hence

$$m\mathbf{p}_0 = \Sigma m_i \mathbf{p}_i. \tag{III}$$

We have called the vector $-m_i\mathbf{p}_i$ the force of inertia of the point A_i (pp. 73 and 188).

Therefore: *the sum of the forces of inertia of the points of a system is equal to the force of inertia of the total mass of the system situated at the centre of mass.*

Remark. Forming the projections on the axes of the coordinate system, we obtain from equations (II) and (III):

$$mx_0^{\cdot} = \Sigma m_i x_i^{\cdot}, \quad my_0^{\cdot} = \Sigma m_i y_i^{\cdot}, \quad mz_0^{\cdot} = \Sigma m_i z_i^{\cdot}, \tag{II'}$$

$$mx_0^{\cdot\cdot} = \Sigma m_i x_i^{\cdot\cdot}, \quad my_0^{\cdot\cdot} = \Sigma m_i y_i^{\cdot\cdot}, \quad mz_0^{\cdot\cdot} = \Sigma m_i z_i^{\cdot\cdot}. \tag{III'}$$

Resultant of a system of weights. Let a system of points $A_1, A_2, \ldots$ of masses $m_1, m_2, \ldots$ be situated in a gravitational force field. Let us denote the gravitational acceleration vector by $\mathbf{g}$ and the centre of mass by S. Let O be an arbitrary point. As before, let us put $\mathbf{r}_0 = \overline{OS}$ and $\mathbf{r}_i = \overline{OA_i}$ for $i = 1, 2, \ldots$ The total moment of the weights with respect to O is $\mathbf{M} = (m_1\mathbf{g} \times \mathbf{r}_i) + (m_2\mathbf{g} \times \mathbf{r}_2) + \ldots$; therefore

$$\mathbf{M} = \mathbf{g} \times (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots),$$

whence by (I), p. 194,

$$\mathbf{M} = \mathbf{g} \times m\mathbf{r}_0 = m\mathbf{g} \times \mathbf{r}_0. \tag{2}$$

In particular, if the point O coincides with S, then $\mathbf{r}_0 = 0$, whence by (2) $\mathbf{M} = 0$.

Therefore: *the total moment of the weights of the points of a system with respect to the centre of mass is zero.*

Since the weights form a system of parallel forces having the same direction, this system has a resultant (p. 26). The resultant passes through

the centre of mass because the total moment with respect to the centre of mass is zero.

Dynamic properties of the centre of mass. Let the forces $\mathbf{P}_i$ act on the material points m_i of a given system. Let us denote the acceleration of the point m_i by $\mathbf{p}_i$, and the acceleration of the centre of mass of this system of points with respect to an inertial system of coordinates by $\mathbf{p}_0$. By formula (III), p. 195, we have $m\mathbf{p}_0 = \Sigma m_i\mathbf{p}_i$, where $m = \Sigma m_i$. Since $m_i\mathbf{p}_i = \mathbf{P}_i$ it follows that $m\mathbf{p}_0 = \Sigma\mathbf{P}_i$, whence

$$m\mathbf{p}_0 = \mathbf{P}, \text{ where } \mathbf{P} = \Sigma\mathbf{P}_i. \tag{IV}$$

Therefore: *the centre of mass of a system of points moves so as if the total mass of the system were concentrated there and the sum of all the forces acted there.*

Equation (IV) can be written in the form $\mathrm{d}(m\mathbf{v}_0) / \mathrm{d}t = \mathbf{P}$.

Hence: *the derivative of the momentum of a system is equal to the sum of all the acting forces.*

If the sum of the forces acting on the points of a system is equal to zero, i. e. if $\mathbf{P} = 0$, then by (IV) we have $m\mathbf{p}_0 = 0$, i. e. $\mathbf{p}_0 = 0$. If the sum $\mathbf{P}$ is constantly zero, then $\mathbf{p}_0 = 0$ constantly, and hence the velocity $\mathbf{v}_0$ of the centre of mass is constant. The centre of mass is then at rest or in uniform motion along a straight line. Let us note that by (II), p. 194, $m\mathbf{v}_0 = \Sigma m_i\mathbf{v}_i$; hence in this case the total momentum (or quantity of motion) of the system is a constant vector.

Therefore: *if the sum of the forces acting on a system of points is constantly equal to zero, then the centre of mass is at rest or in uniform motion along a straight line and the total momentum of the system is a constant vector.*

This theorem is known as the *principle of conservation of momentum or of quantity of motion.*

As we know (p. 187), the sum of the internal forces is always zero; therefore the sum of all the forces acting on the points of a system is equal to the sum of the external forces. We can therefore replace the sum of the forces by the sum of the external forces in the theorems given.

Let us denote the sum of the external forces by $\mathbf{P}^{(e)}$. By (IV)

$$m\mathbf{p}_0 = \mathbf{P}^{(e)}. \tag{V}$$

If no external forces are acting on a system, then $\mathbf{p}_0 = 0$, and hence $\mathbf{v}_0 = \text{const}$. We can therefore say that *the internal forces cannot change the velocity of the centre of mass either as to magnitude or as to direction.*

Let us consider the solar system (i. e. the system composed of the sun and planets). The forces with which the fixed stars attract the bodies of the solar system can be neglected since these forces are very small because of the immense distances of the fixed stars from the solar system. We can therefore assume that only the internal forces with which the bodies attract each other according to Newton's law (p. 89) act on the bodies of the solar system. It follows from this that relative to the fixed stars the centre of mass of the solar system is at rest or moves with uniform motion along a straight line.

Suppose that we are inquiring into the motion of a system of points in a gravitational field. The sum of the weights is $m_1g + m_2g + \ldots = mg$. The centre S of mass will therefore move like a material point of mass m under the influence of the weight mg, i. e. along a straight line or a parabola until the moment when at least one of the points of the system touches the ground. For at this moment a new external force appears resulting from the collision of the point with the earth.

Examples. 1. The centre of mass of a projectile travels along a parabola even when the projectile explodes and bursts. The motion of the centre of mass will not be disturbed by this, since the explosion takes place under the influence of internal forces. Only when one of the fragments falls to the earth will the motion of the centre of mass undergo a change.

2. If a person is on a smooth horizontal plane (e. g. on ice) the external forces are the reaction of the plane and his weight; both forces are directed vertically. If the person was at rest initially, then as long as other external forces do not appear, the centre of mass will only be able to move vertically. The motions which a person executes by means of muscular action occur under the influence of internal forces, and hence cannot influence the motion of the centre of mass in the horizontal direction. Therefore, if there were no friction, walking would be impossible.

If at some moment a certain part of a system of points changes its momentum under the influence of internal forces, then the momentum of the remaining part experiences simultaneously a change equal in magnitude and direction, but opposite in sense. This is so because the internal forces cannot change the total momentum. Denoting the masses of the first and second parts by m' and m'', and the changes of the velocities of the centres of these masses by $\mathbf{v}_0'$ and $\mathbf{v}_0''$, we obtain $m'\mathbf{v}_0' + m''\mathbf{v}_0'' = 0$, whence $|\mathbf{v}_0'| / |\mathbf{v}_0''| = m'' / m'$.

Therefore: *the change of the velocities of the centres of mass is inversely proportional to the masses of both parts of the system.*

This explains the recoil of a cannon after it has been fired. Similarly, if a person starts to run along the deck of a boat, the boat begins to move in the opposite direction. The velocities of the boat and the person will be inversely proportional to their masses.

Example I. One end of a heavy rod AB (of constant density) rests on a smooth horizontal plane Π (Fig. 135). The external forces are the

weight and the reaction at the point A; both forces are vertical. Therefore, if the rod was at rest initially, then under the influence of the external forces, whose sum is directed vertically, the rod will move in such a way that the centre S of its mass will fall vertically.

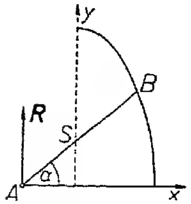

Fig. 135.

Let us choose the intersection of the vertical plane (passing through the rod) with the plane Π as the x-axis. As the y-axis let us take the vertical along which the centre of gravity moves. Put $AB = l$ and denote the angle which AB makes with the x-axis by α. Denoting the coordinates of the point B by x and y, we obtain: $x = \frac{1}{2}l \cos\alpha$, and $y = l \sin\alpha$, whence

$$(x / \tfrac{1}{2}l)^2 + (y / l)^2 = 1.$$

The end of the rod B will therefore move along an ellipse with axes l and $2l$.

Example 2. Let a system of points $A_1, A_2, \ldots$ of masses $m_1, m_2, \ldots$ move in a central field (p. 101) of elastic forces (p. 110) proportional to the masses. Let O be the centre of the field. Set $\overline{OA}_1 = \mathbf{r}_1$, $\overline{OA}_2 = \mathbf{r}_2$, etc. Denoting the forces acting on the points $A_1, A_2, \ldots$ by $\mathbf{P}_1, \mathbf{P}_2, \ldots$ and putting $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots$, we obtain $\mathbf{P}_1 = -\lambda^2 m_1\mathbf{r}_1$, $\mathbf{P}_2 = -\lambda^2 m_2\mathbf{r}_2$ etc., whence $\mathbf{P} = -\lambda^2(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots)$. Denoting the center of the total mass m by S and setting $\overline{OS} = \mathbf{r}_0$, we obtain by (I), p. 194,

$$\mathbf{P} = -\lambda^2 m\mathbf{r}_0.$$

Therefore the centre of mass will move just like a material point os mass m subjected to the action of an elastic force $\mathbf{P}$. The centre of masf will therefore move with plane harmonic motion along a straight line or an ellipse (p. 113).

§ 3. Moment of momentum. Angular momentum with respect to a point. Let a system of points $A_1, A_2, \ldots$ of masses $m_1, m_2, \ldots$ and total mass m be given. Let us consider a system of momenta i. e. of vectors $m_1\mathbf{v}_1, m_2\mathbf{v}_2, \ldots$ with initial points at $A_1, A_2, \ldots$

The total moment of a system of momenta with respect to an arbitrary point A is called the *angular momentum* or the *moment of momentum* of the system with respect to A.

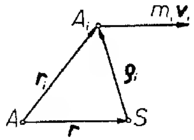

Fig. 136.

Therefore the angular momentum $\mathbf{K}$ with respect to A is

$$\mathbf{K} = \Sigma\mathrm{Mom}_A(m_i\mathbf{v}_i).$$

Setting $\mathbf{r}_i = \overline{AA}_i$, we obtain

$$\mathbf{K} = \Sigma(m_i\mathbf{v}_i \times \mathbf{r}_i). \tag{1}$$

If ξ, η, ζ are the coordinates of the point A and x_i, y_i, z_i the coordinates of the point A_i, then the projections of the angular momentum on the coordinate axes are

$$\begin{aligned} K_x &= \Sigma m_i[y_i^{\cdot}(z_i - \zeta) - z_i^{\cdot}(y_i - \eta)], \\ K_y &= \Sigma m_i[z_i^{\cdot}(x_i - \xi) - x_i^{\cdot}(z_i - \zeta)], \\ K_z &= \Sigma m_i[x_i^{\cdot}(y_i - \eta) - y_i^{\cdot}(x_i - \xi)]. \end{aligned} \tag{I}$$

In particular, when $\xi = \eta = \zeta = 0$, we have

$$\begin{aligned} K_x = \Sigma m_i(y_i^{\cdot}z_i - z_i^{\cdot}y_i),\ K_y = \Sigma m_i(z_i^{\cdot}x_i - x_i^{\cdot}z_i), \\ K_z = \Sigma m_i(x_i^{\cdot}y_i - y_i^{\cdot}x_i). \end{aligned} \tag{I'}$$

Let S be the centre of mass (Fig. 136). Put $\overline{AS} = \mathbf{r}_0$ and $\overline{SA}_i = \boldsymbol{\rho}_i$ for $i = 1, 2, \ldots$ We have $\mathbf{r}_i = \boldsymbol{\rho}_i + \mathbf{r}_0$. Therefore by (1) the angular momentum with respect to A is

$$\mathbf{K} = \Sigma m_i\mathbf{v}_i \times (\boldsymbol{\rho}_i + \mathbf{r}_0) = \Sigma m_i\mathbf{v}_i \times \boldsymbol{\rho}_i + (\Sigma m_i\mathbf{v}_i) \times \mathbf{r}_0.$$

The first term of the last member is the angular momentum of the centre of mass. This angular momentum we denote by $\mathbf{K}_0$. Since $\Sigma m_i\mathbf{v}_i = m\mathbf{v}_0$ (where $\mathbf{v}_0$ denotes the velocity of the centre of mass),

$$\mathbf{K} = \mathbf{K}_0 + m\mathbf{v}_0 \times \mathbf{r}_0. \tag{2}$$

Let us note that $m\mathbf{v}_0 \times \mathbf{r}_0$ is the moment with respect to A of the total momentum whose point of application is at the centre of mass.

Formula (2) follows directly from the theorem on p. 20 concerning the change of the total moment of a system of vectors.

Angular momentum in an advancing motion. Let us assume that a system of points moves with an advancing motion with a velocity $\mathbf{v}$, i. e. that all points move with a velocity $\mathbf{v}$.

The angular momentum with respect to an arbitrary point A is therefore according to (2) $\mathbf{K} = \Sigma(m_i\mathbf{v} \times \mathbf{r}_i) = \Sigma(\mathbf{v} \times m_i\mathbf{r}_i) = \mathbf{v} \times \Sigma m_i\mathbf{r}_i$. But by (I), p. 194, $\Sigma m_i\mathbf{r}_i = m\mathbf{r}_0$. Hence $\mathbf{K} = \mathbf{v} \times m\mathbf{r}_0$, or

$$\mathbf{K} = m\mathbf{v} \times \mathbf{r}_0. \tag{3}$$

Therefore: *the angular momentum of a system of points moving with an advancing motion relative to a certain point A is equal to the moment with respect to A of the total momentum whose point of application is at the centre of mass of the system.*

In particular, if the point A coincides with the centre of mass, then we have $\mathbf{r}_0 = 0$, and hence $\mathbf{K} = 0$. Hence: *the angular momentum with respect to the centre of mass in an advancing motion is equal to zero.*

It follows from this (p. 26) that a system of momenta in an advancing motion has a resultant whose origin is at the centre of mass of the system.

Angular momentum in a motion relative to the centre of mass. Let a system of coordinates $O(x, y, z)$ move with an advancing motion with a velocity $\mathbf{u}$. Denoting the relative velocities of the points $A_1, A_2, \ldots$ by $\mathbf{w}_1, \mathbf{w}_2, \ldots$, the relative velocity of the centre of their mass S by $\mathbf{w}_0$, the angular momentum of the relative motion by $\mathbf{K}_r$, and the angular momentum of the absolute motion of the system of these points with respect to O by $\mathbf{K}_a$, we obtain

$$\mathbf{K}_r = m_1\mathbf{w}_1 \times \overline{OA_1} + m_2\mathbf{w}_2 \times \overline{OA_2} + \ldots$$

Since $\mathbf{w}_1 = \mathbf{v}_1 - \mathbf{u}$, $\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{u}$, $\ldots$,

$$\mathbf{K}_r = (m_1\mathbf{v}_1 \times \overline{OA_1} + m_2\mathbf{v}_2 \times \overline{OA_2} + \ldots) - \mathbf{u} \times (m_1\overline{OA_1} + m_2\overline{OA_2} + \ldots).$$

The expression enclosed in the first parenthesis is equal to $\mathbf{K}_a$; the expression enclosed in the second parenthesis is $m \cdot \overline{OS}$ (p. 194). Therefore

$$\mathbf{K}_r = \mathbf{K}_a - m\mathbf{u} \times \overline{OS}. \tag{4}$$

In particular, if the origin O of the moving system of coordinates is chosen at the centre of mass S of the system of points $A_1, A_2, \ldots$, then $\overline{OS} = 0$, whence by (4) $\mathbf{K}_r = \mathbf{K}_a$.

Therefore, if we are investigating the motion of a system with respect to the centre of mass, then *the angular momenta with respect to the centre of mass in relative and absolute motion are equal.*

Angular momentum with respect to an axis. The total moment of momenta of a system of points with respect to a certain axis is called the *angular momentum of a system with respect to this axis.*

Formulae (I) represent the angular momenta of a system of points with respect to axes parallel to the x, y, z axes and passing through the point A, while formulae (I′) represent the angular momenta with respect to the x, y and z axes. Let us consider the angular momentum with respect to the y-axis

$$K_y = \Sigma m_i(z_i^{\cdot} x_i - x_i^{\cdot} z_i).$$

Denoting by S_i the areal velocity (p. 47) of the motion which the projection of the point A_i executes in the vertical xz-plane, we obtain from formula (II), p. 48, $S_i = \frac{1}{2}(z_i^{\cdot} x_i - x_i^{\cdot} z_i)$; therefore

$$K_y = 2\,\Sigma m_i S_i.$$

Hence: *the angular momentum with respect to a certain axis is equal to twice the sum of the products of the masses and the areal velocities of the motions which the projections of the points execute on a plane perpendicular to this axis:*

$$K = 2\,\Sigma m_i S_i. \tag{5}$$

If we introduce the polar coordinates r, φ in the plane perpendicular to the axis, then by (I), p. 47, we obtain $S_i = \frac{1}{2} r_i^2 \varphi_i^{\cdot}$, and hence

$$K = \Sigma m_i r_i^2 \varphi_i^{\cdot}. \tag{6}$$

Let a system of points rotate about a certain axis with an angular velocity ω. We then have $\varphi_1^{\cdot} = \varphi_2^{\cdot} = \ldots = \omega$. Therefore $K = \Sigma m_i r_i^2 \omega = \omega\, \Sigma m_i r_i^2$. Since $\Sigma m_i r_i^2 = I$, where I is the moment of inertia with respect to the axis of rotation,

$$K = I\omega. \tag{7}$$

Therefore: *if a system of points rotates about a certain axis, then the angular momentum with respect to the axis of rotation is equal to the product of the angular velocity and the moment of inertia with respect to the axis of rotation.*

Dynamic properties of angular momentum. Let A be an arbitrary point which is either fixed or in motion (Fig. 137). Denote the vectors $\overline{AA_i}$ by $\mathbf{r}_i$, the vectors $\overline{OA_i}$ by $\boldsymbol{\rho}_i$ (where O is the origin of the inertial system of coordinates), the vector $\overline{OA}$ by $\boldsymbol{\rho}$, and the velocity of the point A by $\mathbf{u}$. Therefore:

$$\boldsymbol{\rho}^{\cdot} = \mathbf{u} \quad \text{and} \quad \boldsymbol{\rho}_i^{\cdot} = \mathbf{v}_i \qquad (i = 1, 2, \ldots). \tag{8}$$

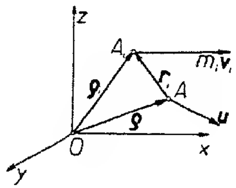

Fig. 137.

Let $\mathbf{K}$ be the angular momentum with respect to A. By (1), p. 199, $\mathbf{K} = \Sigma(m_i\mathbf{v}_i \times \mathbf{r}_i)$. Denoting the accelerations of the points A_i by $\mathbf{p}_i$, we obtain after differentiating with respect to t,

$$\mathbf{K}^{\cdot} = \Sigma(m_i\mathbf{p}_i \times \mathbf{r}_i) + \Sigma(m_i\mathbf{v}_i + \mathbf{r}_i^{\cdot}). \tag{9}$$

But $\mathbf{r}_i = \boldsymbol{\rho}_i - \boldsymbol{\rho}$; hence $\mathbf{r}_i^{\cdot} = \boldsymbol{\rho}_i^{\cdot} - \boldsymbol{\rho}^{\cdot}$. Therefore by (8) $\mathbf{r}_i^{\cdot} = \mathbf{v}_i - \mathbf{u}$, and $\Sigma(m_i\mathbf{v}_i \times \mathbf{r}_i^{\cdot}) = \Sigma(m_i\mathbf{v}_i \times \mathbf{v}_i) - \Sigma(m_i\mathbf{v}_i \times \mathbf{u})$, But $\mathbf{v}_i \times \mathbf{v}_i = 0$, and $\Sigma m_i\mathbf{v}_i = m\mathbf{v}_0$, where $\mathbf{v}_0$ denotes the velocity of the centre of mass. Therefore

$$\Sigma(m_i\mathbf{v}_i \times \mathbf{r}_i^{\cdot}) = -m\mathbf{v}_0 \times \mathbf{u}. \tag{10}$$

If the force $\mathbf{P}_i$ acts on the point A_i, then $m_i\mathbf{p}_i = \mathbf{P}_i$, and therefore $m_i\mathbf{p}_i \times \mathbf{r}_i = \mathbf{P}_i \times \mathbf{r}_i = \mathrm{Mom}_A\mathbf{P}_i$. Hence, if $\mathbf{M}$ is the total moment of the

forces with respect to A, then $\Sigma m_i \boldsymbol{p}_i \times \boldsymbol{r}_i = \Sigma \boldsymbol{P}_i \times \boldsymbol{r}_i = \mathrm{Mom}_A \boldsymbol{P}_i = \boldsymbol{M}$. This formula together with formula (10) gives by (9)

$$\boldsymbol{K}^{\cdot} = \boldsymbol{M} - m\boldsymbol{v}_0 \times \boldsymbol{u}. \tag{II}$$

The expression $\boldsymbol{v}_0 \times \boldsymbol{u}$ will be zero if we assume that the point A is at rest (hence that $\boldsymbol{u} = 0$), or that A coincides with the centre of mass (hence that $\boldsymbol{u} = \boldsymbol{v}_0$). In both instances we obtain

$$\boldsymbol{K}^{\cdot} = \boldsymbol{M}. \tag{III}$$

Therefore: *the derivative of the angular momentum (with respect to a fixed point or a centre of mass) is equal to the total moment of the acting forces.*

Forming the projections on a fixed axis or on one passing through the centre of mass and not changing its direction, we conclude from formula (III) that *the derivative of the angular momentum with respect to a fixed axis (or to one passing through the centre of mass and not changing its direction) is equal to the total moment of the forces with respect to this axis.*

In particular, if the total moment of the forces with respect to a certain fixed point or with respect to the centre of mass is constantly zero, then the derivative of the angular momentum is zero, i. e. the angular momentum is a constant vector.

Therefore: *if the total moment of the forces (with respect to a certain fixed point or the centre of mass) is constantly zero, then the angular momentum is a constant vector.*

The preceding theorem is known as *the principle of conservation of angular momentum.*

A similar theorem is obtained for the angular momentum with respect to an axis.

Therefore: *if the moment of the forces with respect to a certain fixed axis (or one passing through the centre of mass and not changing its direction) is constantly zero, then the angular momentum with respect to this axis is constant.*

The angular momentum by formula (5), p. 201, is $K = 2\Sigma m_i S_i$, where S_i denote the areal velocities of the motions executed by the projections on the plane Π perpendicular to the axis. Therefore, if the angular momentum is constant, then

$$\Sigma m_i S_i = c = \text{const.} \tag{11}$$

Let us note that

$$a_i = \int_{t_0}^{t} S_i \, \mathrm{d}t$$

represents the area swept out in the plane Π by the projection on it of the radius vector r_i of the point A_i from the time t_0 to t. Therefore by (11)

$$\Sigma m_i a_i = c(t - t_0). \tag{12}$$

Therefore: *the sum of the products of the masses and areas swept out by the projections of the radius vectors on the plane perpendicular to the axis is proportional to the time.*

Because of this the principle of conservation of angular momentum is also known as *the principle of conservation of areas.*

As we know (p. 187), the moment of the internal forces with respect to an arbitrary point is zero. Hence the moment of all the acting forces is reduced to the moment of the external forces. Therefore, if the moment of the external forces with respect to a certain fixed point A or the centre of mass is denoted by $\boldsymbol{M}^{(e)}$, then the equality (III) will assume the form

$$\boldsymbol{K}^{\cdot} = \boldsymbol{M}^{(e)}. \tag{III'}$$

In the theorems given previously we can therefore replace the moment of all the acting forces by the moment of the external forces.

If no external forces act on a system of points, then the principle of conservation of areas (of angular momentum) obviously holds, and hence the angular momentum with respect to each fixed point or centre of mass is then a constant vector. Since the angular momentum with respect to each fixed axis (or one passing through the centre of mass and not changing its direction) is then constant, equations (11) and (12) hold for motions which are executed by the projections of the points on an arbitrary fixed plane (or on one moving together with the centre of mass and not changing its direction).

Motion in a gravitational field. Let a system of material points $A_1, A_2, \ldots$ of masses $m_1, m_2, \ldots$ move in a gravitational field. If the only external forces are the weights of the points, then the total moment of the weights with respect to the centre of mass is zero (p. 195) and hence the angular momentum with respect to the centre of mass is constant.

Let us assume that a system of coordinates with its origin at the centre of gravity is moving with an advancing motion. In order to obtain the relative motion it is necessary to add to the acting forces the forces of transport (the force of Coriolis is zero because, by hypothesis, the system of coordinates is moving with an advancing motion).

Denote the gravitational acceleration vector by $\boldsymbol{g}$. Since the centre of gravity has an acceleration $\boldsymbol{g}$, the acceleration of transport is also $\boldsymbol{g}$. Therefore the force of transport for the individual points is $-m_1\boldsymbol{g}$,

$-m_2\mathbf{g}, \ldots$, respectively. We see then that the forces of transport are balanced by the weights of the points. Therefore the relative motion will be such as if the force of gravity were not acting. If there are no external forces besides the weights, then the angular momentum with respect to the centre of mass in relative motion will be a constant vector, and by (4), p. 200, will be equal to the angular momentum with respect to the centre of mass in absolute motion.

Example 1. Two material points A and B of masses m_1 and m_2 joined by a rigid massless rod are moving in a gravitational field. Therefore the forces $\boldsymbol{R}$ and $-\boldsymbol{R}$ of weight and reactions of the rod act on the points A and B. The reactions behave just like internal forces because they act along the line joining these points, are equal in magnitude, and have opposite senses. The centre of mass will therefore move (like a material point of mass $m_1 + m_2$ under the influence of gravity) with a vertical acceleration g along a straight line or along a parabola.

Let us denote the velocities of the points A and B by $\mathbf{v}_1$ and $\mathbf{v}_2$, and the centre of mass by S. The angular momentum with respect to the centre of mass is

$$\boldsymbol{K} = m_1\mathbf{v}_1 \times \overline{SA} + m_2\mathbf{v}_2 \times \overline{SB}. \tag{12}$$

Since

$$SA = \frac{m_2}{m_1 + m_2} AB, \qquad SB = \frac{m_1}{m_1 + m_2} AB,$$

(p. 156),

$$\overline{SA} = -\frac{m_2}{m_1 + m_2}\overline{AB}, \qquad \overline{SB} = \frac{m_1}{m_1 + m_2}\overline{AB},$$

from which we obtain after substituting in (12)

$$\boldsymbol{K} = \frac{m_1 m_2}{m_1 + m_2}(\mathbf{v}_2 - \mathbf{v}_1) \times \overline{AB}. \tag{13}$$

Let us assume that $\boldsymbol{K} \neq 0$; therefore $\boldsymbol{K} \perp \overline{AB}$. Since $\boldsymbol{K}$ is a constant vector, the segment AB (Fig. 138) is always parallel during motion to a certain plane Π perpendicular to the angular momentum $\boldsymbol{K}$.

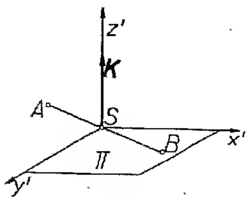

Fig. 138.

Let us choose the centre of mass as the origin of the coordinate system (x', y', z') moving with an advancing motion. Assume that the $x'y'$-plane is always parallel to Π. Therefore the z'-axis is parallel to $\boldsymbol{K}$. The rod AB therefore always remains in the $x'y'$-plane. It follows from this that the relative motion of

the rod AB will be a rotation about the z'-axis (because the point S of the rod is motionless relative to the frame (x', y', z')). Since the angular momentum in relative motion is constant, the angular momentum with respect to the z'-axis will be, by formula (7), p. 201,

$$K_{z'} = I_{z'}\omega = \text{const}, \tag{14}$$

where the moment of inertia

$$I_{z'} = m_1 AS^2 + m_2 BS^2 = \frac{m_1 m_2}{m_1 + m_2} AB^2,$$

and ω denotes the angular velocity. Therefore $\omega = \text{const}$.

Hence the relative motion will be a rotation in the $x'y'$-plane about the centre of mass S with a constant angular velocity.

Rotation of a system about an axis. Let us assume that no external forces act on a system of material points U. Suppose that the system was at rest initially, and then some part of the system U_1 began to rotate about a certain fixed axis l under the influence of internal forces. Let us denote the moment of inertia of this part of the system with respect to l by I_1, and the angular velocity by ω_1. Then its angular momentum with respect to the axis of rotation is $K_1 = I_1\omega_1$.

Since the total angular momentum of a system must be zero, because the internal forces cannot change the angular momentum, the remaining part of the system U_2 must execute a motion whose angular momentum with respect to the l-axis is $K_2 = -K_1$, such that the sum of both angular momenta is zero (i. e. so that $K_1 + K_2 = 0$). Suppose that the motion of the other part is also a rotation about the l-axis (this case occurs if we assume e. g., that both parts can only rotate about l). If we denote the moment of inertia of the part U_2 with respect to l by I_2 and its angular velocity by ω_2, then $K_2 = I_2\omega_2$.

Since $K_1 + K_2 = 0$,

$$I_1\omega_1 + I_2\omega_2 = 0. \tag{15}$$

The preceding equation expresses the relation between the angular velocities of both parts of the system U. Since

$$\omega_1 / \omega_2 = -I_2 / I_1, \tag{16}$$

both parts of the system rotate in opposite directions and their angular velocities are in magnitude inversely proportional to the moments of inertia.

Denote by φ_1 and φ_2 the angles through which both parts U_1 and U_2 of the system U have turned in the time t. Since $\dot\varphi_1 = \omega_1$, and $\dot\varphi_2 = \omega_2$,

it follows by (13), that $I_1\dot{\varphi}_1 + I_2\dot{\varphi}_2 = 0$, whence $I_1\varphi_1 + I_2\varphi_2 = c$. Assuming that $\varphi_1 = 0$ and $\varphi_2 = 0$ at the moment $t = 0$, we obtain $I_1\varphi_1 + I_2\varphi_2 = 0$, i. e.

$$\varphi_1 / \varphi_2 = -I_2 / I_1. \tag{17}$$

The angles of rotation are therefore in magnitude also inversely proportional to the moments of inertia of the two parts of the system.

If at a certain moment $\varphi_1 - \varphi_2 = 2k\pi$, where k is an arbitrary integer, then the position of the system is such as if it had turned through an angle φ_1.

We see then that the action of internal forces is sufficient for turning a system of points about an axis through an arbitrary angle φ. Such a rotation can occur for instance in the following manner: one part of the system turns through an angle φ, and the other part through an angle $2\pi - \varphi$ in the opposite direction.

This explains the fact that when a cat falls it can turn in the air in such a way as to fall on all fours.

If we start to turn a material wheel about a vertical axis with an angular velocity ω_1 on the deck of a boat, then the boat begins to turn in the opposite direction with an angular velocity ω_2; both velocities will satisfy relations (15) and (16), where I_1 and I_2 denote the moments of inertia of the wheel and boat respectively.

Example 2. A piece of paper rests on a smooth horizontal plane; the paper is pierced by a pin at the point O so that it can only turn about this point. An insect A crawls over the paper (Fig. 139).

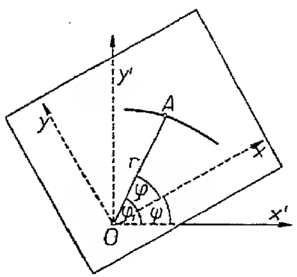

Fig. 139.

The external forces acting on the system consisting of the paper and insect are: the reaction of the pin with its origin at the point O, the weight of the paper and that of the insect as well as the reaction of the horizontal plane; these orces have a vertical direction. The moment of these forces with respect to the vertical z-axis passing through O is therefore equal to zero, and because of this the angular momentum with respect to the z-axis is constant.

Let us assume that the insect and the piece of paper were at rest at $t = 0$. The angular momentum with respect to the z-axis will therefore be constantly zero.

Select a fixed coordinate system (x', y') in the horizontal plane and

a moving system (x, y) on the piece of paper, both having a common origin at O. Denote the angle between OA and x' by φ_1, the angle between OA and x by φ, and the angle through which the paper has turned by ψ, i. e. the angle between x and x'. Finally, let m denote the mass of the insect, I the moment of inertia of the paper with respect to O and let $r = OA$. Then the angular momentum with respect to the z-axis is $mr^2\dot{\varphi}_1 + I\dot{\psi} = 0$, and since $\varphi_1 = \varphi + \psi$,

$$mr^2\dot{\varphi} + (mr^2 + I)\,\dot{\psi} = 0. \tag{18}$$

Let us suppose that the insect crawls along a curve whose equation is $r = f(\varphi)$. By (18), and from the fact that $\dot{\psi} / \dot{\varphi} = \mathrm{d}\psi / \mathrm{d}\varphi$, we obtain $\mathrm{d}\psi / \mathrm{d}\varphi = -mr^2 / (mr^2 + I)$. Therefore, integrating from φ_0 to φ, we get

$$\psi - \psi_0 = -\int_{\varphi_0}^{\varphi} \frac{mr^2}{mr^2 + I}\,\mathrm{d}\varphi. \tag{19}$$

The difference $\psi - \psi_0$ represents the angle through which the paper has turned while the insect crawled along the curve $r = f(\varphi)$ from φ_0 to φ.

We see that the angle of rotation does not depend on the velocity of the insect, but only on the curve along which it crawls. In particular, if the insect crawls along the circle $r =$ const, then by (19)

$$\psi - \psi_0 = -\frac{mr^2}{mr^2 + I}(\varphi - \varphi_0). \tag{20}$$

Angular momentum in relative motion. Let the coordinate system $O'(x', y', z')$ move relative to the inertial system of coordinates $O(x, y, z)$. In order to determine the relative motion of the system of material points it is necessary to add the forces of transport and Coriolis to the acting forces. Denote by $\boldsymbol{K}_r$ the angular momentum of the relative motion with respect to the origin O', and by $\boldsymbol{M}$, $\boldsymbol{M}_t$ and $\boldsymbol{M}_C$ the moments of the acting forces, the force of transport, and the force of Coriolis, with respect to the point O'. Since Newton's laws apply to relative motion if we add the forces of transport and Coriolis to the acting forces (p. 135),

$$\dot{\boldsymbol{K}}_r = \boldsymbol{M} + \boldsymbol{M}_t + \boldsymbol{M}_C. \tag{21}$$

This formula becomes simpler in the case when O' coincides with the centre of mass of a system of material points, and the system of coordinates (x, y, z) moves with an advancing motion. This is so because as we have proved (p. 200), in this case $\boldsymbol{K}_r = \boldsymbol{K}$, where $\boldsymbol{K}$ denotes the angular momentum (with respect to the centre of mass) in absolute motion. Therefore $\dot{\boldsymbol{K}}_r = \dot{\boldsymbol{K}}$, and since $\dot{\boldsymbol{K}} = \boldsymbol{M}$, we obtain

$$\dot{\boldsymbol{K}}_r = \boldsymbol{M}. \tag{22}$$

Hence, if we are investigating motion relative to the centre of mass, then *the derivative of the angular momentum* (with respect to the centre of mass) *in relative motion is equal to the moment* (with respect to the centre of mass) *of the acting forces.*

§ 4. Work and potential of a system of points. Work. Let forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ act on the points of a system. The work of the force $\boldsymbol{P}_i$ is expressed (p. 94) by the formula

$$L_i = \int_{C_i} (P_{i_x}\,dx_i + P_{i_y}\,dy_i + P_{i_z}\,dz_i),$$

where C_i denotes the path of the i-th point.

The total work (or briefly the *work*) *of the forces* $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ is defined as the sum of the works of the separate forces.

Therefore the total work is

$$L = \Sigma \int_{C_i} (P_{i_x}\,dx_i + P_{i_y}\,dy_i + P_{i_z}\,dz_i). \tag{I}$$

The total work done by the forces from the time t_0 to t can be represented in the form (IV), p. 95,

$$L = \int_{t_0}^{t} \Sigma (P_{i_x}\dot{x}_i + P_{i_y}\dot{y}_i + P_{i_z}\dot{z}_i)\,dt \tag{II}$$

or ((V), p. 95)

$$L = \int_{t_0}^{t} \Sigma (\boldsymbol{P}_i \boldsymbol{v}_i)\,dt, \tag{II$'$}$$

where $\boldsymbol{v}_i$ denote the velocities of the points, and $\boldsymbol{P}_i\boldsymbol{v}_i$ is a scalar product.

Work equal to zero. The cases in which the work of the forces acting on a system is zero are very important. We shall give several examples.

Example 1. For a rigid system of material points (p. 190) the following theorem holds:

The work of the internal forces in a rigid system is zero.

Proof. Let us first consider two points of the rigid system A_1 and A_2 (Fig. 140). Put:

$$\boldsymbol{r}_1 = \overline{OA_1}, \quad \boldsymbol{r}_2 = \overline{OA_2}, \quad \boldsymbol{r} = \overline{A_1A_2}, \tag{1}$$

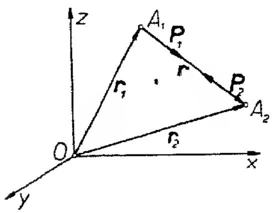

Fig. 140.

where O is the origin of the coordinate system. If $\boldsymbol{v}_1, \boldsymbol{v}_2$ are the velocities of the points A_1, A_2, then

$$\boldsymbol{v}_1 = \dot{\boldsymbol{r}}_1, \quad \boldsymbol{v}_2 = \dot{\boldsymbol{r}}_2. \tag{2}$$

By (1) we have $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$, whence $\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_2 - \dot{\boldsymbol{r}}_1$, and therefore

$$\dot{\boldsymbol{r}} = \boldsymbol{v}_2 - \boldsymbol{v}_1. \tag{3}$$

Since $A_1A_2 = \text{const}$, $\boldsymbol{r}^2 = \text{const}$. Differentiating with respect to the time t, we obtain $2\boldsymbol{r}\dot{\boldsymbol{r}} = 0$. Hence by (3)

$$\boldsymbol{r}(\boldsymbol{v}_2 - \boldsymbol{v}_1) = 0. \tag{4}$$

Denote the force which the point A_2 (or A_1) exerts on the point A_1 (or A_2) by $\boldsymbol{P}_1$ (or $\boldsymbol{P}_2$). In virtue of the law of action and reaction

$$\boldsymbol{P}_1 = -\boldsymbol{P}_2. \tag{5}$$

Since the forces $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ have the direction of the vector $\overline{A_1A_2} = \boldsymbol{r}$, we can assume that

$$\boldsymbol{P}_1 = \lambda\boldsymbol{r} \quad \text{and} \quad \boldsymbol{P}_2 = -\lambda\boldsymbol{r}, \tag{6}$$

where λ is a factor of proportionality depending on time (because the magnitude of the forces $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ can change in the course of time). By (II$'$) the work of the forces $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ is

$$L = \int_{t_0}^{t_1} (\boldsymbol{P}_1\boldsymbol{v}_1 + \boldsymbol{P}_2\boldsymbol{v}_2)\,dt. \tag{7}$$

From equations (6) we get $\boldsymbol{P}_1\boldsymbol{v}_1 + \boldsymbol{P}_2\boldsymbol{v}_2 = \lambda(\boldsymbol{r}\boldsymbol{v}_1 - \boldsymbol{r}\boldsymbol{v}_2) = \lambda\boldsymbol{r}(\boldsymbol{v}_1 - \boldsymbol{v}_2)$, and hence by (4) $\boldsymbol{P}_1\boldsymbol{v}_1 + \boldsymbol{P}_2\boldsymbol{v}_2 = 0$. It follows from this and (7) that

$$L = 0.$$

We have therefore proved that the work of the internal forces with which any two points of a rigid system react on each other is zero. The sum of the works of all the internal forces is hence also zero, q. e. d.

Example 2. Two material points A_1 and A_2 are connected by an inextensible (massless) string passing through a fixed point O (Fig. 141). Let us assume that there is no friction. We shall prove that the work of the forces exerted by the string on the points of the system is equal to zero.

Choose the point O as the origin of the system of coordinates. Let x_1, y_1, z_1 be the coordinates of the point A_1, and x_2, y_2, z_2 the coordinates of the point A_2. Put

$$r_1 = OA_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}, \quad \text{and} \quad r_2 = OA_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}. \tag{8}$$

Since the length of the string $l = \text{const}$, and $r_2 + r_1 = l$,

$$\dot{r}_1 + \dot{r}_2 = 0. \tag{9}$$

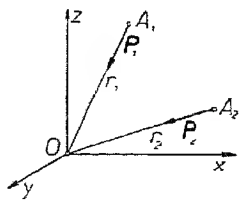

Fig. 141.

Denote the forces which the string exerts on the points A_1 and A_2 by $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. We have (p. 190)

$$|\boldsymbol{P}_1| = |\boldsymbol{P}_2|. \tag{10}$$

Putting $P = -|\boldsymbol{P}_1| = -|\boldsymbol{P}_2|$, we obtain:

$$P_{1_x} = P\frac{x_1}{r_1}, \quad P_{1_y} = P\frac{y_1}{r_1}, \quad \text{and} \quad P_{1_z} = P\frac{z_1}{r_1};$$

hence

$$P_{1_x}\dot{x}_1 + P_{1_y}\dot{y}_1 + P_{1_z}\dot{z}_1 = P(x_1\dot{x}_1 + y_1\dot{y}_1 + z_1\dot{z}_1)\,/\,r_1 = P\dot{r}_1,$$

and analogously

$$P_{2_x}\dot{x}_2 + P_{2_y}\dot{y}_2 + P_{2_z}\dot{z}_2 = P\dot{r}_2.$$

By (II), p. 208, the work of the forces $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ is then

$$L = \int_{t_0}^{t}[P\dot{r}_1 + P\dot{r}_2]\,dt = \int_{t_0}^{t}P[\dot{r}_1 + \dot{r}_2]\,dt.$$

From (9) we therefore get $L = 0$.

Example 3. Let us suppose that some body K moves in such a way that it constantly remains tangent to a certain surface Σ. Assuming that the forces of reaction of the surface have their points of application at the points of tangency, we see that the reactions change their points of application if the body comes in contact with the surface Σ at different points each time. Let us suppose that the work of the forces of reaction is in this case expressed by formula (II′), p. 208, where $\boldsymbol{v}_i$ denote the velocities of the points of tangency at which the reactions $\boldsymbol{P}_i$ have their points of application at a given time.

If the velocities of the points of tangency are always equal to zero, then we say that the body K *rolls* on the surface Σ.

Therefore: *when a body rolls the work of the forces of reaction is zero.*

Potential of a system. If the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ depend only on the position of the system of points, then the forces are said to form a *force field*.

Since the position of a system is defined by the coordinates $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ of its points, the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ are functions of the variables $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ Consequently

$$P_{i_x} = F_i(x_1, y_1, z_1, \ldots), \quad P_{i_y} = \Phi_i(x_1, y_1, z_1, \ldots), \quad P_{i_z} = \Psi_i(x_1, y_1, z_1, \ldots).$$

If the total work in a force field does not depend on the path described by a system of points, but only on the initial and final positions of these points, then the force field is called a *conservative* or *potential field.*

Let us consider an arbitrary position S_0 of the system of points defined by the coordinates $x_1^0, y_1^0, z_1^0, x_2^0, y_2^0, z_2^0, \ldots$ Denote by V the work done by the forces in displacing the system from the position S_0 to the position S whose coordinates are $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ If the field is a potential field, then the work V will not depend on the path described. Therefore V will be a function of the variables $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ The function V is known as the *potential* or the *force function.*

The following formulae can be proved by the same reasoning as in the case of one material point (p. 99):

$$P_{i_x} = \frac{\partial V}{\partial x_i}, \quad P_{i_y} = \frac{\partial V}{\partial y_i}, \quad P_{i_z} = \frac{\partial V}{\partial z_i} \qquad (i = 1, 2, \ldots). \quad \text{(III)}$$

Conversely, if there exists a function V satisfying equations (III), then the field is a potential field and V is a potential.

If a system is displaced from a position whose potential is V_1 to a position whose potential is V_2, then the work is

$$L = V_2 - V_1.$$

If each one of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ forms a potential field whose potentials are $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots$, then the field is a potential field whose potential is

$$V = V_1 + V_2 + \ldots \qquad (11)$$

Potential of the force of gravity. Let a system of points of masses $m_1, m_2, \ldots$ move in a gravitational field. If we assume that the z-axis is directed vertically upwards, then (p. 100)

$$V_1 = -m_1gz_1, \quad V_2 = -m_2gz_2, \quad \ldots$$

Therefore by (11) the field will be a potential field whose potential is

$$V = -(m_1z_1 + m_2z_2 + \ldots)\,g. \qquad (12)$$

Denoting the coordinate of the center of mass of the system by z_0, we shall have $m_1z_1 + m_2z_2 + \ldots = mz_0$, where m denotes the total mass of the system (p. 152). Therefore according to (12)

$$V = -mgz_0. \qquad (13)$$

Let us note that mg is the weight of the entire system (i. e. the sum of the weights of the separate points).

If in one position of the system the centre of gravity has coordinates $z_0^{(1)}$ and in the other $z_0^{(2)}$, then the work done by the force of gravity is $L = (-mgz_0^{(2)}) - (-mgz_0^{(1)})$, whence

$$L = mg(z_0^{(1)} - z_0^{(2)}). \qquad (14)$$

Hence: *the work in a gravitational field depends only on the difference of the levels of the centre of gravity of the system, and does not depend on the paths of its individual points.*

The work of the weights is therefore equal to the work that would be done by the total weight of the system whose point of application would be at the centre of its mass.

Potential of the internal forces. Let a system of points $A_1, A_2, \ldots, A_n$ of masses $m_1, m_2, \ldots, m_n$ be given. Assume that the internal forces with which two arbitrary points of the system react on each other depend in magnitude only on the distances between these points, i. e. if $\mathbf{P}^{ij}$ denotes the force with which the point $A_j(x_j, y_j, z_j)$ reacts on the point $A_i(x_i, y_i, z_i)$, and r_{ij} the distance between the points A_i and A_j (Fig. 142), then $|\mathbf{P}^{ij}|$ is a function of the distance r_{ij}, i. e.

$$|\mathbf{P}^{ij}| = f_{ij}(r_{ij}), \tag{15}$$

where

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}. \tag{16}$$

Let P_{ij} denote the projection of the force $\mathbf{P}^{ij}$ on the direction of $\overline{A_jA_i}$. Then:

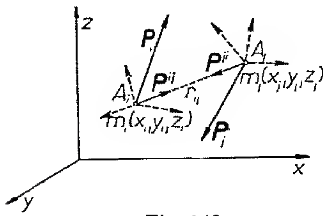

Fig. 142.

1° $|P_{ij}| = |\mathbf{P}^{ij}|$.

2° $P_{ij} \leqq 0$, if the points A_i and A_j attract each other, and $P_{ij} \geqq 0$ if the points A_i and A_j repel each other.

Since by the law of action and reaction $\mathbf{P}^{ij} = -\mathbf{P}^{ji}$, it follows that $P_{ij} = P_{ji}$, whence by (15) and condition 1°

$$P_{ij} = \pm f_{ij}(r_{ij}) \quad \text{or} \quad P_{ij} = F_{ij}(r_{ij}). \tag{17}$$

From the definition of the number P_{ij} it follows that the projections of the force $\mathbf{P}^{ij}$ on the axes of the system (x, y, z) can be written in the form:

$$P_x^{ij} = P_{ij}\frac{x_i - x_j}{r_{ij}}, \quad P_y^{ij} = P_{ij}\frac{y_i - y_j}{r_{ij}}, \quad P_z^{ij} = P_{ij}\frac{z_i - z_j}{r_{ij}}. \tag{18}$$

Let us put

$$V_{ij} = \int P_{ij}\, dr_{ij} = \int F_{ij}(r_{ij})\, dr_{ij}. \tag{19}$$

Since $P_{ij} = P_{ji}$ and $r_{ij} = r_{ji}$,

$$V_{ij} = V_{ji}. \tag{20}$$

We have

$$\frac{\partial V_{ij}}{\partial x_i} = \frac{dV_{ij}}{dr_{ij}} \cdot \frac{\partial r_{ij}}{\partial x_i} = P_{ij}\frac{x_i - x_j}{r_{ij}},$$

and therefore by (18):

$$\partial V_{ij} / \partial x_i = P_x^{ij}, \quad \partial V_{ij} / \partial y_i = P_y^{ij}, \quad \partial V_{ij} / \partial z_i = P_z^{ij}. \tag{21}$$

Let us set

$$V = \tfrac{1}{2}\Sigma V_{ij} = \tfrac{1}{2}\Sigma \int P_{ij}\, dr_{ij}, \tag{22}$$

where the summation is extended over all number pairs i, j, such that $i \neq j$, $i \leqq n$ and $j \leqq n$.

Let $\mathbf{P}_i$ denote the sum of all the internal forces acting on the point A_i. Hence

$$\mathbf{P}_i = \sum_{j=1}^{n} \mathbf{P}^{ij} \quad \text{for} \quad j \neq i. \tag{23}$$

Let us calculate the partial derivative $\partial V / \partial x_i$. The variable x_i appears only in the functions V_{ij} and V_{ji}, where $j \neq i$. In virtue of this and (22)

$$\frac{\partial V}{\partial x_i} = \tfrac{1}{2}\sum_{j=1}^{n}\left[\frac{\partial V_{ij}}{\partial x_i} + \frac{\partial V_{ji}}{\partial x_i}\right] \qquad \text{for } i \neq j.$$

From equations (20) and (21) it therefore follows that

$$\frac{\partial V}{\partial x_i} = \tfrac{1}{2}\sum_{j=1}^{n} 2P_x^{ij} = \sum_{j=1}^{n} P_x^{ij} \qquad \text{for } i \neq j,$$

whence by (23):

$$\frac{\partial V}{\partial x_i} = P_{i_x}, \quad \frac{\partial V}{\partial y_i} = P_{i_y}, \quad \frac{\partial V}{\partial z_i} = P_{i_z}.$$

Hence V is a potential. We have therefore proved the following theorem:

If the internal forces with which the points of a system react on each other depend only on the distances of these points, then the internal forces form a potential field whose potential is given by formula (22).

Example 4. Let the points of a system attract each other according to Newton's law. Then $P_{ij} = -Km_im_j / r_{ij}^2$ for $i \neq j$, and hence according to (19) $V_{ij} = \int P_{ij}\, dr_{ij} = Km_im_j / r_{ij}$, whence by (22)

$$V = \tfrac{1}{2}K\, \Sigma m_im_j / r_{ij}, \tag{24}$$

where the summation extends over every number pair i, j such that $i \neq j$, $i \leqq n$ and $j \leqq n$.

In particular, for two points we have

$$V = Km_1m_2 / r_{12}. \tag{25}$$

Example 5. Let the points of a system attract each other with forces proportional to the distances. Then $P_{ij} = -\lambda_{ij}^2 r_{ij}$ for $i \neq j$, where λ_{ij} is a factor of proportionality depending on the pair of points m_i, m_j. Therefore according to (19) $V_{ij} = \int P_{ij}\, dr_{ij} = -\frac{1}{2}\lambda_{ij}^2 r_{ij}^2$, from which by (22)

$$V = -\tfrac{1}{4}\Sigma\lambda_{ij}^2 r_{ij}^2. \tag{26}$$

§ 5. Kinetic energy of a system of points. Let there be given a system of points $m_1, m_2, \ldots$, having velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots$ at a certain moment t.

The *kinetic energy of a system* of points at the time t is defined as the sum of the kinetic energies of the separate points.

If we set $v_1 = |\mathbf{v}_1|$, $v_2 = |\mathbf{v}_2|, \ldots$, then the kinetic energy of the system will be

$$E = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 + \ldots = \tfrac{1}{2}\Sigma m_i v_i^2. \tag{I}$$

Kinetic energy of a system in an advancing motion. If a system moves with an advancing motion with a velocity $\mathbf{v}$ (i. e. if each of its points has this same velocity $\mathbf{v}$), then, putting $v = |\mathbf{v}|$, we have $E = \frac{1}{2}\Sigma m_i v^2 = \frac{1}{2}v^2\Sigma m_i$, or

$$E = \tfrac{1}{2}mv^2, \tag{II}$$

where m denotes the total mass of the system.

Kinetic energy in a rotating motion about an axis. Let a system of points rotate about an axis l with an angular velocity ω. Denoting the distances of the points of the system from the axis of rotation by $r_1, r_2, \ldots$, we have $v_i = r_i\omega$, and therefore $E = \frac{1}{2}\Sigma m_i v_i^2 = \frac{1}{2}\Sigma m_i r_i^2\omega^2 = \frac{1}{2}\omega^2\Sigma m_i r_i^2$. Since $\Sigma m_i r_i^2$ is the moment of inertia of the system with respect to the axis of rotation, setting $\Sigma m_i r_i^2 = I$, we obtain

$$E = \tfrac{1}{2}I\omega^2. \tag{III}$$

Theorem of König. Let an arbitrary system of coordinates with origin at O move with an advancing motion relative to a frame of reference. Denote the velocity of the point O by $\mathbf{u}$, the absolute velocities by $\mathbf{v}_i$, and the relative velocities of the material points m_i $(i = 1, 2, \ldots)$ by $\mathbf{w}_i$. Since $\mathbf{u}$ is the velocity of transport, it follows that

$$\mathbf{v}_i = \mathbf{u} + \mathbf{w}_i, \tag{1}$$

whence $\mathbf{v}_i^2 = (\mathbf{u} + \mathbf{w}_i)^2 = \mathbf{u}^2 + \mathbf{w}_i^2 + 2\mathbf{u}\mathbf{w}_i$. Putting $v_i = |\mathbf{v}_i|$, $w_i = |\mathbf{w}_i|$, and $u = |\mathbf{u}|$, we obtain

$$E = \tfrac{1}{2}\Sigma m_i v_i^2 = \tfrac{1}{2}\Sigma m_i u^2 + \tfrac{1}{2}\Sigma m_i w_i^2 + \Sigma m_i \mathbf{u}\mathbf{w}_i.$$

If we set $m = \Sigma m_i$, then we get

$$E = \tfrac{1}{2}mu^2 + \tfrac{1}{2}\Sigma m_i w_i^2 + \mathbf{u}\Sigma m_i \mathbf{w}_i. \tag{2}$$

Denote the absolute and relative velocities of the centre of mass by $\mathbf{v}_0$ and $\mathbf{w}_0$, respectively. By (1) we have $\mathbf{v}_0 = \mathbf{u} + \mathbf{w}_0$. Since $\Sigma m_i \mathbf{w}_i = m\mathbf{w}_0$, it follows from (2) that $E = \frac{1}{2}mu^2 + \frac{1}{2}\Sigma m_i w_i^2 + m\mathbf{u}\mathbf{w}_0$ or, writing $\mathbf{v}_0 - \mathbf{u}$ instead of $\mathbf{w}_0$,

$$E = \tfrac{1}{2}mu^2 + \tfrac{1}{2}\Sigma m_i \omega_i^2 + m\mathbf{u}(\mathbf{v}_0 - \mathbf{u}). \tag{IV}$$

The first term of this sum denotes the energy of the advancing motion of the system of points moving with a velocity $\mathbf{u}$. The second term denotes the kinetic energy of the relative motion, where the velocities $\mathbf{w}_i$ can be considered as the velocities of the points m_i relative to the point O which moves with a velocity $\mathbf{u}$.

Therefore, if we assume that the motion of the system consists of an advancing motion with a velocity of the arbitrary point O and a relative motion with respect to this point O, then *the kinetic energy of the system is equal to the sum of the kinetic energy of the advancing motion, the kinetic energy of the relative motion, and the product of the total mass of the system by the scalar product of the velocity of the point O and the relative velocity of the centre of mass.*

This theorem is known as the *theorem of König.*

In particular, if the centre of mass is chosen as the point O, then $\mathbf{u} = \mathbf{u}_0$, whence by (IV)

$$E = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}\Sigma m_i w_i^2. \tag{3}$$

Therefore, if the motion of a system is considered as a motion consisting of an advancing motion whose velocity is that of the centre of mass and a relative motion with respect to the centre of mass, then *the kinetic energy of the system is equal to the sum of the kinetic energy of the advancing motion and the kinetic energy of the relative motion.*

Principle of the equivalence of work and kinetic energy. Let the forces $\mathbf{P}_i$ act on the material points m_i. Denote the velocity of the point m_i at the time t by $\mathbf{v}_i$, its velocity at the time t_0 by $\mathbf{v}_i^{(0)}$, and the work of the force $\mathbf{P}_i$ during the time from t_0 to t by $L_{t_0 t}^{(i)}$. Therefore ((3), p. 105) $\frac{1}{2}m_i v_i^2 - \frac{1}{2}m_i(v_i^{(0)})^2 = L_{t_0 t}^{(i)}$, whence

$$\tfrac{1}{2}\Sigma m_i v_i^2 - \tfrac{1}{2}\Sigma m_i (v_i^{(0)})^2 = \Sigma L_{t_0 t}^{(i)}.$$

Setting: $E = \frac{1}{2}\Sigma m_i v_i^2$, $E_0 = \frac{1}{2}\Sigma m_i (v_i^{(0)})^2$, $L_{t_0 t} = \Sigma L_{t_0 t}^{(i)}$, we obtain

$$E - E_0 = L_{t_0 t}. \tag{V}$$

In this formula E denotes the kinetic energy of the system at the time t, and E_0 at the timo t_0; the expression L_{t_0t} represents the sum of the works of the separate forces acting on the system, i. e. the total work which these forces did in the time from t_0 to t.

Therefore: *the increase in the kinetic energy of a system of material points is equal to the total work of the forces acting on the points of the system.*

This theorem is known as *the principle of the equivalence of work and kinetic energy.*

If the total work of the acting forces is zero, i. e. $L_{t_0t} = 0$, then by (V) $E - E_0 = 0$, or

$$E = E_0.$$

Therefore: *if the total work of the forces acting on the points of a system is constantly zero, then the kinetic energy of the system is constant.*

The above theorem is known as *the principle of conservation of kinetic energy.*

Let us assume that a system of acting forces possesses a potential. If we denote the potential at the time t by V, and the potential at the time t_0 by V_0, then $L_{t_0t} = V - V_0$, and hence by (V) $E - E_0 = V - V_0$ or

$$E - V = E_0 - V_0.$$

The magnitude $U = -V$ is called the *potential energy of the system.* Consequently

$$E + U = E_0 + U_0 = \text{const.} \tag{VI}$$

The sum $E + U$ is called the *total energy of the system.*

Therefore: *if a system of points moves in a potential field, then the total energy of the system is constant.*

These theorems are obviously generalizations of the corresponding theorems proved on p. 105 for one material point.

Kinetic energy in relative motion. Let the system of coordinates $O'(x', y', z')$ move relative to the inertial frame $O(x, y, z)$. Denote by E_r and $E_r^{(0)}$ the kinetic energy in relative motion at the times t and t_0, by L_{t_0t}, $L_{t_0t}^t$ and $L_{t_0t}^C$ the works in relative motion of the acting forces, the forces of transport and Coriolis during the time from t_0 to t. Since Newton's laws apply to relative motion if the forces of transport and Coriolis (p. 135) are added to the acting forces, it follows that

$$E_r - E_r^{(0)} = L_{t_0t} + L_{t_0t}^t + L_{t_0t}^C. \tag{4}$$

Since the acceleration of Coriolis is perpendicular to the relative velocity, the force of Coriolis is also perpendicular to this velocity.

Therefore the work of the forces of Coriolis in relative motion is zero, and because of this we can write

$$E_r - E_r^{(0)} = L_{t_0t} + L_{t_0t}^t. \tag{5}$$

Hence: *the increase in kinetic energy in relative motion is equal to the sum of the works in relative motion of the acting forces and of the forces of transport.*

In particular, let the point O' be situated at the centre of mass S of the system $O'(x', y', z')$ and let this system move with an advancing motion. Since the acceleration of transport is by this assumption equal to the acceleration $\mathbf{p}_0$ of the centre of mass, the forces of transport of the separate points of the system $m_1, m_2, \ldots$ are:

$$\mathbf{P}_{1t} = -m_1\mathbf{p}_0, \quad \mathbf{P}_{2t} = -m_2\mathbf{p}_0, \ \ldots \tag{6}$$

Denoting the relative velocities of the points of the system by $\mathbf{w}_1, \mathbf{w}_2, \ldots$, we obtain

$$L_{t_0t}^t = \int_{t_0}^{t} \mathbf{P}_{1t}\mathbf{w}_1\,dt + \int_{t_0}^{t} \mathbf{P}_{2t}\,\mathbf{w}_2 dt + \ldots,$$

whence according to (6)

$$L_{t_0t}^t = -\int_{t_0}^{t} \mathbf{p}_0(m_1\mathbf{w}_1 + m_2\mathbf{w}_2 + \ldots)\,dt.$$

The relative velocity $\mathbf{w}_0$ of the centre of mass is equal to zero because we have assumed that the centre S of the total mass m is always at the origin of the moving system $O'(x', y', z')$. Therefore $m_1\mathbf{w}_1 + m_2\mathbf{w}_2 + + \ldots = m\mathbf{w}_0 = 0$, whence according to the last formula $L_{t_0t}^t = 0$. Hence by (5)

$$E_r - E_r^{(0)} = L_{t_0t}. \tag{7}$$

Hence: *the increase in kinetic energy in relative motion with respect to the centre of mass is equal to the work in relative motion of the acting forces.*

Example I. A system of material points moves in a gravitational force field. Let the centre of mass S be the origin of the coordinate system (x', y', z') moving with an advancing motion. Assume that the z'-axis is vertical and has a downward sense.

Since the weights of the separate points have the direction of the z'-axis, then (just as in the absolute system, p. 212) the work of the weights in relative motion is equal to the work done by the total weight whose initial point is at the centre of mass. As the center of mass is at rest relative to the system (x', y', z'), for by hypothesis it is constantly at the

origin of this system, the work of the weights in relative motion is zero. Therefore the increase in kinetic energy in relative motion is equal to the work of the remaining forces (excluding the weights) which act on the points of the system.

In particular, if the weights are the only forces acting on the points of the system, then the kinetic energy of this system in relative motion is constant.

Let us suppose, e. g., that at the moment $t = 0$ we have released freely a material point of mass m_1, and after T seconds, another point of mass m_2. After the time $t > T$ the velocities of the points m_1 and m_2 are:

$$v_1 = gt \text{ and } v_2 = g(t - T). \tag{8}$$

The velocity of the centre of mass v_0 is obtained from the equation

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)\, v_0.$$

Therefore $v_0 = (m_1 v_1 + m_2 v_2) / (m_1 + m_2)$. The relative velocities of the points m_1 and m_2 are $w_1 = v_1 - v_0$ and $w_2 = v_2 - v_0$; hence $w_1 = m_2(v_1 - v_2) / (m_1 + m_2)$, and $w_2 = m_1(v_2 - v_1) / (m_1 + m_2)$, from which by (8) $w_1 = m_2 gT / (m_1 + m_2)$, and $w_2 = -m_1 gT / (m_1 + m_2)$.

The kinetic energy in relative motion is therefore

$$E_r = \tfrac{1}{2} m_1 w_1^2 = + \tfrac{1}{2} m_2 w_2^2 = m_1 m_2 g^2 T^2 / 2(m_1 + m_2) = \text{const.}$$

We see then that the kinetic energy in relative motion with respect to the centre of mass is constant.

Example 2. Two points A and B of masses m_1 and m_2, connected by a massless inextensible string, move in a vertical plane in such a way that the point A must remain constantly on the horizontal axis x and the point B on the vertical axis z.

Let us denote the length of the string by l and the coordinates of the points A, B by x, z (Fig. 143). The point A is acted upon by the reaction $\mathbf{R}_1$ perpendicular to the x-axis (we assume that there is no friction), the weight $\mathbf{Q}_1$, and the reaction $\mathbf{P}_1$ of the string; the point B is acted upon by the reaction $\mathbf{R}_2$ perpendicular to the z-axis, the weight $\mathbf{Q}_2$, and the reaction $\mathbf{P}_2$ of the string.

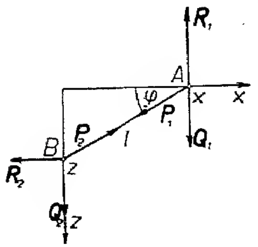

Fig. 143.

Of these forces, the following do no work: the reactions $\mathbf{R}_1$, $\mathbf{R}_2$, and the weight $\mathbf{Q}_1$, because these forces are perpendicular to the path. The forces $\mathbf{P}_1$ and $\mathbf{P}_2$ also do no work, since $\mathbf{P}_1 = -\mathbf{P}_2$ and furthermore the distance between

the points A, B is constant (p. 208). Only the weight $\mathbf{Q}_2$ therefore does work.

Suppose that at the time $t_0 = 0$ the points A, B had the coordinates x_0, z_0 and a zero velocity. The kinetic energy at the instant t is

$$E = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \dot{z}^2.$$

The work of the force $\mathbf{Q}_2$ is equal to $m_2 g(z - z_0)$ if the z-axis is given a downward sense. Therefore by the principle of equivalence of work and kinetic energy

$$\tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \dot{z}^2 = m_2 g(z - z_0). \tag{9}$$

Denote by φ the angle which the string makes with the x-axis at the time t, and by φ_0 the angle at the time $t_0 = 0$. We then have:

$$x = l \cos\varphi, \quad z = l \sin\varphi, \quad z_0 = l \sin\varphi_0,$$

whence

$$\dot{x} = -l\dot{\varphi} \sin\varphi, \quad \dot{z} = l\dot{\varphi} \cos\varphi. \tag{10}$$

Hence according to (9)

$$\tfrac{1}{2} l^2 \dot{\varphi}^2 (m_1 \sin^2\varphi + m_2 \cos^2\varphi) = m_2 g l(\sin\varphi - \sin\varphi_0). \tag{11}$$

From the above equation we can, knowing φ, calculate $\dot{\varphi}$, and then from equations (10) determine the velocities $\dot{x}$ and $\dot{y}$. Knowing φ, we can also determine the reactions $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{P}_1$, $\mathbf{P}_2$. Assume for simplicity's sake that $m_1 = m_2 = m$, and $\varphi_0 = 0$. We obtain from (11)

$$\tfrac{1}{2} l \dot{\varphi}^2 = g \sin\varphi. \tag{12}$$

Differentiating with respect to t, we get $l\dot{\varphi}\ddot{\varphi} = g\dot{\varphi} \cos\varphi$, whence

$$\ddot{\varphi} = g \cos\varphi / l. \tag{13}$$

Denoting the acceleration of the point A by $\mathbf{p}_1$, we obtain

$$m_1 \mathbf{p}_1 = \mathbf{R}_1 + \mathbf{Q}_1 + \mathbf{P}_1. \tag{14}$$

Forming the projection on the x-axis and putting $P = |\mathbf{P}_1| = |\mathbf{P}_2|$, we get

$$m\ddot{x} = -P \cos\varphi. \tag{15}$$

Since in virtue of (10)

$$\ddot{x} = -l\ddot{\varphi} \sin\varphi - l\dot{\varphi}^2 \cos\varphi, \tag{16}$$

from equations (15) and (16) we can obtain P, knowing φ, because $\dot{\varphi}$ and $\ddot{\varphi}$ can be calculated from equations (12) and (13). We get

$$P = 3mg \sin\varphi. \tag{17}$$

Forming the projection on the z-axis, we get from (14) $R_1 + mg + P \sin\varphi = 0$, or

$$R_1 = -mg - P \sin\varphi, \tag{18}$$

where R_1 denotes the projections of the force $\mathbf{R}_1$ on the z-axis.

Similarly, for the point B we have $m\mathbf{p}_2 = \mathbf{R}_2 + \mathbf{P}_2 + \mathbf{Q}_2$. Forming the projection on the x-axis and observing that $\mathbf{P}_2 = -\mathbf{P}_1$, we obtain the equation $R_2 + P\cos\varphi = 0$, whence

$$R_2 = -P\cos\varphi. \tag{19}$$

Formulae (17)—(19) determine the dependence of the reactions on the angle φ.

Example 3. Two material points A_1 and A_2 of masses m_1 and m_2, connected by a massless inextensible string passing through a fixed point O, move without friction in a horizontal plane passing through O. In this plane select a coordinate system (x, z) whose origin is at O (see Fig. 140). Since the directions of the forces $\mathbf{P}_1$, $\mathbf{P}_2$, with which the string acts on the points A_1 and A_2, constantly pass through O, the points A_1 and A_2 will move with a central motion with centre at O.

Put $r_1 = OA_1$ and $r_2 = OA_2$; let φ_1, φ_2 denote the angles between the x-axis and the segments OA_1 and OA_2. The areal velocities of the points A_1 and A_2 are equal to $\frac{1}{2}r_1^2\dot\varphi_1$ and $\frac{1}{2}r_2^2\dot\varphi_2$, respectively. Since the areal velocities in a central motion are constant (p. 86), it follows that

$$r_1^2\dot\varphi_1 = c_1, \quad r_2^2\dot\varphi_2 = c_2, \tag{20}$$

where c_1 and c_2 are certain constants.

On p. 209 we proved that the total work of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ is zero. Therefore the kinetic energy of the system of points A_1 and A_2 has a constant value. Denoting the magnitudes of the velocities of the points A_1 and A_2 by v_1 and v_2, we obtain $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = c$, whence

$$m_1v_1^2 + m_2v_2^2 = h, \tag{21}$$

where c and $h = 2c$ are constants. But ((3), p. 47)

$$v_1^2 = \dot r_1^2 + r_1^2\dot\varphi_1^2 \text{ and } v_2^2 = \dot r_2^2 + r_2^2\dot\varphi_2^2;$$

hence by (20), expressing $\dot\varphi_1$ and $\dot\varphi_2$ in terms of r_1 and r_2, we obtain $v_1^2 = \dot r_1^2 + c_1^2/r_1^2$ and $v_2^2 = \dot r_2^2 + c_2^2/r_2^2$, from which by (21)

$$m_1\dot r_1^2 + m_2\dot r_2^2 + m_1c_1^2/r_1^2 + m_2c_2^2/r_2^2 = h. \tag{22}$$

Denote the length of the string by l. Then $r_1 + r_2 = l$ therefore $r_2 = l - r_1$, whence

$$\dot r_2 = -\dot r_1. \tag{23}$$

Substituting in (22), we obtain

$$(m_1 + m_2)\dot r_1^2 + m_1c_1^2/r_1^2 + m_2c_2^2/(l - r_1)^2 = h. \tag{24}$$

The differential equation (24) determines r_1 as a function of the time t. From equations (23) and (20) we obtain r_2, φ_1, φ_2. In order to determine the reactions $\mathbf{P}_1$, $\mathbf{P}_2$, let us note that if $\mathbf{p}_1$ denotes the acceleration of the point A_1, then $m_1\mathbf{p}_1 = \mathbf{P}_1$. Form the projection on the direction of $\overline{OA_1}$. Denoting the projections of $\mathbf{p}_1$ and $\mathbf{P}_1$ on $\overline{OA_1}$ by p_{1r} and P, we get

$$m_1p_{1r} = P. \tag{25}$$

By (II), p. 47, we have $p_{1r} = \ddot r_1 - r_1\dot\varphi_1^2$. Hence according to (20)

$$p_{1r} = \ddot r_1 - c_1^2/r_1^3. \tag{26}$$

In order to determine $\ddot r_1$, let us differentiate equation (24). We get

$$\dot r_1[(m_1 + m_2)\ddot r_1 - m_1c_1^2/r_1^3 + m_2c_2^2/(l - r_1)^3] = 0. \tag{27}$$

If $\dot r_1 \neq 0$, then in virtue of (25)—(27)

$$P = -\frac{m_1m_2}{m_1 + m_2}\left[\frac{c_2^2}{(l - r_1)^3} + \frac{c_1^2}{r_1^3}\right]. \tag{28}$$

From formula (28) we can obtain the reaction P knowing only r_1. Knowing P, we know $\mathbf{P}_1$ and $\mathbf{P}_2$ because $|\mathbf{P}_1| = |\mathbf{P}_2|$.

§ 6. Problem of two bodies. Let two material points of masses M and m attract each other according to Newton's law with a force of magnitude

$$P = KmM/r^2,$$

where r denotes the distance of these points. On p. 106 we examined the motion of the point m under the assumption that the point M is motionless. We proved that Kepler's law obtain in this case. Now we shall not assume that the point M is motionless, but that both points are unconstrained. Therefore under the influence of their mutual attraction, both points m and M will move. Obviously, their centre of mass will be at rest or in uniform straight line motion, because according to the law of action and reaction the sum of the forces acting on the points m and M is equal to zero. We can therefore place the origin of the inertial frame at the centre of gravity of both points.

Let x_1, y_1, z_1 be the coordinates of the point M, and x_2, y_2, z_2 those of the point m. Newton's equations of motion for the points M and m will have the form:

$$M\ddot x_1 = \frac{KmM}{r^2}\,\frac{x_2 - x_1}{r}, \quad M\ddot y_1 = \frac{KmM}{r^2}\,\frac{y_2 - y_1}{r}, \quad M\ddot z_1 = \frac{KmM}{r^2}\,\frac{z_2 - z_1}{r}, \tag{1}$$

$$m\ddot{x}_2 = -\frac{KmM}{r^2}\frac{x_2 - x_1}{r}, \quad m\ddot{y}_2 = -\frac{KmM}{r^2}\frac{y_2 - y_1}{r},$$
$$m\ddot{z}_2 = -\frac{KmM}{r^2}\frac{z_2 - z_1}{r}. \tag{1'}$$

Since the centre of gravity of the system M, m is at the origin of the coordinate system, $Mx_1 + mx_2 = 0$, $My_1 + my_2 = 0$, and $Mz_1 + mz_2 = 0$, whence $x_2 = -Mx_1 / m$, $y_2 = -My_1 / m$, and $z_2 = -Mz_1 / m$. Therefore:

$$x_2 - x_1 = -\frac{M+m}{m}x_1, \quad y_2 - y_1 = -\frac{M+m}{m}y_1,$$
$$z_2 - z_1 = -\frac{M+m}{m}z_1. \tag{2}$$

Hence

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \frac{M+m}{m}r_1, \tag{3}$$

where $r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ denotes the distance of the point M from the centre of gravity. Substituting in equations (1) the expressions from formulae (2) and (3), we obtain:

$$M\ddot{x}_1 = -\frac{Km^3M}{(M+m)^2 r_1^2}\cdot\frac{x_1}{r_1}, \quad M\ddot{y}_1 = -\frac{Km^3M}{(M+m)^2 r_1^2}\cdot\frac{y_1}{r_1},$$
$$M\ddot{z}_1 = -\frac{Km^3M}{(M+m)^2 r_1^2}\cdot\frac{z_1}{r_1}. \tag{4}$$

Comparing these equations with equations (I), p. 106, we see that the motion of the point M is such as if this point were attracted by a motionless mass $m^3 / (M+m)^2$ situated at the origin of the system.

Therefore: *if two points M and m attract each other according to Newton's law, then each one of them, for example M, moves relative to the centre of gravity (of both points) so as if a motionless mass $m^3 / (M+m)^2$ were situated at the centre of gravity and attracted the point M according to Newton's law.*

Hence the investigation of a motion relative to the centre of mass of two points is reduced to the case considered on p. 106.

It follows from this that both points move along a conic at whose focus is found the centre of gravity of these points. The paths of both points are therefore plane paths.

Let us still examine the motion of the point m relative to the point M. Let us place at M the origin of the coordinate system (x', y', z') which

moves together with M with an advancing motion. Denoting the coordinates of m with respect to this coordinate system by ξ, η, ζ, we obtain

$$\xi = x_2 - x_1, \quad \eta = y_2 - y_1, \quad \zeta = z_2 - z_1. \tag{5}$$

Multiplying equations (1) by m / M and subtracting from (1′) we get in virtue of (5)

$$m\ddot{\xi} = -\frac{K(M+m)\,m}{r^2}\frac{\xi}{r}, \quad m\ddot{\eta} = -\frac{K(M+m)\,m}{r^2}\frac{\eta}{r},$$
$$m\ddot{\zeta} = -\frac{K(M+m)\,m}{r^2}\frac{\zeta}{r}. \tag{6}$$

Comparing equations (6) with equations (I), p. 106, we see that the point m moves relative to the point M so as if M were motionless and its mass were increased by the mass of the point m.

Therefore: *if two points M and m attract each other according to Newton's law, then the relative motion of m with respect to M is such as if M were motionless and its mass were increased by the mass of the point m.*

In this case also, the investigation of the motion of one material point relative to another is therefore reduced to the case considered on p. 106.

Let us assume that the relative motion of the point m takes place along an ellipse of major axis $2a$, and let the time required to complete one revolution be T. By (10), p. 108, we obtain $a^3 / T^2 = K(M+m) / 4\pi^2$. We see, therefore, that in relative motion the ratio a^3 / T^2 depends on the masses of both bodies. Since we are investigating the motions of the planets relative to the sun, assuming that M denotes the mass of the sun and m the mass of the planet, we see that *Kepler's third law* (p. 87), which refers to the relative motion of a planet with respect to the sun, *is not exact*. For another planet (using a corresponding notation)

$$a_1^3 / T_1^2 = K(M + m_1) / 4\pi^2, \tag{7}$$

whence

$$\frac{a^3 / T^2}{a_1^3 / T_1^2} = \frac{M+m}{M+m_1} = \frac{1 + m/M}{1 + m_1/M}. \tag{8}$$

In the solar system the ratio m / M is expressed in the thousandths and therefore the last fraction differs little from one. Accurate observations of planetary motions reveal these deviations from Kepler's third law.

Two celestial bodies which rotate about each other (far away from other bodies) are called *double stars*. Assuming that double stars attract each other

according to Newton's law, we can apply to them the conclusions obtained in this §. Observations confirm these conclusions and at the same time the law of universal gravitation from which these conclusions were drawn.

§ 7. Problem of n bodies. Let n material points attract each other mutually with forces acting according to Newton's law of universal gravitation (p. 89). The so-called *problem of n bodies* is concerned with the investigation of motions in such a system of points.

This problem is important for astronomy. The sun and planets form such a system if we neglect the influence of the fixed stars which is very small because of their remoteness from the solar system.

The problem of two bodies with which we were concerned in § 6 is a particular case of the problem of n bodies.

The problem of n bodies is not solved in all generality. Even in the case of three bodies there are many questions still unanswered. By means of the *theory of perturbations*, however, we can determine the motions of the solar system with the desired accuracy.

In the problem of n bodies we have to deal only with internal forces. Therefore from the theorem on the centre of mass (p. 196), it follows that the centre of mass of a system is at rest or in uniform straight line motion. We can therefore choose the origin of the inertial system of coordinates at the centre of mass. With respect to such a chosen system of coordinates the momentum of the system of n points will be constantly zero (p. 195).

From the theorem concerning angular momentum (p. 202), it follows that the angular momentum of a system of points is constant. Hence the plane passing through the centre of mass and perpendicular to the angular momentum does not change its position.

In the case of the solar system the centre of mass lies in the sun (on account of the great mass of the sun as compared with the remaining planets).

The plane passing through the centre of mass of the solar system and perpendicular to the angular momentum was called the *invariable plane* by Laplace.

This plane does not change its position in space relative to the inertial system of coordinates whose origin is in the sun. According to calculations carried out by Laplace, the invariable plane forms an angle $\alpha = 1.7689°$ with the ecliptic.

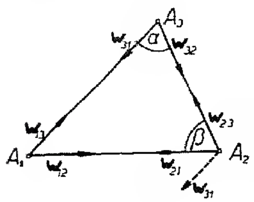

Fig. 144.

Problem of three bodies. Let there be given three material points A_1, A_2, A_3 of masses m_1, m_2, m_3 (Fig. 144). Denote the force with which m_j attracts m_i by $\boldsymbol{P}_{ij}$; let $\boldsymbol{w}_{ij}$ be the force with which a unit of mass situated at A_j attracts a unit of mass situated at A_i. According to Newton's law we therefore have

$$\boldsymbol{P}_{ij} = m_i m_j \boldsymbol{w}_{ij}. \tag{1}$$

From the law of action and reaction it follows that $\boldsymbol{P}_{ij} = -\boldsymbol{P}_{ji}$; hence

$$\boldsymbol{w}_{ij} = -\boldsymbol{w}_{ji}. \tag{2}$$

Denote the acceleration of the point m_i in an inertial system of coordinates by $\boldsymbol{p}_i$. Then $m_1\boldsymbol{p}_1 = \boldsymbol{P}_{12} + \boldsymbol{P}_{13} = m_1 m_2 \boldsymbol{w}_{12} + m_1 m_3 \boldsymbol{w}_{13}$, whence

$$\boldsymbol{p}_1 = m_2 \boldsymbol{w}_{12} + m_3 \boldsymbol{w}_{13}. \tag{3}$$

Similarly

$$\boldsymbol{p}_2 = m_1 \boldsymbol{w}_{21} + m_3 \boldsymbol{w}_{23}. \tag{4}$$

By (3) and (4), and because that in view of (2), $\boldsymbol{w}_{12} = -\boldsymbol{w}_{21}$, $\boldsymbol{w}_{13} = -\boldsymbol{w}_{31}$, we obtain:

$$\boldsymbol{p}_2 - \boldsymbol{p}_1 = (m_1 + m_2)\, \boldsymbol{w}_{21} + m_3(\boldsymbol{w}_{23} + \boldsymbol{w}_{31}). \tag{5}$$

The difference $\boldsymbol{p}_2 - \boldsymbol{p}_1$ represents the relative acceleration of the point A_2 with respect to A_1, i. e. the acceleration of A_2 relative to the coordinate system, whose origin is A_1, moving with an advancing motion. Put $\boldsymbol{p} = \boldsymbol{p}_2 - \boldsymbol{p}_1$. From equation (5) we get

$$m_2\boldsymbol{p} = m_2(m_1 + m_2)\, \boldsymbol{w}_{21} + m_2 m_3(\boldsymbol{w}_{23} + \boldsymbol{w}_{31}). \tag{6}$$

The right side of equation (6) represents the relative force of the point A_2 in motion relative to A_1.

The first one of its terms, i. e. $m_2(m_1 + m_2)\ \boldsymbol{w}_2$, represents the relative force which would act on the point A_2 if there were no third point A_3 (i. e. if there were $m_3 = 0$). This force would have (in agreement with the theorem given on p. 223) a direction towards A_1 and would be such as if the mass of A_1 were increased by the mass of A_2.

The second term of the sum, i. e. $m_2 m_3(\boldsymbol{w}_{23} + \boldsymbol{w}_{31})$, is called the *force of perturbation*; it is due to the action of the point A_3.

Example 1. Let m_1 denote the mass of the earth, m_2 the mass of the moon and m_3 the mass of the sun.

Approximately, the mass of the sun is $\frac{1}{3} \cdot 10^6$ times the mass of the earth, the distance of the earth from the sun is 400 times the distance of the earth from the moon, and finally the mass of the moon is $\frac{1}{80}$ of the mass of the earth. Hence:

$$m_3 = \tfrac{1}{3} \cdot 10^6 m_1, \quad m_2 = \tfrac{1}{80} m_1, \quad A_1A_3 = 400 A_1A_2. \tag{7}$$

From the triangle $A_1A_2A_3$ we obtain

$$399 A_1A_2 \leq A_2A_3 \leq 401 A_1A_2. \tag{8}$$

Because of the great distance of the sun from the earth and the moon (as compared with the distance of the moon from the earth), $\mathbf{w}_{23}$ and $\mathbf{w}_{31}$ will differ little from each other in magnitude and direction and they will have opposite senses. The absolute value of the sum $\mathbf{w}_{23} + \mathbf{w}_{31}$ is therefore small. Making use of (7) and (8), it can be shown that

$$\frac{m_2 m_3 |\mathbf{w}_{23} + \mathbf{w}_{31}|}{m_2(m_1 + m_2)|\mathbf{w}_{21}|} \leq 1 \cdot 5 \cdot 10^{-2}. \tag{9}$$

We see by (6), therefore, that the force of perturbation due to the sun is small, and we can neglect it in the first approximation.

Hence: *In the first approximation the relative motion of the moon with respect to the earth is obtained by neglecting the attraction of the sun.*

An approximate investigation of the relative motion in the given case is therefore reduced to the problem of two bodies. This also refers to other planets having satellites.

Example 2. Let A_1 be the centre of the earth, A_2 a point on the surface of the earth and A_3 the centre of the moon. Let m_1, m_2, and m_3 denote the masses of the earth, the point A_2 and the moon, respectively. Assume that the points A_1, A_2, and A_3, are collinear.

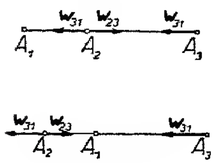

Fig. 145.

The vectors $\mathbf{w}_{23}$ and $\mathbf{w}_{31}$ have opposite senses. If A_2 lies between A_1 and A_3, then $|\mathbf{w}_{32}| > |\mathbf{w}_{31}|$ (Fig. 145a), and hence $\mathbf{w}_{23} + \mathbf{w}_{31}$ has the sense of $\mathbf{w}_{23}$. If A_1 lies between A_3 and A_2 (Fig. 145b), then $|\mathbf{w}_{23}| < |\mathbf{w}_{31}|$, and hence $\mathbf{w}_{23} + \mathbf{w}_{31}$ has the sense of $\mathbf{w}_{31}$. In both cases the force of perturbation of the moon $m_2 m_3(\mathbf{w}_{23} + \mathbf{w}_{31})$ is directed vertically upwards with respect to the earth. The action of this force explaines the tides.

Example 3. Let m_1 denote the mass of some planet, m_2 the mass of its satellite, a the mean distance of the planet from its satellite and T the time of one revolution of the satellite about the planet in relative motion (with respect to the planet). In virtue of (7), p. 223, we have

$$a^3 / T^2 = K(m_1 + m_2) / 4\pi^2. \tag{10}$$

If m_1', m_2', a', and T' denote the corresponding magnitudes for another planet and its satellite, then analogous to (10)

$$a'^3 / T'^2 = K(m_1' + m_2') / 4\pi^2. \tag{11}$$

By (10) and (11), $(m_1 + m_2) / (m_1' + m_2') = a^3 T'^2 / a'^3 T^2$. Neglecting

the masses of the satellites m_2 and m_2' because they are usually small as compared with the masses of the planets, we get

$$m_1 / m_1' = a^3 T'^2 / a'^3 T^2. \tag{12}$$

Therefore: *the ratio of the masses of two planets can be obtained from the observation of the motions of their satellites.*

Remark. We can also assume that m_1' denotes the mass of the sun, $m_2' = m_1$, a' the mean distance of the planet from the sun, and T' its period. Under these assumptions formula (12) represents the ratio of the mass of the given planet (possessing a satellite) to the mass of the sun.

§ 8. Motion of bodies of variable mass. Let us now investigate the motion of a body whose mass changes because particles leave the body (or new ones join it) during motion.

An example is that of a moving waggon into which sand is being poured (or from which sand is running out). A rocket is another example. When the fuel within a rocket burns, gases are expelled which propel the rocket. The mass of the rocket diminishes, therefore, by the mass of the escaping gases.

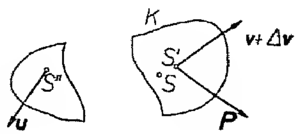

Fig. 146.

Let us assume that a body consists of a great number of small particles which can be considered as material points. Denote by m the mass of the body, by $\mathbf{v}$ the velocity of its centre of mass S at the time t, and by $m + \Delta m$ and $\mathbf{v} + \Delta\mathbf{v}$ the mass and velocity of the centre S at the time $t + \Delta t$. Finally, let $\mathbf{P}$ denote the sum of the forces acting on the body at the time t (Fig. 146).

The mass of the particles leaving the body during the interval Δt is $(-\Delta m)$; let $\mathbf{u}$ and $\mathbf{u} + \Delta\mathbf{u}$ denote the velocities (at the times t and $t + \Delta t$) of the centre of mass S'' of the particles leaving the body.

Let us consider the system U of all the particles of which the body is composed at the time t. The momentum of this system at the time t is $\mathbf{H} = m\mathbf{v}$, and at the time $t + \Delta t$ it will be $\mathbf{H}' = (m + \Delta m)(\mathbf{v} + \Delta\mathbf{v}) + t + (-\Delta m)(\mathbf{u} + \Delta\mathbf{u})$. Hence the increase in the momentum is

$$\mathbf{H}' - \mathbf{H} = m\,\Delta\mathbf{v} - \Delta m(\mathbf{u} + \Delta\mathbf{u} - \mathbf{v}) + \Delta m\,\Delta\mathbf{v}.$$

Dividing by Δt and passing to the limit, we obtain

$$\frac{d\mathbf{H}}{dt} = m\frac{d\mathbf{v}}{dt} - \frac{dm}{dt}(\mathbf{u} - \mathbf{v}). \tag{1}$$

Since the derivative of the momentum is equal to the sum of all the acting forces (p. 196), $d\mathbf{H} / dt = \mathbf{P}$. Therefore from (1)

$$m\mathbf{v}^{\cdot} - m^{\cdot}(\mathbf{u} - \mathbf{v}) = \mathbf{P}. \tag{2}$$

The above equation can be written in the form $m\mathbf{v}^{\cdot} + m^{\cdot}\mathbf{v} = m^{\cdot}\mathbf{u} + \mathbf{P}$, whence

$$d(m\mathbf{v}) / dt = m^{\cdot}\mathbf{u} + \mathbf{P}. \tag{3}$$

Formulae (2) and (3) apply equally to the case when new particles join the body. In equations (2) and (3) the vector $\mathbf{u}$ represents the velocity of the centre of mass S'' of the particles leaving or joining the body.

Substituting $\mathbf{u} - \mathbf{v} = \mathbf{w}$ in equation (2), we obtain

$$m\mathbf{v}^{\cdot} - m^{\cdot}\mathbf{w} = \mathbf{P}. \tag{4}$$

The vector $\mathbf{w}$ represents the relative velocity of the centre of mass of the particles leaving the body with respect to the centre of mass of the body.

Example. The motion of a rocket. Denote the mass of a rocket by m, its velocity by $\mathbf{v}$, the relative velocity (with respect to the rocket) of the gases escaping from the rocket by $\mathbf{w}$ (Fig. 147), and the sum of the external forces acting on the body (such as gravity, the resistance of the air, etc.) by $\mathbf{P}$. With this notation formula (4) obtains.

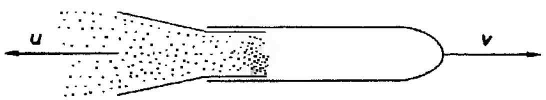

Fig. 147.

Let us suppose at first that the rocket moves in a horizontal plane along a straight line which we shall select as the x-axis, giving it a sense agreeing with the direction of the rocket. Put $v = |\mathbf{v}|$ and $w = |\mathbf{w}|$. Assume that $\mathbf{P} = 0$ (and hence that the force of gravity is balanced by the reaction of the plane; the resistance of the air and friction are neglected). Since $\mathbf{v}$ and $\mathbf{w}$ have opposite senses, by (4) $mv^{\cdot} + m^{\cdot}w = 0$, whence

$$v^{\cdot} = -\frac{m^{\cdot}}{m}w. \tag{5}$$

The relative velocity w of the escaping gases can be considered as constant. Integrating equation (5), we obtain

$$v = -w \ln m + c. \tag{6}$$

Assume that $m = m_0$ and $v = 0$ at $t = 0$. Then according to equation (6), $0 = -w \ln m_0 + c$, and hence $c = w \ln m_0$. Therefore by (6)

$$v = w \ln \frac{m_0}{m}, \tag{7}$$

whence $m_0 / m = e^{v/w}$ or

$$m = m_0 e^{-v/w}. \tag{8}$$

Let us suppose that the rocket attained a velocity $v = 100$ km/h $= 27$ m/sec. We can assume that $w = 1000$ m/sec is the velocity of the escaping gas. Therefore by (8) $m = m_0 e^{-0.027} = 0.973 m_0$, whence $m_0 - m = 0.03 m_0$.

Hence in order to realize a velocity of 100 km/h, it is necessary to burn an amount of fuel equal to 3% of the mass of the rocket.

Let the rocket now move vertically upwards. Assume that the z-axis is directed vertically upwards and let us retain the previous notation. We obtain from (4) (neglecting air resistance) $mv^{\cdot} + m^{\cdot}w = -mg$, whence

$$v^{\cdot} = -w\frac{m^{\cdot}}{m} - g. \tag{9}$$

Integrating (9) and assuming $v = 0$ and $m = m_0$ at $t = 0$ we obtain as previously

$$v = w \ln \frac{m_0}{m} - gt. \tag{10}$$

In order that the rocket may not fall back to earth and that it may penetrate interplanetary space it would be necessary to give it a velocity $v \geq 12$ km/sec (p. 110). From equation (10) we obtain

$$v \leq w \ln \frac{m_0}{m},$$

whence $e^{v/w} \leq m_0 / m$ or $m e^{v/w} \leq m_0$, and therefore

$$m_0 - m \geq m(e^{v/w} - 1).$$

Putting $v = 12$ km/sec and $w = 1000$ m/sec $= 1$ km/sec, we get

$$m_0 - m \geq 160000\, m.$$

In this inequality m denotes the mass of the rocket after attaining a velocity $v = 12$ km/sec, and $m_0 - m$ the mass of the propelling fuel burned. If we assume that $m = 1$ kg, then $m_0 - m \geq 160000$ kg.

It is therefore necessary to burn 160000 kg of fuel in order that 1 kg of mass excape into space.

Hence in order to make an interplanetary journey in a rocket having together with its passengers a mass of one ton, it would be necessary to take along 160000 tons of fuel — which is obviously impossible. This shows that at the present state of technical sciences such a journey cannot be made. The matter would be pushed forward if w (the velocity of the escaping gases), which to-day is close to 2000 m/sec, could be markedly increased.

CHAPTER VI[1)]

STATICS OF A RIGID BODY

I FREE BODY

§ 1. Rigid body. A material body which despite the action of forces does not sustain any deformations (i. e. in which the mutual distances of the points of the body do not undergo a change) is called a *rigid body*.

Rigid bodies are not found in nature, since every body becomes deformed more or less under the influence of the action of forces. However, if some body under the influence of forces experiences only small deformations not exceeding a certain limit, then we can take as a model of such a body a rigid body, and the conclusions that we shall draw will be approximately in agreement with experience (provided the forces are not large). From this arises the great importance of the theory of a rigid body for practical applications.

We shall consider in turn statics, kinematics and dynamics of a rigid body.

In the theory of a rigid body we shall meet, in addition to rigid material solids, rigid material surfaces and lines (p. 168) as models of bodies in which one or two dimensions are small in comparison with those remaining. Examples of such bodies are plates, rods, wires, etc.

Rigid systems of material points. It often proves useful to look upon a rigid body as a collection (system) of a large number of material points. We assume then, that the material points act on each other with certain forces which ensure that the system of points is rigid, i. e. that the mutual distances of its points do not undergo a change. These forces are called *internal forces*.

We assume that Newton's law of action and reaction (p. 173) applies to internal forces, i. e. that two points act on each other with

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1) For the understanding of this chapter the information included in chapters I and III (from p. 69 to 75) and the theorems on centre of gravity in chapter IV, §§ 1, 2, 6, 7 and 8, are sufficient.

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forces equal in magnitude and oppositely directed along the straight line joining these points.

In addition to internal forces, other forces, called *external forces*, can act on the points of a system.

Therefore, if a rigid body is considered as a rigid system of material points, then the forces acting on a rigid body are external forces acting on the points of the system

One might question whether it is admissible to consider a rigid body as a system of material points. This assumption can be justified, however, in the following manner: by subdividing the rigid body into very many small pieces and replacing each one of them by a material point of the same mass, we obtain a rigid system of material points representing the given body with considerable approximation.

Although the assumption that a rigid body is a collection of material points is not correct, we shall make use of it since it simplifies reasoning and leads to satisfactory results. Properly, however, the theory of a rigid body and the theory of rigid systems of material points should be treated separately.

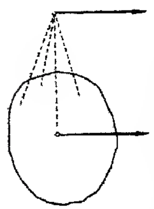


Fig. 148.

§ 2. Force. Point of application of a force. In the theory of a rigid body we assume that the point of application (origin) of the force acting on a rigid body may belong to the body or not; in the latter case we assume, however, that the point of application is rigidly attached to the body (we can imagine e. g., that the point of application is joined to the body by means of rigid massless rods) (Fig. 148). The action of the force will therefore be such as if the point of application belonged to the body.

Moment with respect to a point. If the force $\mathbf{P}$ acts at the point A whose coordinates are x, y, z, then the moment of the force with respect to the point O whose coordinates are x_0, y_0, z_0 has the projections:

$$M_x = P_y(z - z_0) - P_z(y - y_0), \quad M_y = P_z(x - x_0) - P_x(z - z_0), \quad M_z = P_x(y - y_0) - P_y(x - x_0), \tag{1}$$

on the axes of the system (p. 17, (I)).

In particular, if O is the origin of the system, i. e. if $x_0 = y_0 = z_0 = 0$, we get:

$$M_x = P_y z - P_z y, \quad M_y = P_z x - P_x z, \quad M_z = P_x y - P_y x. \tag{2}$$

From the definition of the moment (p. 15) it follows that

$$|\mathbf{M}| = |\mathbf{P}|h, \tag{3}$$

where h denotes the distance of the point O from the position of the force $\mathbf{P}$ (i. e. from the line on which $\mathbf{P}$ lies); this distance is called the *arm of the force P with respect to the point O*.

The moment of the force $\mathbf{P}$ with respect to the axis l is obtained by selecting an arbitrary point O on l and then forming the projection on the axis l of the moment of the force P with respect to O (p. 18).

If a sense is given on the line l, then the moment of the force $\mathbf{P}$ with respect to the axis l will be defined by giving its component with respect to this axis. This component is also called (if an error is precluded) the moment of the force $\mathbf{P}$ with respect to the axis l.

If the axis l passes through the point $O(x_0, y_0, z_0)$ and forms with the axes of the coordinate system the angles α, β, γ, then denoting by $\mathbf{M}$ the moment of the force $\mathbf{P}$ with respect to O, and by M_l the moment with respect to the axis l, we get

$$M_l = M_x \cos\alpha + M_y \cos\beta + M_z \cos\gamma \tag{4}$$

or, in virtue of (1),

$$\begin{aligned} M_l = P_x[(y - y_0)\cos\gamma - (z - z_0)\cos\beta] + \\ + P_y[(z - z_0)\cos\alpha - (x - x_0)\cos\gamma] + \\ + P_z[(x - x_0)\cos\beta - (y - y_0)\cos\alpha]. \end{aligned} \tag{5}$$

In particular, if the point O, through with the axis l passes, is the origin of the coordinate system, i. e. if $x_0 = y_0 = z_0 = 0$, we obtain

$$M_l = P_x[y\cos\gamma - z\cos\beta] + P_y[z\cos\alpha - x\cos\gamma] + P_z[x\cos\beta - y\cos\alpha]. \tag{6}$$

The projections M_x, M_y, M_z, in formulae (1) and (4) are the moments of the force $\mathbf{P}$ with respect to axes parallel to the axes x, y, z, and passing through O, whereas in formulae (2) they are the moments with respect to the axes x, y, z.

If we denote the distance of the axis l from the force $\mathbf{P}$ by d (more exactly: from the position of the force $\mathbf{P}$, i. e. the line on which $\mathbf{P}$ lies), and the angle between l and $\mathbf{P}$ by α (Fig. 149), we obtain (p. 18, formula (III))

$$|M_l| = |\mathbf{P}|d\sin\alpha. \tag{7}$$

If, in particular, $\mathbf{P} \perp l$, or $\alpha = \frac{1}{2}\pi$, then

$$|M_l| = |\mathbf{P}|d. \tag{8}$$

The sign of the moment M_l is obtained from the following rule:

$M_l > 0$ if the force $\mathbf{P}$ tries to turn the body about the axis l counterclockwise (with respect to a person whose feet are at an

arbitrary point O of the axis l, and whose head points in the direction of the axis l); in the contrary case $M_l < 0$.

By means of the above rule and formula (7) we can determine M_l, knowing $|\mathbf{P}|$, d, and α.

If the force $\mathbf{P}$ and the point O lie in a certain plane Π (Fig. 150), then the moment $\mathbf{M}$ of the force $\mathbf{P}$ with respect to O is perpendicular to the plane Π. Consequently $\mathbf{M}$ is equal to the moment of the force $\mathbf{P}$ with respect to the axis l, perpendicular to Π and passing through O:

$$|\mathbf{M}| = |M_l|.$$

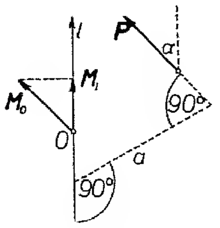


Fig. 149.

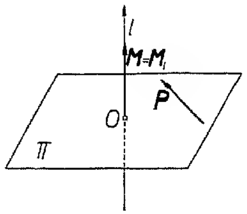


Fig. 150.

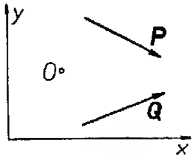


Fig. 151.

If we consider, for example, a system of forces lying in the xy-plane, then assuming that O also lies in xy, we have $M_x = 0$ and $M_y = 0$. The moment with respect to an axis parallel to z, i. e. M_z, is then called briefly the *moment with respect to O* and we denote it simply by M. Therefore

$$M = P_x(y - y_0) - P_y(x - x_0) \quad \text{or} \quad M = P_x y - P_y x. \tag{8}$$

Let us suppose, for example, that we have drawn the x and y axes as in Fig. 151. Therefore the z-axis should be taken directed vertically downwards. Hence if we want to determine the moment of the force $\mathbf{P}$ with respect to some point O, it is necessary to remember that $M > 0$ if the force tries to turn the piece of paper about O clockwise (i. e. as in Fig. 151); in the contrary case $M < 0$, as for the force $\mathbf{Q}$.

Given the arm h, we can therefore obtain M from formula (3), determining the sign in the manner given above.

Equilibrium of forces. If a rigid body is at rest we say that it is *in equilibrium*. The forces acting on a rigid body which remains in equilibrium are said to *balance one another* (to be in equilibrium) or to *annul one another*.

Statics is concerned with the investigation of conditions which forces in equilibrium must satisfy.

It is necessary to note the difference that exists between the equilibrium of a body and the equilibrium of forces. A body is in equilibrium then, and only then, when it is at rest. If a body is in equilibrium, then the system of forces acting on it is in equilibrium. Conversely, however, if a system of forces acting on a body is in equilibrium, it does not follow necessarily that the body is in equilibrium, since it can move e. g. with a uniformly advancing motion.

At this time we shall deduce conditions for the equilibrium of forces independently of the principles of dynamics by assuming certain hypotheses which are rather obvious. We shall show later (in chapter IX) that the conditions for equilibrium follow from the so-called *principle of virtual work*.

§ 3. Hypotheses for the equilibrium of forces. In order to deduce the conditions for the equilibrium of a rigid body, we shall assume the following hypotheses:

I. *To a system of forces acting on a rigid body which is in equilibrium we can add (or remove from the system) without disturbing equilibrium:*

a) *two forces equal in magnitude and acting along the same line, but oppositely directed* (Fig. 152a);

b) *several forces having a common point of application and whose sum is zero* (Fig. 152b).

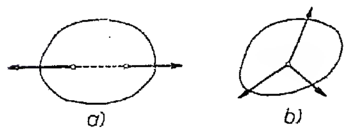


Fig. 152.

II. *Zero forces balance one another*; in other words: *if no forces act on a rigid body, then the body can remain in equilibrium.*

These hypotheses can be verified experimentally. We shall deduce from them the necessary and sufficient conditions for the equilibrium of forces. For the time being we shall be concerned with certain corollaries resulting from the assumed hypotheses.

§ 4. Transformation of systems of forces. Making use of the definition of elementary transformations (p. 28), we can formulate hypothesis I as follows:

I′. *If a rigid body is in equilibrium, we can perform arbitrary elementary transformations on the system of acting forces without disturbing equilibrium.*

Change of the point of application of a force. From theorem 1, p. 28 it follows that

1° *the point of application of a force can be chosen anywhere on its line of action.*

In the case of equilibrium the action of a force will therefore be defined if we give its magnitude, direction, sense, and position; the point of application of the force is immaterial. In virtue of the theorem on p. 18. we conclude from this that the action of the force $\boldsymbol{P}$ will be determined if we give its projections and the projections of its moment $\boldsymbol{M}$ with respect to an arbitrary point. The projections:

$$P_x, P_y, P_z, \quad M_x, M_y, M_z, \tag{1}$$

therefore define the action of a force on a rigid body. Let us note that since $\boldsymbol{M} \perp \boldsymbol{P}$, the scalar product $\boldsymbol{M} \cdot \boldsymbol{P}$ is zero, whence

$$M_x P_x + M_y P_y + M_z P_z = 0. \tag{2}$$

In general, therefore, five of the numbers (1) are sufficient to define a force; the sixth can be determined from equation (2).

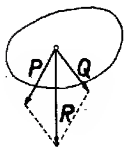


Fig. 153.

Law of composition and resolution of forces. From theorems 2 and 3, p. 30, we conclude that:

2° *several forces acting at one point can be replaced by their sum acting at this same point;*

3° *each force can be replaced by several forces having the same origin as the given force and a sum equal to the given force.*

These theorems are known as the *law of composition and resolution of forces.*

Equipollent systems. From hypothesis I and theorem 3, p. 30, we conclude that:

4° *a system of forces acting on a rigid body can be replaced by an arbitrary equipollent system.*

In other words: *equipollent systems of forces act on a rigid body in the same manner*; hence the importance of the notion of the equipollence of systems. It is easy to see that theorem 4° includes theorems 1°, 2°, and 3°.

As we know, two systems of forces are equipollent if they have equal sums and equal total moments with respect to one point (p. 22). By theorem 4° the action of a system of forces on a rigid body will therefore be defined if we give the sum $\boldsymbol{R}$ and the total moment $\boldsymbol{M}$ of the system of forces with respect to an arbitrary point.

Let the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$, whose points of application $A_1, A_2, \ldots$ have coordinates $x_1, y_1, z_1,\ x_2, y_2, z_2, \ldots$, act on a rigid body. Denoting the sum by $\boldsymbol{R}$ and the total moment with respect to the origin of tho system by $\boldsymbol{M}$, we obtain from formula (2), p. 232,

$$R_x = \Sigma P_{i_x}, \quad R_y = \Sigma P_{i_y}, \quad R_z = \Sigma P_{i_z}, \tag{3}$$

$$M_x = \Sigma(P_{i_y} z_i - P_{i_z} y_i),\ M_y = \Sigma(P_{i_z} x_i - P_{i_x} z_i),\ M_z = \Sigma(P_{i_x} y_i - P_{i_y} x_i).$$

The action of a system of forces is hence defined by means of the six numbers $R_x, R_y, R_z, M_x, M_y, M_z$.

The parameter of the system (p. 21) is $K = \boldsymbol{R} \cdot \boldsymbol{M}$, i. e.

$$K = R_x M_x + R_y M_y + R_z M_z. \tag{4}$$

Force couple. A system consisting of two forces equal in magnitude, parallel, but oppositely directed, is called a *force couple* (p. 23). The moment of a couple does not depend on the choice of the point with respect to which the moment is determined (p. 23). Since the sum of the forces of a couple is zero, two couples are equipollent if they have equal moments. Therefore the action of a force couple on a rigid body is defined by giving its moment.

A force couple tries to turn a body. The action of a couple does not undergo a change if the couple is arbitrarily translated and rotated in its plane (without changing the sense of the moment). A couple can also be arbitrarily translated in space without a change of the sense of its moment so that in every position it remains in a parallel plane (Fig. 154).

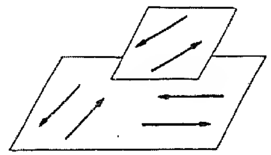


Fig. 154.

A couple whose moment is equal to zero is equipollent to a zero vector. Such a couple is also called a *zero couple.*

Reduction of a system of forces. The theorems concerning the reduction of systems (§ 16, p. 23) enable one to determine the simplest system of forces equipollent to the given one (i. e. the simplest system of forces by which one can replace the given system). In particular, the theorem on reduction can be stated as follows:

Every system of forces acting on a rigid body can be replaced:

a) *either by one force equal to the sum of the forces of the system and acting at an arbitrary point O, and a force couple whose moment is equal to the moment of the system with respect to O,*

b) *or by two forces, one of which acts at an arbitrarily chosen point.*

The theorems given on pp. 25 and 26 can be stated in a similar manner.

Let a force $\boldsymbol{P}$ whose origin is at the point A act on a rigid body. Let us choose an arbitrary point O. From the theorem on reduction it follows (if

the system is assumed to be the force $\mathbf{P}$) that *the force* $\mathbf{P}$ *can be replaced by an equal force acting at* O, *and by a force couple whose moment is equal to the moment of the force* $\mathbf{P}$ *with respect to* O.

Plane system of forces. If a system of forces lies in one plane, then their system is called a *plane system*. By theorem 3, p. 26, *a plane system of forces either has a resultant or is equipollent to a force couple.*

From the table given on p. 25 we see that a plane system has a resultant if the sum of the forces of a system is different from zero, or if the sum as well as the total moment are equal to zero; on the other hand, if the sum is zero and the total moment is different from zero, then the system is equipollent to a couple.

In the xy-plane let there be given a plane system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots,$ acting at the points $A_1, A_2, \ldots$ whose coordinates are $x_1, y_1, x_2, y_2, \ldots$

The projections of the forces P_{i_z} on the z-axis as well as the coordinates z_i of the points A_i are zero. Therefore, denoting the sum of the forces by $\mathbf{R}$, and the total moment with respect to the origin of the system by $\boldsymbol{M}$, we obtain from formulae (3), p. 237:

$$R_z = 0, \quad M_x = 0, \quad M_y = 0.$$

Hence the action of a plane system of forces is determined by three numbers: R_x, R_y, and M_z.

From formulae (3), p. 237, we also obtain (writing M instead of M_z):

$$R_x = \Sigma P_{i_x}, \quad R_y = \Sigma P_{i_y}, \quad M = \Sigma(P_{i_x} y_i - P_{i_y} x_i). \tag{5}$$

Parallel system of forces. From theorem 4, p. 26, it follows that *a parallel system of forces has a resultant or is equipollent to a force couple.*

Let the parallel forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ (Fig. 155) have origins at the points $A_1, A_2, \ldots$ whose coordinates are $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ Let us assume that the sum of the forces $\mathbf{R}$ is different from zero. Consequently the given system has a resultant.

Let us select the sense of an arbitrary force of the system, e. g. the sense of the force $\mathbf{P}_1$ as positive. Let us denote for $i = 1, 2, \ldots$ by P_i the number whose absolute value is equal to $|\mathbf{P}_i|$ and whose sign is positive or negative depending on whether $\mathbf{P}_i$ has a positive sense (i. e. agreeing with the sense of $\mathbf{P}_1$) or not. We define R similarly. We have $R = \Sigma P_i$.

On p. 28 we proved that the resultant $\mathbf{R}$ passes through a certain point O called the *centre of forces*. The coordinates x_0, y_0, z_0, of the centre of forces are obtained from formula (4), p. 28, by putting $a_i = P_i$:

$$x_0 = \Sigma P_i x_i \,/\, R, \quad y_0 = \Sigma P_i y_i \,/\, R, \quad z_0 = \Sigma P_i z_i \,/\, R. \tag{6}$$

If the forces are rotated about their points of application through the same angle so that they still remain parallel (as e. g. the dotted vectors in Fig. 155), then the centre of forces does not undergo a change. This follows from formulae (6) because the coordinates x_0, y_0, z_0, depend only on P_i, x_i, y_i, and z_i, and do not depend on the direction of the forces. The new resultant will therefore also pass through O.

If the points of application of the forces lie in one plane (or on one line), then the centre of forces also lies on this plane (or on this line).

For assuming that the points of application lie in the plane Π and choosing this plane as the xy-plane, we obtain $z_1 = z_2 = \ldots = 0$; from formulae (6) we therefore get $z_0 = 0$, which means that the centre of forces lies in the plane Π.

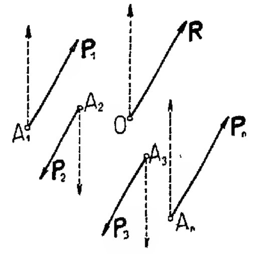


Fig. 155.

Similarly, if the points of application lie on one line l, then choosing it as the x-axis, we have $y_1 = y_2 = \ldots = 0$, and $z_1 = z_2 = \ldots = 0$; hence by (6) $y_0 = 0$ and $z_0 = 0$; consequently the centre of forces lies on the line l.

Let the material points whose masses are $m_1, m_2, \ldots$ be acted upon by forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ which are parallel, have the same sense, and are in magnitude proportional to the masses of the individual points. Putting $P_1 = |\mathbf{P}_1|$, $P_2 = |\mathbf{P}_2|$, $\ldots$, we obtain:

$$P_1 = km_1, \quad P_2 = km_2, \ldots \quad R = P_1 + P_2 + \ldots = km, \tag{7}$$

where k is the factor of proportionality, and $m = m_1 + m_2 + \ldots$ From formulae (6) and (7) we get after substitution:

$$x_0 = \Sigma m_i x_i \,/\, m, \quad y_0 = \Sigma m_i y_i \,/\, m, \quad z_0 = \Sigma m_i z_i \,/\, m.$$

Comparing these equalities with formulae (I), p. 152, we see that the centre of forces is the centre of mass of a given system of material points.

Therefore: *the centre of mass of a system of material points is the centre of forces which are parallel, have the same sense and are in magnitude proportional to the masses of the points on which they act.*

Gravitational forces. Let a rigid body be situated in a gravitational field. Consider the body as a system of material points of masses $m_1, m_2, \ldots,$ we can assume that the weights of the separate points are parallel forces, having the same sense (vertically downwards). The weights therefore have a resultant (Fig. 156).

The magnitudes of the weights of the separate points are $Q_1 = m_1 g$, $Q_2 = m_2 g, \ldots$ (where g denotes the acceleration of gravity). Consequently

the magnitudes of the weights are proportional to the masses of the points. Therefore in virtue of the preceding theorem, the centre of the gravitational forces is the centre of mass of the body. The magnitude of the resultant is

$$Q = m_1 g + m_2 g + \ldots = (m_1 + m_2 + \ldots)\, g = mg,$$

where m denotes the mass of the body.

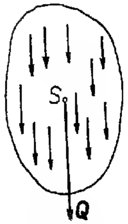


Fig. 156.

Therefore: *in every position of a body the resultant of the gravitational forces passes through the centre of gravity of the body. The weight of the body* (i. e. the resultant of the gravitational forces acting on its separate points) *is*

$$Q = mg, \tag{8}$$

where m denotes the mass of the body, and g the acceleration of gravity.

On the basis of the above theorem we can replace the action of the force of gravity by one force situated at the centre of gravity of the body.

Systems of couples. A system consisting of several couples has a zero sum. From the table on p. 25 it follows that such a system is equipollent to a couple or to a zero vector (i. e. to a zero couple). Let $\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots$ denote the moments of the individual couples. Then the total moment will be $\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2 + \ldots$ From the theorem on reduction (p. 24) we therefore obtain the following theorem:

A system consisting of several couples is equipollent to one couple whose moment is equal to the total moment of the system.

Let us note that a force couple (whose moment is different from zero) cannot be equipollent to one force. For in view of the fact that the sum of the forces of the couple is zero, this force would have to be zero and its moment different from zero, which is impossible.

Example 1. The centres of the sides of a plane polygon $A_1, A_2, \ldots, A_n$ (Fig. 157) are acted upon by forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, lying in the plane of the polygon, forming with its sides $\overline{A_1A_2}, \overline{A_2A_3}, \ldots, \overline{A_nA_1}$ an angle φ, and directed towards the exterior and proportional in magnitude to the sides of the polygon.

It is easy to see that the sum of the forces is zero, because forming the sum $\boldsymbol{P}_1 + \boldsymbol{P}_2 + \ldots + \boldsymbol{P}_n$, we obtain a polygon similar to the given one, but turned through an angle φ relative to it.

The system of forces is therefore equipollent to a couple or zero.

Let us select an arbitrary system of coordinates $O(x, y)$ and denote by $x_1, y_1,\ x_2, y_2, \ldots$ the coordinates of the points $A_1, A_2, \ldots$ The point of application of the force $\boldsymbol{P}_1$ has the coordinates $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$. Therefore the moment of the force $\boldsymbol{P}_1$ with respect to O is ((8), p. 234)

$$M_1 = \tfrac{1}{2}[P_{1_x}(y_1 + y_2)] - \tfrac{1}{2}[P_{1_y}(x_1 + x_2)]. \tag{9}$$

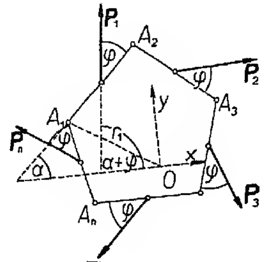


Fig. 157.

By hypothesis

$$|\boldsymbol{P}_1| = \lambda d_1, \tag{10}$$

where $d_1 = \overline{A_1A_2}$ and λ is the factor of proportionality. If $\overline{A_1A_2}$ forms an angle α with the x-axis, then the force $\boldsymbol{P}_1$ forms an angle $\alpha + \varphi$ with the x-axis. Therefore:

$$P_{1_x} = |\boldsymbol{P}_1| \cos(\alpha + \varphi), \quad P_{1_y} = |\boldsymbol{P}_1| \sin(\alpha + \varphi).$$

Hence in virtue of (10), $P_{1_x} = \lambda d_1(\cos\alpha\cos\varphi - \sin\alpha\sin\varphi)$. But $d_1\cos\alpha = x_2 - x_1$, and $d_1 \sin\alpha = y_2 - y_1$; consequently

$$P_{1_x} = \lambda[(x_2 - x_1)\cos\varphi - (y_2 - y_1)\sin\varphi].$$

Similarly

$$P_{1_y} = \lambda[(y_2 - y_1)\cos\varphi + (x_2 - x_1)\sin\varphi].$$

Substituting in (9), we obtain

$$M_1 = \tfrac{1}{2}\lambda[2(y_1x_2 - y_2x_1)\cos\varphi + (y_1^2 + x_1^2 - y_2^2 - x_2^2)\sin\varphi]. \tag{11}$$

Putting $OA_1 = r_1$, $OA_2 = r_2, \ldots$ and denoting by $p_1, p_2, \ldots$ the areas of the triangles OA_1A_2, $OA_2A_3, \ldots$, we get $r_1^2 = x_1^2 + y_1^2$, $r_2^2 = x_2^2 + y_2^2, \ldots$, $p_1 = \frac{1}{2}(y_1x_2 - y_2x_1)$, etc. Hence by (11)

$$M_1 = \tfrac{1}{2}\lambda[4p_1\cos\varphi + (r_1^2 - r_2^2)\sin\varphi]. \tag{12}$$

Similar expressions are obtained for the moments of the remaining forces.

The total moment of the forces with respect to O is $M = M_1 + M_2 + \ldots$ Hence according to (12)

$$M = \tfrac{1}{2}\lambda[4(p_1 + p_2 + \ldots + p_n)\cos\varphi + (r_1^2 - r_2^2 + r_2^2 - r_3^2 + \ldots + r_n^2 - r_1^2)\sin\varphi].$$

Since $p = p_1 + p_2 + \ldots + p_n$ is the area of the polygon $A_1A_2 \ldots A_n$,

$$M = 2\lambda p\cos\varphi. \tag{13}$$

The total moment is therefore proportional to the area of the polygon and to the cosine of the angle φ.

In particular, if the forces are perpendicular to the sides of the polygon, then $\varphi = \frac{1}{2}\pi$ and $\cos\varphi = 0$, whence in virtue of (13) $M = 0$, i. e. the forces form a system equipollent to zero.

On the other hand, if the forces are directed along the sides (i. e. if $\varphi = 0$), we have by (13) $M = 2\lambda p$, and hence the moment is then proportional to the area of the polygon.

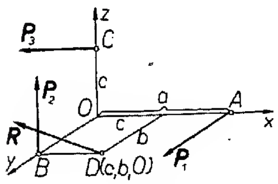


Fig. 158.

Example 2. The points $A(a, 0, 0)$, $B(0, b, 0)$, and $C(0, 0, c)$, on the axes of the coordinate system (x, y, z) are the points of application of the forces $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, parallel to the axes of the system, equal in magnitude and having senses as shown in the Fig. 158. What relation exists among the coordinates a, b, c if the system has a resultant?

Let us put $P = |\mathbf{P}_1| = |\mathbf{P}_2| = |\mathbf{P}_3|$. The sum of the forces $\mathbf{R}$ therefore has the projections

$$R_x = -P, \quad R_y = P, \quad R_z = P. \tag{14}$$

Let us calculate the total moment $\mathbf{M}$ with respect to O. The moment of the forces $\mathbf{P}_1$ and $\mathbf{P}_3$ with respect to the x-axis is zero; the moment of the force $\mathbf{P}_2$ with respect to the x-axis is $-Pb$. Hence $M_x = -Pb$; similarly $M_y = Pc$ and $M_z = -Pa$. The parameter of the system is $K = \mathbf{R}\cdot\mathbf{M} = R_xM_x + R_yM_y + R_zM_z$; therefore

$$K = P^2(b + c - a).$$

If the system has a resultant, then $K = 0$ (p. 26). Consequently

$$b + c - a = 0. \tag{15}$$

Equation (15) constitutes the s u f f i c i e n t condition and, as is easily seen from the table on p. 25, also the n e c e s s a r y condition that the system have a resultant, because $\mathbf{R} \neq 0$.

As the point of application of the resultant we can take the point $D(x, y, z)$ with respect to which the total moment is zero.

Let us denote the total moment with respect to D by $\mathbf{M}'$. We have:

$$M'_x = -Pz - P(b - y), \quad M'_y = -Px + P(c - z),$$
$$M'_z = -P(a - x) + Py.$$

Assuming that the moment with respect to D is zero, we get:

$$y - z = b, \quad z + x = c, \quad x + y = a.$$

On account of (15) these equations are dependent. Two of them are the equations of the line on which the resultant lies. Putting $z = 0$, for example, we obtain $x = c$, and $y = b$. Therefore we can take the point $D(c, b, 0)$ as the point of application of the resultant.

Example 3. Parallel forces $\mathbf{P}$ and $\mathbf{Q}$ act at the points A and B, $\mathbf{P} + \mathbf{Q} \neq 0$. Determine the center of forces.

The center of forces lies on the line AB (p. 239). Let us choose it as the x-axis, taking the point A as the origin of the x-axis and giving it a sense such that the point B lies on its positive part. Let us put $P = |\mathbf{P}|$ and denote by Q the number whose absolute value is equal to $|\mathbf{Q}|$, while the sign is + or — depending on whether $\mathbf{Q}$ has a sense which agrees, or does not agree, with that of the force $\mathbf{P}$. Putting $AB = d$ and denoting the coordinate of the centre of forces O by x_0, we get from formula (6), p. 238,

$$x_0 = Qd\,/\,R, \quad \text{where} \quad R = P + Q.$$

If the forces $\mathbf{P}$ and $\mathbf{Q}$ have the same sense (Fig. 159a), then $Q > 0$, and consequently $0 < Q/R < 1$, whence $0 < x_0 < d$. The centre of forces is therefore situated between the points A and B.

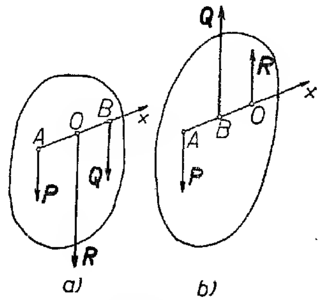


Fig. 159.

On the other hand, if the forces $\mathbf{P}$ and $\mathbf{Q}$ have opposite senses and e. g. $|\mathbf{P}| < |\mathbf{Q}|$ (Fig. 159b), then $Q < 0$, and $R < 0$, whence $x_0 > 0$. Furthermore $|R| < |Q|$; consequently $x_0 > d$. The centre of forces hence lies beyond the point B.

It is easy to verify that in both cases $AO\,/\,BO = |\mathbf{Q}|\,/\,|\mathbf{P}|$.

Therefore: *the centre of two parallel forces* (whose sum $\neq 0$) *is situated on the line joining the points of application of these forces.*

If the forces have the same senses, then their centre lies between the points of application; in the opposite case it lies beyond the point of application of that force whose absolute value is greater.

The distances of the centre of forces from the points of application are inversely proportional to the magnitudes of these forces.

Example 4. Forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ whose origins are $A_1, A_2, \ldots$, and forces $\mathbf{Q}_1, \mathbf{Q}_2, \ldots$ whose origins are $B_1, B_2, \ldots$, act on a rigid rod AB. All forces are parallel to each other and perpendicular to the rod, and the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ have a sense opposite to that of the forces $\mathbf{Q}_1, \mathbf{Q}_2, \ldots$ (Fig. 160).

Let $\mathbf{R} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{Q}_1 + \mathbf{Q}_2 + \ldots$ Let us denote by $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$ the absolute values of the forces, and by $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ the corresponding lengths of the segments $AA_1, AA_2, \ldots$ and $AB_1, AB_2, \ldots$ Let us assume the sense of the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ as positive. Put

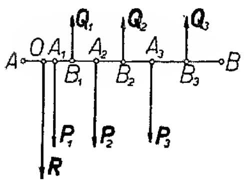


Fig. 160.

$$R = P_1 + P_2 + \ldots - Q_1 - Q_2 - \ldots \quad (16)$$

Obviously $|R| = |\mathbf{R}|$. If $R > 0$, the sum $\mathbf{R}$ has a sense agreeing with the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ However, if $\mathbf{R} < 0$, then $\mathbf{R}$ has a sense agreeing with the forces $\mathbf{Q}_1, \mathbf{Q}_2, \ldots$

Let us calculate the total moment M of the forces with respect to A. Denoting by $M_1, M_2, \ldots$ and $M_1^1, M_2^1, \ldots$ the moments of the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ and $\mathbf{Q}_1, \mathbf{Q}_2, \ldots$ with respect to A, we have (according to the agreement concerning the sign of the moment assumed on p. 233):

$$M_1 = P_1 a_1,\ M_2 = P_2 a_2,\ \ldots,\quad M_1^1 = -Q_1 b_1,\ M_2^1 = -Q_2 b_2,\ \ldots$$

Consequently

$$M = P_1 a_1 + P_2 a_2 + \ldots - Q_1 b_1 - Q_2 b_2 - \ldots \quad (17)$$

Let us assume that $\mathbf{R} = 0$. The system of forces is therefore equipollent to a couple of moment M according to formula (17).

If $M > 0$, the couple will tend to turn the rod clockwise, if $M < 0$ — counterclockwise. Finally, if $M = 0$, the system will be equipollent to zero.

Let us now assume that $\mathbf{R} \neq 0$. The system therefore has a resultant.

Let O be the origin of the resultant $\mathbf{R}$ lying on the line AB. Set $d = \pm AO$, taking the $+$ sign if the point O is on the same side of the point A as the origin of the force, and the $-$ sign in the contrary case. The moment of the resultant with respect to O, as is easily verified, is Rd according to our convention. Since the moment of the resultant is equal to the total moment, we get from (17)

$$d = \frac{1}{R}(P_1 a_1 + P_2 a_2 + \ldots - Q_1 b_1 - Q_2 b_2 - \ldots).$$

§ 5. Conditions for equilibrium of forces. We shall now prove the following

Theorem I. *In order that a system of forces acting on a rigid body be in equilibrium, it is necessary and sufficient that the sum of the forces and the total moment be zero, i. e. that the system of forces be equipollent to zero.*

Proof. We shall prove at first that the condition is necessary. Let us assume that the rigid body is a rigid system of material points $A_1, A_2, \ldots$ and that it is in equilibrium under the action of a given system of forces. Let us consider an arbitrary point A_i. Denote by $\mathbf{P}_i$ the sum of the external forces, and by $\mathbf{W}_i$ the sum of the internal forces acting at A_i. Since the point A_i is in equilibrium (because the entire rigid system is in equilibrium), it follows that $\mathbf{P}_i + \mathbf{W}_i = 0$. Consequently

$$\Sigma(\mathbf{P}_i + \mathbf{W}_i) = 0, \quad (1)$$

where the sum Σ extends over all the points A_i of the given rigid system.

From the law of action and reaction it follows that the sum of the forces with which two points react on each other is zero. Since all the internal forces can be grouped in pairs, the sum of the internal forces is zero or $\Sigma \mathbf{W}_i = 0$, whence by (1)

$$\Sigma \mathbf{P}_i = 0. \quad (2)$$

It follows from this that the sum of the external forces, i. e. the forces acting on the rigid body in equilibrium, is zero.

Let us now choose an arbitrary point O. Since the forces $\mathbf{P}_i$ and $\mathbf{W}_i$ have the common origin A_i, and moreover $\mathbf{P}_i + \mathbf{W}_i = 0$, it follows that (p. 17) $\mathrm{Mom}_O \mathbf{P}_i + \mathrm{Mom}_O \mathbf{W}_i = 0$, whence

$$\Sigma(\mathrm{Mom}_O \mathbf{P}_i + \mathrm{Mom}_O \mathbf{W}_i) = 0. \quad (3)$$

The total moment of the internal forces with which two points react on each other is — as is easily verified — zero. Consequently the total moment of all the internal forces is zero. Therefore $\Sigma \mathrm{Mom}_O \mathbf{W}_i = 0$, whence by (3)

$$\Sigma \mathrm{Mom}_O \mathbf{P}_i = 0. \quad (4)$$

We have proved, therefore, that the sum as well as the total moment of the forces acting on a rigid body is zero. This proves the necessity of the condition.

Let us now assume that the given system of forces is equipollent to zero. Since a system equipollent to zero is equipollent to a zero force, it is in equilibrium by hypothesis II (p. 235). The condition is therefore at the same time sufficient, q. e. d.

From theorem I it follows that if a system of forces acting on a rigid body is not equipollent to zero, then the body cannot be in equilibrium. In particular, a rigid body cannot remain in equilibrium under the action of a system of forces consisting of:

a) one force different from zero,

b) one force couple of moment different from zero,

c) one force different from zero and one couple of moment different from zero.

As we know (p. 22), a system of forces is equipollent to zero if the total moments with respect to three non-collinear points are zero. On the basis of theorem I we obtain from this the following

Theorem II. *In order that a system of forces acting on a rigid body be in equilibrium, it is necessary and sufficient that the moments of the system with respect to three non-collinear points be equal to zero.*

In applications we frequently find the following theorem useful:

Theorem III. *If a system consisting of three forces is in equilibrium, then these forces lie in one plane and are either parallel or their prolongations intersect in one point.*

Theorem III follows from the theorem of chapter I, § 14 (p. 22), in the proof of which it was shown that the vectors lie in one plane.

Analytic form of the conditions for equilibrium. Let us choose an arbitrary system of coordinates $O(x, y, z)$. Let the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ act on a rigid body at the points $A_1, A_2, \ldots$ whose coordinates are $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ Let us denote by $\mathbf{R}$ the sum, and by $\mathbf{M}$ the total moment, of the system with respect to O. According to theorem I (p. 244) the equations:

$$\mathbf{R} = 0, \quad \mathbf{M} = 0 \tag{5}$$

constitute the necessary and sufficient conditions for equilibrium. Forming the projections on the axes of the system, we obtain from (5) and from formulae (3), p. 237:

$$\Sigma P_{i_x} = 0, \quad \Sigma P_{i_y} = 0, \quad \Sigma P_{i_z} = 0, \tag{I}$$

$$\Sigma(P_{i_y} z_i - P_{i_z} y_i) = 0, \ \Sigma(P_{i_z} x_i - P_{i_x} z_i) = 0, \ \Sigma(P_{i_x} y_i - P_{i_y} x_i) = 0. \tag{II}$$

Equations (I) are called the *condition of projections* and equations (II) the *condition of moment.*

Equations (I) and (II) are the analytic form of the conditions for the equilibrium of a system of forces. From these equations we can determine in general six unknowns.

Plane systems of forces. The conditions for equilibrium obviously apply also to a plane system of forces.

Let the forces lie in the xy-plane. Since $P_{i_z} = 0$ and $z_i = 0$, the conditions for equilibrium (I) and (II) assume the form:

$$\Sigma P_{i_x} = 0, \quad \Sigma P_{i_y} = 0, \tag{I'}$$

$$\Sigma(P_{i_x} y_i - P_{i_y} x_i) = 0. \tag{II'}$$

In the case of a plane system we therefore obtain three equations. From them we can in general determine three unknowns.

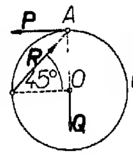
Example 1. A heavy sphere is in equilibrium under the action of three forces (Fig. 161): the weight $\mathbf{Q}$ (acting at the centre of the sphere O), the horizontal force $\mathbf{P}$ (acting at the point A situated on the surface of the sphere at the end of the vertical diameter) and the force $\mathbf{R}$ (acting at the point B situated on the surface of the sphere at the end of the horizontal diameter). The weight $\mathbf{Q}$ is given. Determine the forces $\mathbf{P}$ and $\mathbf{R}$.

The forces $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$, are in equilibrium; therefore by theorem III they lie in a plane and their directions intersect in one point which is the point A. Consequently the force $\mathbf{R}$ has the direction of the line BA which forms an angle of 45° with the horizontal. Since $\mathbf{Q} + \mathbf{P} + \mathbf{R} = 0$, knowing the force $\mathbf{Q}$ and the directions of the forces $\mathbf{P}$ and $\mathbf{R}$, we can form a triangle of forces (Fig. 162). From this triangle we obtain

$$P = Q, \quad R = Q / \cos 45° = \sqrt{2} Q,$$

where P, Q and R, denote the absolute values of the forces.

Example 2. The vertices of a square $ABCD$ of side a are the points of application of four forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$, lying in the plane of the square and forming with the sides the angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (Fig. 163). Give the conditions for equilibrium.



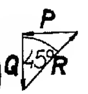


Fig. 161. Fig. 162.

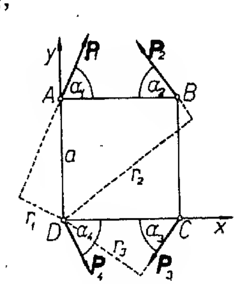


Fig. 163.

Let us denote the absolute values of the forces by P_1, P_2, P_3, P_4. Let us select the x and y axes along the sides of the square. Forming the projections of the forces on the x and y axes, we get in the case of equilibrium (when the forces have senses as shown in Fig. 163):

$$P_1 \cos\alpha_1 - P_2 \cos\alpha_2 - P_3 \cos\alpha_3 + P_4 \cos\alpha_4 = 0, \tag{6}$$

$$P_1 \sin\alpha_1 + P_2 \sin\alpha_2 - P_3 \sin\alpha_3 - P_4 \sin\alpha_4 = 0. \tag{7}$$

Denoting the arms of the forces with respect to the vertex D by r_1, r_2, r_3, r_4, we obtain:

$$r_1 = a \cos \alpha_1, \quad r_2 = \sqrt{2}a \sin(\alpha_2 + 45°), \quad r_3 = a \sin \alpha_3.$$

Moreover $r_4 = 0$. Since in the case of equilibrium the total moment with respect to D is zero (taking the sign of the moment according to the rule given on p. 233) we get after dividing by a

$$P_1 \cos \alpha_1 - P_2\sqrt{2} \sin(\alpha_2 + 45°) + P_3 \sin \alpha_3 = 0. \tag{8}$$

Equations (6), (7), and (8), constitute the necessary and sufficient condition for equilibrium.

Example 3. A rod AB lying in the horizontal xy-plane is acted upon by the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, lying in this plane and acting at the points $A_1, A_2, \ldots, A_n$ (Fig. 164). Give the conditions which the forces must satisfy in order that the rod be in equilibrium for every position in the xy-plane, if we assume that the forces do not change their magnitudes, directions, senses, or points of application (on the rod).

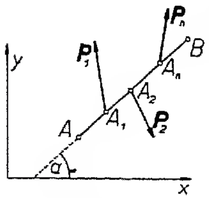


Fig. 164.

Let us consider an arbitrary position of the rod AB in the xy-plane. Denote by x_0, y_0 the coordinates of the point A, by $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ the coordinates of the points $A_1, A_2, \ldots$, and by α the angle which the rod AB makes with the x-axis. Let us put: $d_1 = AA_1, d_2 = AA_2, \ldots, d_n = AA_n$. We have

$$x_i = x_0 + d_i \cos \alpha, \; y_i = y_0 + d_i \sin \alpha \qquad \text{for } i = 1, 2, \ldots, n. \tag{9}$$

From the conditions of equilibrium (I′) and (II′), p. 247, we obtain

$$\Sigma P_{i_x} = 0, \quad \Sigma P_{i_y} = 0, \tag{10}$$

$$\Sigma(P_{i_x} y_i - P_{i_y} x_i) = \Sigma[P_{i_x}(y_0 + d_i \sin \alpha) - P_{i_y}(x_0 + d_i \cos \alpha)] = 0. \tag{11}$$

Condition (11) can be written in the form

$$y_0 \Sigma P_{i_x} - x_0 \Sigma P_{i_y} + \sin \alpha \, \Sigma P_{i_x} d_i - \cos \alpha \, \Sigma P_{i_y} d_i = 0, \tag{12}$$

whence by (10)

$$\sin \alpha \, \Sigma P_{i_x} d_i - \cos \alpha \, \Sigma P_{i_y} d_i = 0. \tag{13}$$

Since relation (13) has to hold for every angle α, we get for $\alpha = \frac{1}{2}\pi$ and then for $\alpha = 0$

$$\Sigma P_{i_x} d_i = 0, \quad \Sigma P_{i_y} d_i = 0. \tag{14}$$

Equations (10) and (14) are the necessary and sufficient conditions in order that the rod be in equilibrium for every position in the plane.

For if conditions (10) and (14) hold, it is easy to see that condition (13) holds, and consequently by (10) conditions (12) and (11) also hold.

Conditions (10) and (11) are, as we have seen, the necessary and sufficient conditions for equilibrium by (I′) and (II′).

§ 6. Graphical statics. String polygon. The problems which one meets in statics often lead to long and tedious computations. However, there exist graphical methods which enable one to obtain in many cases approximate solutions which are sufficiently accurate for applications.

These methods are of great importance in engineering for they lead more rapidly to the goal omitting intricate computations.

That part of theoretical statics which deals with graphical methods is called *graphical statics*.

Here we shall become acquainted only with some graphical methods as, for example, the graphical determination (by means of a string polygon) of the resultant of a plane system of forces and certain applications of these methods. Later (in § 16) we shall become acquainted with graphical methods serving to determine the stress in the bars of a frame.

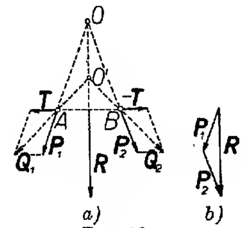


Fig. 165.

Composition of forces. Having two forces $\mathbf{P}_1$ and $\mathbf{P}_2$ whose directions intersect at the point O, we determine the sum $\mathbf{R}$ (Fig. 165a), and then we draw the resultant through the point O (Fig. 165b).

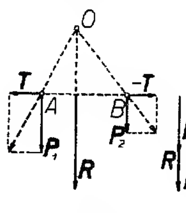
If the point O lies outside the limits of the drawing, we can proceed as follows: we add two forces $\mathbf{T}$ and $-\mathbf{T}$ acting at the points A and B (i. e. at the initial points of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$) and along the line AB. The system $\mathbf{T}, -\mathbf{T}, \mathbf{P}_1, \mathbf{P}_2$, is obviously equipollent to the system $\mathbf{P}_1, \mathbf{P}_2$, because the forces $\mathbf{T}$ and $-\mathbf{T}$ annul each other, and consequently the resultant of the new system of four forces is the same as before.

The forces $\mathbf{T}$ and $\mathbf{P}_1$ are replaced by the force $\mathbf{Q}_1 = \mathbf{T} + \mathbf{P}_1$ with its origin at A; similarly the forces $-\mathbf{T}$ and $\mathbf{P}_2$ are replaced by the force $\mathbf{Q}_2 = -\mathbf{T} + \mathbf{P}_2$ with its origin at B. The resultant $\mathbf{R}$ passes through the point of intersection O' of the forces $\mathbf{Q}_1, \mathbf{Q}_2$.

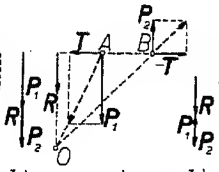
This construction can also be applied to the case of two parallel forces not forming a couple (Fig. 166a, 166b and 167a, 167b).

In this way we can obtain the resultant (or the resultant couple) of the system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ (Fig. 168a and 168b). We first form the resultant $\mathbf{R}_1$ of two of these forces (e. g. the forces $\mathbf{P}_1$ and $\mathbf{P}_2$) and we obtain a system consisting of only $n-1$ forces.

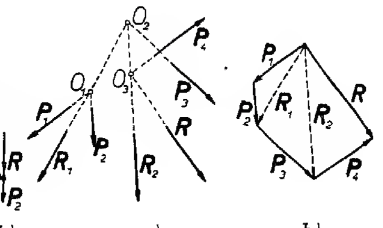
A method that we shall become acquainted with presently will lead us to the goal more quickly.



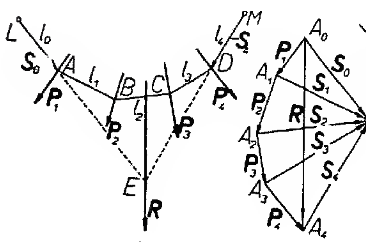
a) b)
Fig. 166.



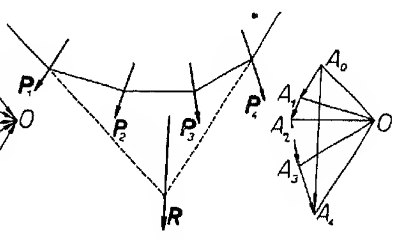
a) b)
Fig. 167.



a) b)
Fig. 168.



a) b)
Fig. 169.



a) b)
Fig. 170.

String polygon. Let us assume that we have to find the resultant of the system of forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$ (Fig. 169a).

We first form the sum $\mathbf{R} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_4$. The polygon obtained is called the *polygon of forces.*[1])

Let us denote (in the polygon of forces) by A_0 the origin of the force $\mathbf{P}_1$, and by A_1, A_2, A_3, A_4, the termini of the forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$. Let us now select an arbitrary point O outside the polygon of forces. This point is called the *pole.*

We connect the pole O with the points $A_0, A_1, \ldots, A_4$. From an arbitrary point A situated on the direction of the force $\mathbf{P}_1$ we draw the lines l_0 and l_1 parallel to the lines OA_0 and OA_1, respectively. The line l_1 is prolonged to the point B of its intersection with the direction of the

[1]) In Fig. 169a, 170a, and those appearing farther on, only the p o s i t i o n s of the forces are given. The m a g n i t u d e s of the forces are indicated in the force polygons (Fig. 169b, 170b, etc.).

force $\mathbf{P}_2$. The line l_1 will cut the direction of the force $\mathbf{P}_2$, because l_1 is parallel to OA_1, and OA_1 is not parallel to $\mathbf{P}_2$ (Fig. 169b).

From the point B we draw the line $l_2 \parallel OA_2$ to the point C of its intersection with the direction of the force $\mathbf{P}_3$. From the point C we draw the line $l_3 \parallel OA_3$ to the point D of its intersection with the direction of the force $\mathbf{P}_4$. From the point D we draw the line $l_4 \parallel OA_4$.

We now determine the point of intersection E of the lines l_0 and l_4. The resultant $\mathbf{R}$ passes through the point E. Since $\mathbf{R}$ is known from the polygon of forces, this resultant can be drawn.

We shall now justify the above construction.

Let us denote the vectors $\overline{A_0O}, \overline{A_1O_1}, \ldots, \overline{A_4O}$ by $\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_4$, respectively. From the force polygon (Fig. 169b) we obtain:

$$\begin{aligned} &\mathbf{P}_1 + \mathbf{S}_1 + (-\mathbf{S}_0) = 0, \quad \mathbf{P}_2 + \mathbf{S}_2 + (-\mathbf{S}_1) = 0, \\ &\mathbf{P}_3 + \mathbf{S}_3 + (-\mathbf{S}_2) = 0, \quad \mathbf{P}_4 + \mathbf{S}_4 + (-\mathbf{S}_3) = 0. \end{aligned} \tag{1}$$

Let us add to the system of forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$ the forces $\mathbf{S}_0$ and $-\mathbf{S}_0$ lying on the line l_0, the forces $\mathbf{S}_1$ and $-\mathbf{S}_1$ lying on l_1, etc., finally the forces $\mathbf{S}_4$ and $-\mathbf{S}_4$ lying on l_4. The system added is obviously equipollent to zero, because the forces $\mathbf{S}_0, -\mathbf{S}_0, \mathbf{S}_1, -\mathbf{S}_1$, etc. annul each other in pairs. The resultant of the enlarged system is therefore the same as before.

In virtue of (1) the forces $\mathbf{P}_1, \mathbf{S}_1$, and $-\mathbf{S}_0$, annul one another because their sum is zero and their directions intersect at A. These forces can therefore be removed. Similarly we can remove the forces $\mathbf{P}_2, \mathbf{S}_2$, and $-\mathbf{S}_1$, next, $\mathbf{P}_3, \mathbf{S}_3$, and $-\mathbf{S}_2$, etc., and finally the forces $\mathbf{P}_4, \mathbf{S}_4$, and $-\mathbf{S}_3$. The remaining forces $\mathbf{S}_0$ and $-\mathbf{S}_4$ consequently form a system equipollent to the given one. The resultant $\mathbf{R}$ therefore passes through the point of intersection E of the forces $\mathbf{S}_0$ and $-\mathbf{S}_4$ (i. e. of the lines l_0, l_4).

The segments l_0, l_1, l_2, l_3, l_4 form a so-called *string polygon*; l_0 and l_4 are called its *extreme sides.*

Therefore: *the resultant passes through the point of intersection of the extreme sides of the string polygon.*

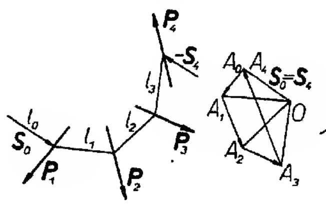
The name of string polygon arises from the fact that a weightless and inextensible string fastened at the points L and M on the lines l_0 and l_4 in the directions $-S_0$ and S_4 (in other respects arbitrary), and assuming the position of the polygon $LABCDM$, will be in equilibrium under the action of the forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$, whose points of application are at the points A, B, C, D, respectively.

In practice superfluous notations are omitted and the drawing is as in Fig. 170 as well as in the Fig. 171 and 172.

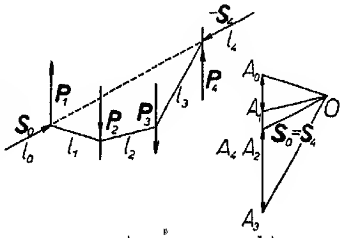
Fig. 171a represents a system of forces whose sum is zero. We then say that *the polygon of forces is closed* (Fig. 171b).

Drawing the string polygon we see that its extreme sides do not intersect (are parallel). We then say that *the string polygon is not closed.*

The system of forces in this case is equipollent to the system of forces $\mathbf{S}_0$ and $-\mathbf{S}_4$, which, as is seen from the polygon of forces, form a couple. The system is then equipollent to the force couple $\mathbf{S}_0$ and $-\mathbf{S}_4$ lying on the extreme sides of the string polygon.



a) b)
Fig. 171.



a) b)
Fig. 172.

Therefore: *if the force polygon is closed and the string polygon is not closed, then the system is equipollent to a force couple.*

In Fig. 172a we see a system of forces for which the polygon of forces (Fig. 172b) is closed and the extreme sides of the string polygon lie on one line. We then say that *the string polygon is closed.*

A system of forces in this case is equipollent to the system of forces $\mathbf{S}_0$ and $-\mathbf{S}_4$ lying on the extreme sides of the string polygon and hence on one straight line. Since (as is seen from the polygon of forces) $\mathbf{S}_0 = \mathbf{S}_4$, the forces $\mathbf{S}_0$ and $-\mathbf{S}_4$ balance each other; the given system is then equipollent to zero.

Therefore: *if the polygon of forces and the string polygon are closed, then the system of forces is equipollent to zero.*

Resultant of a part of a system. Having a string polygon of a certain system of forces, one can easily determine the resultant of an arbitrary part of the system consisting of the forces following each other in the polygon of forces.

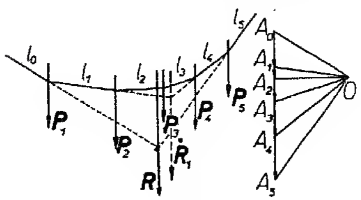


Fig. 173.

For example, let the system of parallel forces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4, \mathbf{P}_5$, be given (Fig. 173). Let the resultant $\mathbf{R}$ of the entire system and the resultant $\mathbf{R}_1$ of the forces $\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$ be determined. From Fig. 173 we see that the string polygon for the system $\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4$ is a part of the string polygon for the entire system.

§ 7. Some applications of the string polygon. **Determination of the reactions at the points of support of a beam.** A system of parallel forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5$, is given. Determine two forces $\mathbf{R}_1, \mathbf{R}_2$, parallel to the preceding and forming together with them a system equipollent to zero. The lines k_1 and k_2 on which the forces $\mathbf{R}_1$ and $\mathbf{R}_2$ are to lie are given.

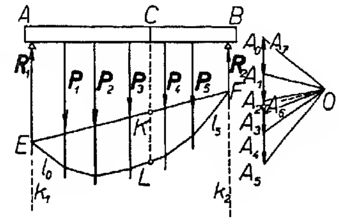


Fig. 174.

A problem like this occurs in the case of a rigid horizontal beam supported at the points A and B and acted upon by vertical forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5$ (Fig. 174). If there is no friction, the reactions at the points A and B are vertical and in equilibrium with the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5$ (p. 263).

In order to determine the forces $\mathbf{R}_1$ and $\mathbf{R}_2$, we draw the string polygon for the given system of forces in the order $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5, \mathbf{R}_2, \mathbf{R}_1$. In the polygon of forces the line A_6O joining the terminus of the force $\mathbf{R}_2$ with the pole O is for the moment unknown. Since the polygon of forces is closed, the point A_7, i. e. the terminus of the force $\mathbf{R}_1$, coincides with the initial point of the force $\mathbf{P}_1$, i. e. with the point A_0.

We draw the string polygon starting from the line $l_0 \parallel A_0O$ until we get to the line $l_5 \parallel A_5O$.

Let us denote by E, F the points of intersection of the lines l_0 and l_5 with the directions of the forces $\mathbf{R}_1$ and $\mathbf{R}_2$, i. e. with the given lines k_1 and k_2. Drawing in the polygon of forces the line OA_6 parallel to the line EF, we obtain the forces $\mathbf{R}_1 = \overline{A_5A_6}$ and $\mathbf{R}_2 = \overline{A_6A_7}$. For it is easy to see that by continuing the drawing of the string polygon for the forces $\mathbf{R}_1$ and $\mathbf{R}_2$ so determined, we obtain a closed string polygon.

Determination of the moment of forces. If we have to determine the moment of the force $\mathbf{P}$ with respect to a certain point A, we first draw the string polygon from an arbitrary point B situated on the direction of the force $\mathbf{P}$, as in Fig. 175.

Next, we pass through A a line parallel to $\mathbf{P}$. We denote the points of intersection of this line with the sides of the string polygon by L, K. From the similarity of triangles A_0A_1O and KLB we obtain $KL : |\mathbf{P}| = h : w$,

where h and w are the altitudes of these triangles. From this $|\mathbf{P}|h = KL \cdot w$. Denoting the moment of the force $\mathbf{P}$ with respect to A by $\mathbf{M}$, we have $|\mathbf{M}| = |\mathbf{P}|h$, or

$$|\mathbf{M}| = KL \cdot w. \tag{1}$$

Therefore: *the moment of the force* $\mathbf{P}$ *with respect to the point* A *is* (in absolute value) *proportional to the segment which the sides of the string polygon cut off from the line passing through* A *and parallel to* $\mathbf{P}$; *the factor of proportionality is the distance of the pole from the force in the polygon of forces.*

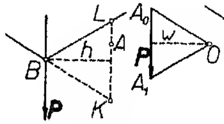


Fig. 175.

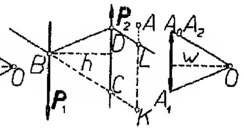


Fig. 176.

Let there now be given a force couple $\mathbf{P}_1$, $\mathbf{P}_2$ (where $\mathbf{P}_1 = -\mathbf{P}_2$). We draw the string polygon of this couple as in Fig. 176. Let us pass through an arbitrary point A a line parallel to the forces and denote by K and L the points of intersection of this line with the extreme sides of the string polygon.

We shall prove that if we denote the moment of the couple by $\mathbf{M}$ and the distance of the pole from the forces $\mathbf{P}_1$ and $\mathbf{P}_2$ in the polygon of forces by w, then formula (1) will hold.

For let us consider the triangle BCD, where B is a point on the direction of the force $\mathbf{P}_1$ from which we started to draw the string polygon, while C and D are the points of intersection of the direction of the force $\mathbf{P}_2$ with the extreme sides of the string polygon.

Let h denote the altitude of the triangle BCD. From the similarity of the triangles BCD and A_0A_1O we have $CD : |\mathbf{P}_1| = h : w$, from which $|\mathbf{P}_1|h = CD \cdot w$. Since $|\mathbf{M}| = |\mathbf{P}_1|h$ and $CD = KL$, we get formula (1).

Consequently: *the moment of a force couple is* (in magnitude) *proportional to the segment which the extreme sides of the string polygon cut off from an arbitrary line parallel to the force couple; the factor of proportionality is the distance of the pole from the force in the force polygon.*

Similar methods of determining the moment are useful when we are dealing with several parallel forces, because then we can take the same w for all the forces (Fig. 177). If we have to determine the moment of a system of parallel forces we determine at first the resultant (or the resultant force couple) and then its moment.

Having drawn the string polygon of a system of parallel forces, we can determine with respect to A the moment of an arbitrary part of the system consisting of the forces following each other in the same order as in the force polygon. This can be done because the string polygon of this part is included in the string polygon of the entire system. In the drawing the segment $K'L'$ is proportional (in absolute value) to the moment of the system of forces $\mathbf{P}_2$, $\mathbf{P}_3$ with respect to A.

Finally, let the system of forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5, \mathbf{R}_1, \mathbf{R}_2$ be given, equipollent to zero (Fig. 174). Through an arbitrary point C we pass a line parallel to the forces. The segment LK of this line lying between the sides of the string polygon is proportional (in absolute value) to the total moment with respect to C of the forces situated on one side of this line (in our case to the moment of the forces $\mathbf{R}_1$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ or to that of the forces $\mathbf{P}_4$, $\mathbf{P}_5$, $\mathbf{R}_2$; the moments of both parts of the system with respect to C are equal in absolute value, since their sum is zero as a consequence of the assumption that the system is equipollent to zero). The factor of proportionality is w, i. e. the distance of the pole O from the forces in the force polygon.

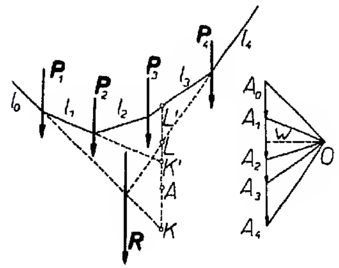


Fig. 177.

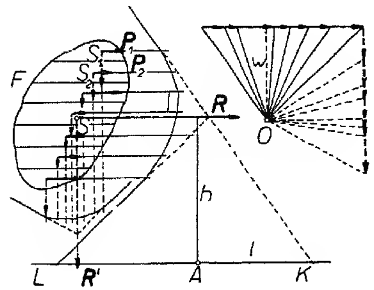


Fig. 178.

Determination of the centre of gravity and of the statical moment of plane figures. In order to determine the centre of gravity of a plane figure F, we divide it into strips by means of parallel lines. If at the centres of gravity $S_1, S_2, \ldots$ of the strips obtained we attach the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ which are parallel, have the same sense, and are proportional in magnitude to the areas $F_1, F_2, \ldots$ of these strips, then the centre of the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ will be the center of gravity of the figure F.

For let us note that the centre of gravity of the figure F is the centre of mass of the system of material points which is obtained if each strip is replaced by a material point whose mass is equal to the mass of the strip

(p. 154); and by the theorem proved on p. 239 the centre of mass of the system of material points obtained is the centre of the system of parallel forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$

Strips that are sufficiently narrow can be considered as trapezoids; the centres of gravity of the trapezoids can be determined according to the construction given on p. 177, Fig. 118. The lines of division of the figure are usually drawn at equal intervals (Fig. 178). Hence the areas of the trapezoids will be proportional to their medians. The magnitudes of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ can therefore be considered as proportional to the medians of the trapezoids.

The resultant $\boldsymbol{R}$ passes through the centre of gravity S; we determine it by means of the string polygon (Fig. 178).

Changing the direction of the forces and determining a new resultant $\boldsymbol{R}'$, we obtain the centre of gravity S as the point of intersection of both resultants $\boldsymbol{R}$ and $\boldsymbol{R}'$.

In order to determine the statical moment of the figure F with respect to a certain line l, we draw the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ parallel to l.

The moment of the resultant $\boldsymbol{R}$ with respect to an arbitrary point A of the line l is in magnitude proportional to the statical moment of the figure F with respect to l.

For denoting by $\boldsymbol{M}$ the moment of the force $\boldsymbol{R}$ with respect to A, by h the distance of A from the direction of the force $\boldsymbol{R}$, by M_s the statical moment of the given figure with respect to l, finally by F the area of the figure, we have

$$|\boldsymbol{M}| = h|\boldsymbol{R}|, \quad |M_s| = hF, \tag{2}$$

where the centre of gravity lies on the direction of the resultant.

Since the magnitudes of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ have been chosen as proportional to the areas $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots$ of the individual strips,

$$|\boldsymbol{P}_1| = \lambda F_1, \quad |\boldsymbol{P}_2| = \lambda F_2, \quad \text{etc.}, \tag{3}$$

where λ is the factor of proportionality. But

$$|\boldsymbol{R}| = |\boldsymbol{P}_1| + |\boldsymbol{P}_2| + \ldots,$$

whence

$$|\boldsymbol{R}| = \lambda(F_1 + F_2 + \ldots) = \lambda F.$$

In virtue of (2), therefore, $|\boldsymbol{M}| = \lambda h F = \lambda |M_s|$, whence

$$|M_s| = |\boldsymbol{M}| \,/\, \lambda. \tag{4}$$

Consequently: *the statical moment of a figure is* (in absolute value) *proportional to the moment of the resultant.*

Statical moments of plane figures can therefore be determined by means of a string polygon.

In Fig. 178, $|\boldsymbol{M}| = w \cdot KL$, whence by (4)

$$|M_s| = w \cdot KL \,/\, \lambda \tag{5}$$

Let us denote by $d_1, d_2, \ldots$ the lengths of the medians of the trapezoids and by a the distance between the lines of division. Therefore $F_1 = ad_1, F_2 = ad_2, \ldots$ Since it was assumed in the drawing that $|\boldsymbol{P}_1| = kd_1, |\boldsymbol{P}_2| = kd_2, \ldots$, where $k = \frac{1}{3}$, it follows that, $|\boldsymbol{P}_1| = kF_1 \,/\, a, |\boldsymbol{P}_2| = kF_2 \,/\, a, \ldots$ By (3) we then have $\lambda = k \,/\, a$, whence by (5)

$$|M_s| = aw \cdot KL \,/\, k = 3aw \cdot KL.$$

Measuring a, w, and KL, in the drawing, we obtain $|M_s|$ from the above formula.

II. CONSTRAINED BODY

§ 8. Conditions of equilibrium. A rigid body is said to be *constrained* if the positions or the motions of this body are subject to certain conditions. These conditions are called *constraints*.

For example, if one point of a body is fixed, the body can turn only about this point. If two points A and B are fixed, the body can turn only about the line AB. Later we shall learn of still other examples of constrained rigid bodies.

When a constrained rigid body is in equilibrium we say that the forces acting on this body *balance one another* or *are in equilibrium.*

A rigid body fixed at the two points A, B (Fig. 179) and being in equilibrium, will remain in equilibrium when we add an arbitrary force $\boldsymbol{P}$ whose origin is at the point C lying on the line AB. This is evident intuitively because the body can turn only about the axis AB, and hence the force $\boldsymbol{P}$ acting on the fixed axis cannot move the body. If the body were free, then it would remain in equilibrium only in the case when $\boldsymbol{P} = 0$.

We see from this that the conditions for the equilibrium of a free body are different from those for a constrained body.

The investigation of the conditions for equilibrium in the case of a constrained rigid body can be reduced to the case of a free body. With this in view we shall assume that besides the given forces the constrained rigid body is acted upon by additional forces called *reactions* which cause the body to maintain the constraints. The reactions arise from those bodies which limit the freedom of the motions of the given constrained rigid body.

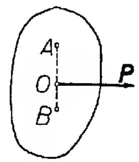


Fig. 179.

For example, if a heavy body rests on a table, then it is not free because it cannot pass through the surface of the table. In this case the reactions are the forces with which the surface of the table presses on the body.

The other forces acting on a constrained rigid body will be called *acting forces* (in order to differentiate them from the reactions). If we introduce the reactions to the acting forces, then we can consider the constrained rigid body as free.

It follows from this that *the necessary and sufficient condition for the equilibrium of the acting forces is that the acting forces balance the reactions.*

However, this condition is not convenient because it involves the forces of reaction which are in general unknown. In some instances, as in the case of a body fixed at one point or two points, we can nevertheless give conditions for the equilibrium of the acting forces without reference to the reactions (p. 270). The condition for equilibrium in which the reactions do not occur is the so-called *principle of virtual work* which we shall consider in chapter IX.

§ 9. Reactions of bodies in contact. Every two rigid bodies (solids, surfaces or lines) which are in contact with each other act on each other with certain forces. These forces are reactions and they arise from the actions of the points of both bodies on each other. Reactions conform to the law of action and reaction.

By the theorem on reduction (p. 237) the forces with which one body acts on the other can always be replaced by a force $\mathbf{R}$ and a couple of moment $\mathbf{M}$. Conversely, by the law of action and reaction, the second body acts on the first body with forces equipollent to the force $-\mathbf{R}$ (with the same origin as $\mathbf{R}$) and a couple of moment $-\mathbf{M}$.

The determination of reactions is very important in problems connected with engineering. So far we do not yet have a theory which solves this matter in its entirety. In practice we make use of certain hypotheses agreeing approximately with experience. We shall consider here only certain problems concerning the reactions of bodies in contact. This matter is taken up fully in textbooks on engineering mechanics.

Experience reveals that in rigid bodies in contact, only those points which are situated near the points of contact act on one another. Let us assume here the simplifying hypothesis that only the points of contact of both bodies act on one another; the reactions will then be the forces acting at the points of contact.

This hypothesis does not hold in all generality. According to this hypothesis, the reactions of two rigid bodies in contact only at one point would be reduced to one force having its origin at the point of contact. On the other hand, experience teaches that in addition to this force there can still appear a force couple whose moment is different from zero, which is contrary to the hypothesis.

For example, if a heavy rigid sphere rests on a rigid horizontal plate, then it can remain in equilibrium even if it is acted upon by a force couple (lying in the horizontal plane) of small moment. In the state of equilibrium the reactions of the plate balance the weight of the sphere as well as the force couple, which would be impossible were the reactions of the plate reduced to only one force acting at the point of tangency.

Normal and tangential reactions. Let two rigid bodies I and II be in contact at the point A (Fig. 180). Let us denote by $\mathbf{R}$ the force with which body II acts on body I at the point A. The force $\mathbf{R}$ has its origin at A. By the law of action and reaction body I acts on body II with a force $-\mathbf{R}$, whose origin is also at A.

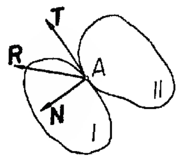
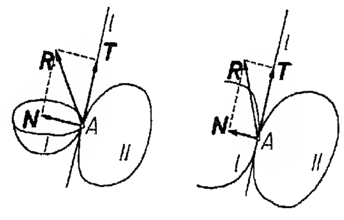


Fig. 180.



a) b)

Fig. 181.

Let body I be a solid or a surface having a tangent plane Π at the point A.

Let us resolve the reaction $\mathbf{R}$ into two components: a component $\mathbf{N}$ perpendicular to the body, i. e. to the plane Π, and a component $\mathbf{T}$ tangent to the body, i. e. lying in Π.

The component $\mathbf{N}$ is called the *normal* reaction, and the component $\mathbf{T}$ the *tangential* reaction or the *friction*. The normal reaction is usually directed with respect to body II to that side in which body I is situated; it is then called the *pressure*. If there is no friction at the points of contact, the bodies in contact are called *smooth*.

Let us consider two more cases:

1° Body I is a surface bounded by a certain curve on which the point of contact A lies, and the bounding curve has a straight line tangent l at A (Fig. 181a).

2° Body I is a curve having a straight line tangent l at A, where A is not the end of this curve (Fig. 181b).

An example of 1° can be a rigid hemisphere bounded by a circumference on which the point A lies; an example of 2° can be an arc of a circumference with the point A lying at its midpoint. In cases 1° and 2°

the normal reaction will be the component of the reaction $\mathbf{R}$ perpendicular to the tangent l; the friction will be the component of the reaction $\mathbf{R}$ lying on the line l.

For two smooth bodies in contact at the point A, the direction and sense of the reaction are determined if one of the bodies is a solid or a surface possessing a tangent plane at A. The direction and sense of the reaction are also determined for bodies 1° and 2° if the lines tangent to them at their point of contact do not coincide. For in this case the reaction must be perpendicular to both tangents. For bodies, one of which is body 1° or 2°, we know only this about the reaction, namely, that it lies in the plane perpendicular to the tangent line l.

Supports. A fixed rigid body (e. g. one attached rigidly to the earth) is called a *support*. In many applications it is necessary to determine the reactions of the supports on other rigid bodies.

If a rigid body resting on supports is in equilibrium, then the forces acting on this body balance the reactions of the supports. If a smooth body rests on smooth supports, then we assume that *reactions* (obviously normal) *are induced which balance the forces acting on the body.*

Because of this hypothesis we can in many cases give the necessary and sufficient conditions for the equilibrium of forces which act on a rigid body resting on smooth supports.

Centre of pressure. Let two smooth bodies I and II be in contact at the points lying in a certain plane Π (Fig. 182). The reactions will therefore be perpendicular to the plane Π. The reactions acting on the body I are consequently parallel; let us assume that they are pressures. Hence they have the same sense. It follows from this that they have a resultant $\mathbf{R}$ which we can assume to be acting at a certain point O of the plane Π. The point O is called the *centre of pressure*.

Obviously the reactions acting on the other body have a resultant $-\mathbf{R}$ and the same centre of pressure.

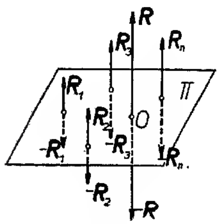


Fig. 182.

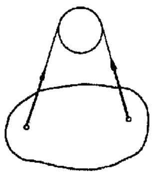


Fig. 183.

Reactions of a string. An inextensible string fastened to a body acts on it only when the string is in tension. If the mass of the string is small (so that it can be neglected) and both ends are fastned to the body, then the string acts at both ends with forces which are equal in magnitude also in the case when it is wound around some smooth body (Fig. 183). The forces with which the string acts at its ends are tangent to the string and have senses in the direction of the string. These forces are called the *tensions of a string.*

Example 1. If a heavy body hanging on a string at the point A is in equilibrium, then the tension $\mathbf{T}$ of the string whose origin is A balances the weight $\mathbf{Q}$ whose origin is at the centre of gravity S.

Consequently $\mathbf{T}+\mathbf{Q}=0$, or

$$|\mathbf{T}|=|\mathbf{Q}|. \tag{1}$$

Moreover, the forces $\mathbf{T}$ and $\mathbf{Q}$ must act along one line. The string is therefore directed vertically and its prolongation passes through the centre of gravity (Fig. 184). Hence, hanging the body in succession from two points and drawing the directions of the string in the body, we obtain as the point of intersection the centre of gravity of the body.

Example 2. If a body hanging by two strings at the points A and B is in equilibrium, then the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ with its origins at A and B balance the weight $\mathbf{Q}$ whose origin is at the centre of gravity S (Fig. 185). Consequently

$$\mathbf{T}_1+\mathbf{T}_2+\mathbf{Q}=0. \tag{2}$$

Fig. 184.

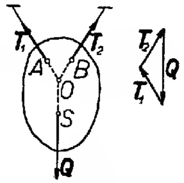


Fig. 185.

Therefore by the theorem given on p. 246 the directions of the forces either intersect at the point O or the forces are parallel. In both cases we can determine the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ by taking the moment with respect to an arbitrary point, e. g. with respect to the point A. Denoting by a_2 and d the arms of the forces $\mathbf{T}_2$ and $\mathbf{Q}$ with respect to A, we get $|\mathbf{T}_2|a=|\mathbf{Q}|d$, or

$$|\mathbf{T}_2|=|\mathbf{Q}|d\,/\,a. \tag{3}$$

Similarly, we obtain $|\mathbf{T}_1|$ by taking the moment with respect to B. In the case when the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ are not parallel, we can determine them graphically by forming the triangle of forces (Fig. 185).

Example 3. A heavy rigid rod hangs at the ends A and B of a massless inextensible string passing through a smooth ring at the point C (Fig. 186a). Determine the tension of the string in the position of equilibrium.

Let us denote the length of the string by l, the angle ACB by φ, and the centre of gravity of the rod by S. Let us put:

$$AB = a, \quad AC = l_1, \quad BC = l_2, \quad AS = b.$$

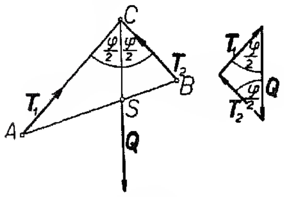


Fig. 186.

Let us suppose that a, b, l, and the weight of the rod $\mathbf{Q}$, are given.

Since the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ of the string balance the weight $\mathbf{Q}$, these forces intersect at the point C (because the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ pass through the point C (p. 246)) and moreover

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{Q} = 0. \tag{4}$$

In addition to this (p. 261)

$$|\mathbf{T}_1| = |\mathbf{T}_2|. \tag{5}$$

When $\varphi = 0$, the rod has a vertical position and the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ have vertical directions. Therefore by (5) $\mathbf{T}_1 = \mathbf{T}_2$, whence by (4)

$$|\mathbf{T}_1| = |\mathbf{T}_2| = \tfrac{1}{2}|\mathbf{Q}|.$$

Let us inquire in what case φ can be different from zero, as in Fig. 186. Let us therefore assume that $\varphi \neq 0$.

Denoting by d_1 and d_2 the distances of the directions of the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ from S and taking the moment with respect to S, we get $|\mathbf{T}_1|d_1 = |\mathbf{T}_2|d_2$; consequently from (5), $d_1 = d_2$. The centre of gravity S is equidistant from the sides AC and BC, i. e. the line CS is the bisector of the angle φ. From a known geometrical theorem concerning the angle bisectors of a triangle, we obtain $AC : BC = AS : BS$, i. e.

$$l_1 : l_2 = b : (a - b). \tag{6}$$

Since

$$l_1 + l_2 = l, \tag{7}$$

solving the system of equations (6) and (7), we get:

$$l_1 = bl / a, \quad l_2 = (a - b)\, l / a. \tag{8}$$

In order that the sides l_1, l_2, a, form a triangle, the inequalities $l_1 + l_2 > a$, $l_1 + a > l_2$, $l_2 + a > l_1$ must hold. They can be written in the form:

$$l_1 + l_2 > a, \quad a > |l_1 - l_2|,$$

whence in virtue of (8)

$$l > a > |a - 2b|\, l / a. \tag{9}$$

The inequalities $a < l < a^2 / |a - 2b|$ must therefore be fulfilled, or, setting $k = b / a$,

$$a < l < a / |1 - 2k|. \tag{10}$$

Hence: *when $\varphi \neq 0$ equilibrium will occur if the length l satisfies condition* (10).

Let us note that if $k = b / a = \frac{1}{2}$ (i. e. if the centre of gravity S falls at the centre of the segment AB), then conditions (9) are satisfied for all $l > a$. In this case, therefore, the position of equilibrium of the rod is always possible when $\varphi \neq 0$.

Angle φ is obtained from the theorem of Carnot

$$a^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos\varphi. \tag{11}$$

From the triangle of forces we get

$$|\mathbf{T}_1| = |\mathbf{T}_2| = \tfrac{1}{2}|\mathbf{Q}| / \cos\tfrac{1}{2}\varphi. \tag{12}$$

After expressing $\cos\frac{1}{2}\varphi$ in terms of a, b, l, from formulae (8) and (11), we obtain by (12)

$$|\mathbf{T}_1| = |\mathbf{T}_2| = |\mathbf{Q}| \frac{l}{a} \sqrt{\frac{(a-b)\,b}{l^2 - a^2}}.$$

In particular, for $b = \frac{1}{2}a$ we get

$$|\mathbf{T}_1| = |\mathbf{T}_2| = |\mathbf{Q}| \frac{l}{2\sqrt{l^2 - a^2}}.$$

Example 4. A horizontal beam rests at the points A and B on two smooth supports. Vertical forces (directed downwards) $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5$ (Fig. 174) act on the beam. Determine the reactions of the supports.

Let us denote by $x_1, x_2, \ldots, x_5$ the distances of the points of application of the forces from A. Put $AB = d$. The reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ at A and B are vertical. Taking the moment with respect to A and denoting by $P_1, P_2, \ldots, P_5, R_1, R_2$ the absolute values of the forces, we obtain $P_1x_1 + P_2x_2 + \ldots + P_5x_5 - R_2d = 0$, whence

$$R_2 = (P_1x_1 + P_2x_2 + \ldots + P_5x_5) / d. \tag{13}$$

Since $R_1 + R_2 = P_1 + P_2 + \ldots + P_5$,

$$R_1 = [P_1(d - x_1) + \ldots + P_5(d - x_5)] / d. \tag{14}$$

The reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ can also be determined by means of the string polygon as on p. 253.

Example 5. A heavy sphere of constant density touches a smooth plane Π inclined at an angle α with the horizontal (Fig. 187). Determine the horizontal force $\mathbf{P}$ maintaining the sphere in equilibrium.

The origin of the weight **Q** of the sphere is at its centre O, and that of the reaction **R** of the plane Π at the point of tangency A; the reaction is perpendicular to Π. The forces **Q** and **R** intersect at the point O. Since the forces **P**, **Q** and **R** balance one another, in virtue of theorem III, p. 246, they intersect at the point O. Moreover $\mathbf{P} + \mathbf{Q} + \mathbf{R} = 0$; hence the forces **P** and **R** are obtained from the triangle of forces. We have

$$R = Q / \cos\alpha, \quad P = Q \tan\alpha,$$

where P, Q and R, denote the absolute values of the corresponding forces.

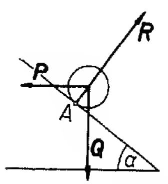


Fig. 187.

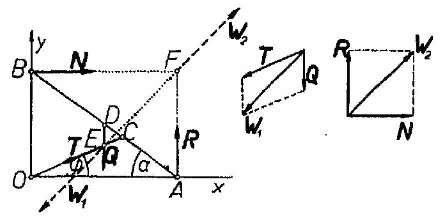


Fig. 188, 189, 190.

Example 6. A heavy rod AB of constant density lies in the vertical plane Π and rests against two smooth planes: a horizontal plane Π_1 and a vertical plane Π_2. Let Ox and Oy be the lines of intersection of the planes Π_1 and Π_2 with the plane Π. The rod AB is tied by an inextensible string OC to the point O. The rod is in equilibrium. Determine the reactions, having been given: $AB = 2l$, the angle α between AB and the x-axis, and the angle φ between OC and the x-axis (Fig. 188).

The force acting on the rod is the weight **Q** acting at the midpoint D of the rod AB. The reactions are the reactions of the planes **R** and **N**, acting at A and B and perpendicular to the planes, as well as the reaction of the string **T** acting at C and directed along the string towards the point O. The acting force balances the reactions. From the condition of projections on the x and y axes we obtain

$$N_x + T_x = 0, \quad R_y + Q_y + T_y = 0, \tag{15}$$

and from the condition of moment with respect to O:

$$-R_y \cdot 2l\cos\alpha + N_x \cdot 2l\sin\alpha - Q_y \cdot l\cos\alpha = 0. \tag{16}$$

Let us denote by R, N, T, Q, the absolute values of the corresponding forces. We obviously have $R_y = R$, $N_x = N$, $Q_y = -Q$ and $T_x = -T\cos\varphi$, $T_y = -T\sin\varphi$. Consequently from equations (15) and (16) we obtain:

$$N - T\cos\varphi = 0, \quad R - Q - T\sin\varphi = 0, \tag{17}$$

$$-2R\cos\alpha + 2N\sin\alpha + Q\cos\alpha = 0. \tag{18}$$

Determining R and N from equations (17) and substituting in (18) we obtain

$$T = \frac{Q\cos\alpha}{2\sin(\alpha - \varphi)}, \tag{19}$$

whence by equations (17)

$$N = \frac{Q\cos\alpha\cos\varphi}{2\sin(\alpha-\varphi)}, \quad R = Q\left(1 + \frac{\cos\alpha\sin\varphi}{2\sin(\alpha-\varphi)}\right). \tag{20}$$

Since $T > 0$, in virtue of (19) $\alpha > \varphi$; hence the point C must lie between A and D.

The problem can also be solved graphically (Fig. 189 and 190).

Denote by E the point of intersection of the forces **T** and **Q**, and by F that of the forces **N** and **R**. The resultant $\mathbf{W}_1$ of the forces **T** and **Q** acts at E, whereas the resultant $\mathbf{W}_2$ of the forces **R** and **N** acts at F. Since the system of forces **N**, **R**, **T**, **Q** is equipollent to zero, the system of forces $\mathbf{W}_1$, $\mathbf{W}_2$ is also equipollent to zero. Hence the forces act along the line EF and $\mathbf{W}_1 + \mathbf{W}_2 = 0$. The force **Q** as well as the directions of the forces **T** and $\mathbf{W}_1 = \mathbf{T} + \mathbf{Q}$ are given; therefore we can determine the forces $\mathbf{W}_1$ and **T** as in Fig. 189. Since $\mathbf{N} + \mathbf{R} = \mathbf{W}_2 = -\mathbf{W}_1$, the forces **N** and **R** are obtained by resolving the force $\mathbf{W}_2$ into components in the directions of the x and y axes (Fig. 190).

Example 7. A heavy rod AB whose centre of gravity is at S rests on a smooth horizontal plane at the point A and on a smooth sphere at the point B (Fig. 191). An inextensible string fastened at A passes over a pulley C and sustains a weight **P** at its other end. Determine the weight **P**, the reaction **R** of the horizontal plane and the reaction **N** of the sphere in the position of equilibrium, having been given $a = AS$, $b = AB$, the angle α between the rod and the plane, and the weight **Q** of the rod.

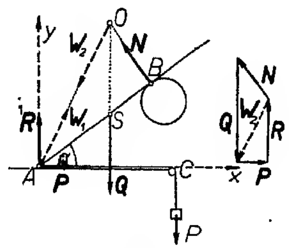


Fig. 191.

Let us choose the axes x and y as in drawing. Since the tension in the string at the point A is **P**, denoting by P, Q, R, N, the absolute values of the forces and forming projections on the x and y axes, we get:

$$P - N\sin\alpha = 0, \quad R - Q + N\cos\alpha = 0. \tag{21}$$

The total moment with respect to A is

$$Qa\cos\alpha - bN = 0. \tag{22}$$

From equations (21) and (22) we obtain

$$N = Q\frac{a\cos\alpha}{b}, \quad P = Q\frac{a\sin 2\alpha}{2b}, \quad R = Q\left(1 - \frac{a}{b}\cos^2\alpha\right).$$

The problem can also be solved graphically. With this in view, let us denote by $\mathbf{W}_1$ the resultant of the forces $\mathbf{R}$ and $\mathbf{P}$, and by $\mathbf{W}_2$ the resultant of the forces $\mathbf{Q}$ and $\mathbf{N}$. The force $\mathbf{W}_1 = \mathbf{R} + \mathbf{P}$ acts at the point A, and $\mathbf{W}_2 = \mathbf{Q} + \mathbf{N}$ at the point O in which the directions of the forces $\mathbf{Q}$ and $\mathbf{N}$ intersect. Since we know the positions of the forces $\mathbf{Q}$ and $\mathbf{N}$, the point O can be determined.

The forces $\mathbf{P}$, $\mathbf{R}$, $\mathbf{Q}$, and $\mathbf{N}$, are in equilibrium; therefore the forces $\mathbf{W}_1$ and $\mathbf{W}_2$ balance each other. Consequently $\mathbf{W}_1 + \mathbf{W}_2 = 0$; moreover the forces $\mathbf{W}_1$ and $\mathbf{W}_2$ lie on one line. This line is obviously the line AO. Since we already know the direction of the force $\mathbf{W}_2$, we can determine $\mathbf{W}_2$ and $\mathbf{N}$ from the relation $\mathbf{W}_2 = \mathbf{Q} + \mathbf{N}$ by drawing the triangle of forces. We have $\mathbf{W}_1 = -\mathbf{W}_2$, and $\mathbf{W}_1 = \mathbf{R} + \mathbf{P}$, hence we obtain the forces $\mathbf{R}$ and $\mathbf{P}$ by resolving the force $\mathbf{W}_1$ in the directions of the forces $\mathbf{R}$ and $\mathbf{P}$.

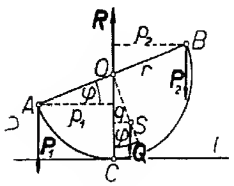


Fig. 192.

Example 8. A heavy rigid wire of constant density, in the form of a semicircle, lies in a vertical plane and rests on a horizontal line l (Fig. 192). Forces $\mathbf{P}_1$ and $\mathbf{P}_2$, directed vertically downwards, act at the ends A and B of the wire. Determine the angle φ which the diameter AB makes with the horizontal in the position of equilibrium, as well as the reaction $\mathbf{R}$ at the point of tangency C (under the assumption that there is no friction).

In the position of equilibrium the forces $\mathbf{P}_1$, $\mathbf{P}_2$, the weight $\mathbf{Q}$ acting at the centre of mass S, and the reaction $\mathbf{R}$ perpendicular to l, balance one another. Since these forces are parallel, (denoting their absolute values by P_1, P_2, Q, and R), we obtain from the condition of projections on the y-axis, which is directed vertically upwards, $-P_1 + R - Q - P_2 = 0$, whence

$$R = P_1 + P_2 + Q. \tag{23}$$

Let us calculate the total moment of the forces with respect to the point of tangency C. From the condition of moment we obtain

$$-P_1p_1 + Qq + P_2p_2 = 0, \tag{24}$$

where p_1, p_2, and q, denote the arms of the forces P_1, P_2, and Q, with respect to C. Putting $r = OB$ (where O is the centre of the diameter AB), we obtain:

$$p_1 = p_2 = r\cos\varphi, \quad q = OS\cdot\sin\varphi, \tag{25}$$

and since $OS = 2r/\pi$ (p. 176),

$$q = \frac{2r\sin\varphi}{\pi}. \tag{26}$$

From (25) and (26) we get, after substituting in (24),

$$(P_2 - P_1)\,r\cos\varphi + \frac{2rQ\sin\varphi}{\pi} = 0,$$

whence

$$\tan\varphi = \frac{(P_1 - P_2)\,\pi}{2Q}. \tag{27}$$

§ 10. Friction. Let two bodies I and II, which are at rest, be in contact at the point A and let them have a common tangent plane Π at this point (Fig. 193). Let us denote by $\mathbf{R}$ the reaction which body II exerts on body I at the point A. If the bodies are not smooth, then the reaction $\mathbf{R}$ is not perpendicular to the plane Π.

Let α be the angle which $\mathbf{R}$ makes with the normal n to Π.

Denoting the normal component by $\mathbf{N}$, the tangential component or *friction* by $\mathbf{T}$, and putting $R = |\mathbf{R}|$, $N = |\mathbf{N}|$, $T = |\mathbf{T}|$, we obtain:

$$T = R\sin\alpha, \quad N = R\cos\alpha, \tag{1}$$

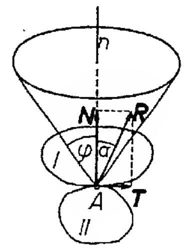


Fig. 193.

whence

$$T = N\tan\alpha. \tag{2}$$

Experiment shows that the angle α cannot exceed a certain limit which depends on the nature of the surfaces I and II.

Let us denote by φ the maximum value of the angle α at the point A for a given pair of bodies I and II in the position of equilibrium. We therefore have $0 \leqq \alpha \leqq \varphi$, i. e. $0 \leqq \tan\alpha \leqq \tan\varphi$, whence by (2)

$$T \leqq N\tan\varphi. \tag{3}$$

Putting $f = \tan\varphi$, we get

$$T \leqq Nf. \tag{4}$$

The number f is called the *static coefficient of friction* for a given pair of bodies I and II at the point of contact A.

Let us consider a cone of revolution whose vertex is at A, and whose axis is the normal n inclined at an angle φ with respect to the generatrices of the cone. This cone is called the *cone of friction* at the point A.

Since $\alpha \leqq \varphi$, *the reaction lies within the cone of friction or on its surface.*

Exactly as for smooth supports (p. 259), we also assume in the case of friction the following principle:

If a rigid body rests on supports and reactions (lying within the cones of friction), which balance the forces acting on the body, are possible, then such reactions are actually induced (if the body was initially at rest).

Example I. A heavy rod AB, lying in a vertical plane, rests against a vertical plane and a horizontal plane (Fig. 194). The coefficients of friction at A and B are f_1 and f_2. Examine the conditions for equilibrium.

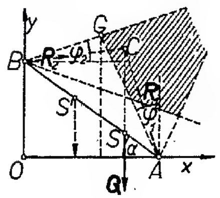


Fig. 194.

Let us consider the cones of friction at A and B. The generatrices of these cones are inclined to the normals at A and B at the angles φ_1 and φ_2, where $\tan\varphi_1 = f_1$, and $\tan\varphi_2 = f_2$. The reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ at A and B must lie within or on the surface of these cones.

Three forces act on the rod AB: $\mathbf{R}_1$, $\mathbf{R}_2$, and the weight $\mathbf{Q}$ acting at the centre of gravity S.

If the rod is in equilibrium, then the directions of these forces pass through one point C (p. 246). This point must obviously lie in a region common to both cones of friction (*vide* shaded region in the figure) because the directions of there actions can only intersect in this region. The direction of the weight must therefore pass through the region common to both cones of friction.

Conversely, if the direction of the weight $\mathbf{Q}$ passes through the region common to the cones of friction, then the rod can remain in equilibrium. For let us choose on the vertical passing through S an arbitrary point C within the region common to the cones of friction. It is easy to see that reactions $\mathbf{R}_1$ and $\mathbf{R}_2$, having directions AC and BC and balancing the weight $\mathbf{Q}$, can occur. Therefore the rod can in this case remain in equilibrium.

If the centre of gravity were at a point S' such that the vertical passing through this point did not cut the region common to the cones of friction, then the equilibrium of the rod would be impossible.

Consequently: *the necessary and sufficient condition for the equilibrium of the rod is that the direction of the weight pass through the region common to the cones of friction.*

Let us put: $AB = l$, $AS = d$, $BS = d'$ and let us denote by α the angle which AB makes with the horizontal. Let us choose the x and y axes of the coordinate system as in the drawing and assume that the rod is in equilibrium. From the conditions of projections and of moment with respect to O we obtain

$$R_{1_x} + R_{2_x} = 0, \quad R_{1_y} + R_{2_y} - Q = 0, \tag{5}$$

as well as $-R_{1_y} l \cos\alpha + R_{2_x} l \sin\alpha + Qd' \cos\alpha = 0$, i. e.

$$-R_{1_y} l + R_{2_x} l \tan\alpha + Qd' = 0. \tag{6}$$

Moreover, we have the following inequalities:

$$|R_{1_x}| \leqq R_{1_y} f_1, \quad |R_{2_y}| \leqq R_{2_x} f_2. \tag{7}$$

The reactions cannot be determined from formulae (5) and (6). Relations (5)—(7) permit us only to give limits which the components of the reactions cannot exceed.

Let us denote by x_0 the abscissa of the point S, and by ξ the abscissa of the point G at which the extreme generatrices of the cones of friction intersect. Equilibrium will result if

$$\xi \leqq x_0. \tag{8}$$

In order to determine ξ, let us write the equations of the lines BG and AG:

$$y = f_2 x + l \sin\alpha, \quad y = -(x - l\cos\alpha)/f_1.$$

The point G is the point of intersection of these lines; hence

$$\xi = \frac{1 - f_1 \tan\alpha}{1 + f_1 f_2} l \cos\alpha. \tag{9}$$

Since $x_0 = d' \cos\alpha$, the inequality (8) assumes the form

$$(1 - f_1 \tan\alpha)/(1 + f_1 f_2) \leqq d'/l. \tag{10}$$

Equilibrium follows if the left side of this inequality is a negative number or zero. In this case $1 - f_1 \tan\alpha \leqq 0$; hence $1/f_1 \leqq \tan\alpha$ or $\cot\varphi_1 \leqq \tan\alpha$; consequently $\frac{1}{2}\pi - \varphi_1 \leqq \alpha$.

Therefore if $\frac{1}{2}\pi - \varphi_1 \leqq \alpha$, then equilibrium follows. On the other hand, if $\frac{1}{2}\pi - \varphi_1 > \alpha$, then the left side of the inequality (10) will be positive and equilibrium will not occur for too small d'.

It is easy to verify these results in the drawing (p. 268).

Example 2. A beam I passes loosely through a groove in beam II. A force $\boldsymbol{P}$ parallel to beam I acts on beam II. The beams are pressed to each other at the points A and B (Fig. 195).

Let us determine the cones of friction at A and B. If the direction of the force $\boldsymbol{P}$ passes through the region common to the cones of friction, then reactions $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ balancing the force $\boldsymbol{P}$ and acting on the beam II will appear at the points A and B. Then beam II will not move. The beam *sticks fast*.

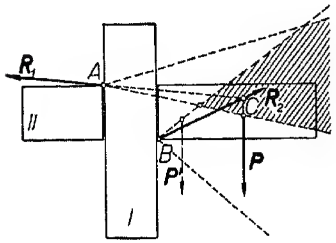


Fig. 195.

From the drawing it is easy to see that beam II will move if the force $\boldsymbol{P}'$ has its origin near beam I and is parallel to it. For then the direction of the force $\boldsymbol{P}'$ will not pass through the common part of both cones of friction; equilibrium will therefore be impossible.

§ 11. Conditions for equilibrium not involving the reaction.

The condition for the equilibrium of forces acting on a rigid body given on p. 257 expresses the relation that obtains between the acting forces and the reactions. We now give several examples in which the conditions for the equilibrium of the acting forces can be made to refer only to the acting forces without including the reaction.

Body with one fixed point. Let a rigid body have one fixed point, e. g. the point O. We can therefore assume that the body is free and that the point O is acted upon by a reaction $\boldsymbol{R}$ holding the point O (Fig. 196).

Let us further assume that the body is in equilibrium under the action of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$. These forces consequently balance the reaction $\boldsymbol{R}$. From the conditions of equilibrium it follows that the sum of the forces and the moment with respect to the point O are equal to zero, i. e.

$$\boldsymbol{R} + \Sigma \boldsymbol{P}_i = 0, \tag{1}$$

$$\Sigma \mathrm{Mom}_O \boldsymbol{P}_i = 0. \tag{I}$$

The reaction $\boldsymbol{R}$ (being a force whose origin is at O) does not appear in equation (I) because its moment with respect to O is zero. From equation (1) we can determine $\boldsymbol{R}$. We have

$$\boldsymbol{R} = -\Sigma \boldsymbol{P}_i. \tag{2}$$

Equation (I) constitutes the n e c e s s a r y condition which the acting forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, must satisfy in the case of equilibrium.

We shall now prove that condition (I) is also a s u f f i c i e n t condition for equilibrium.

Let us assume that the system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, satisfies equation (I), i. e. that the moment of this system of forces with respect to O is zero. It follows from this that the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, have a resultant $\boldsymbol{P} = \Sigma \boldsymbol{P}_i$ whose origin is at O (p. 26). Now the force $\boldsymbol{P}$ acting at O cannot move the given body which is at rest, because the point O is fixed. The system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$ (equipollent to the force $\boldsymbol{P}$), is therefore in equilibrium.

Hence: *A necessary and sufficient condition that a system of forces acting on a rigid body fixed at one point O be in equilibrium is that the total moment of the acting forces with respect to the point O be zero* (or that the system have a resultant passing through O).

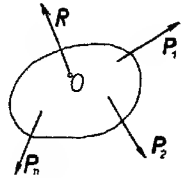


Fig. 196.

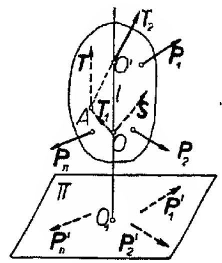


Fig. 197.

Body with a fixed axis. Let a rigid body have a certain fixed line l (it is sufficient for this purpose, for example, to fix two points of this line). We can assume that the body is free and that the points lying on the axis are acted upon by forces of reaction which cause the axis to be fixed.

Let us assume that the body is in equilibrium under the action of the forces $\{\boldsymbol{P}_i\}$. Hence the forces $\{\boldsymbol{P}_i\}$ balance the forces of reaction. From the conditions of equilibrium it follows that the total moment of these forces with respect to the axis l is equal to zero. Since the moment of the forces of reaction with respect to the axis l is zero (because the reactions are forces whose origins lie on the axis), the total moment of the forces $\{\boldsymbol{P}_i\}$ with respect to the axis l is zero, i. e.

$$\Sigma \mathrm{Mom}_l \boldsymbol{P}_i = 0. \tag{II}$$

The forces of reaction do not appear in equation (II). This equation is consequently the n e c e s s a r y condition that the given system of forces $\{\boldsymbol{P}_i\}$ must satisfy in order that the body be in equilibrium.

We shall now prove that condition (II) is also a s u f f i c i e n t condition for equilibrium.

Let us assume, then, that an arbitrary system of forces $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n$ satisfies condition (II). Let us select an arbitrary point O on the axis l (Fig. 197). By the theorem on reduction (p. 237), the given system is

equipollent to a system composed of two forces $\boldsymbol{S}$ and $\boldsymbol{T}$, of which $\boldsymbol{S}$ has its origin at O. Since the total moment of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, with respect to l is zero, the total moment of the forces $\boldsymbol{S}$ and $\boldsymbol{T}$ with respect to l is also zero. Since $\mathrm{Mom}_l \boldsymbol{S} = 0$ (because the force $\boldsymbol{S}$ acts at the point O lying on l), we must also have $\mathrm{Mom}_l \boldsymbol{T} = 0$. Therefore the force $\boldsymbol{T}$ either cuts l or is parallel to l.

Let us consider an arbitrary point $O' \neq O$ on the axis l. Let A be the origin of the force $\boldsymbol{T}$. Since the force $\boldsymbol{T}$ lies in the plane passing through l and A, we can resolve $\boldsymbol{T}$ into two forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ having directions OA and $O'A$, and then translate their points of application to O and O'. In this manner we have shown that the system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, is equipollent to a system of forces acting at the points of the axis l.

Since it is obvious that the forces acting at the points of the axis l, which is fixed by hypothesis, cannot move a body being at rest, the given system of acting forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$, is in equilibrium.

Therefore: *the necessary and sufficient condition that a system of forces acting on a rigid body having a fixed axis be in equilibrium is that the total moment of the system of forces with respect to this axis be zero.*

Remark. Let Π be an arbitrary plane perpendicular to the line l, and O_1 the point of intersection of the plane Π with the axis l. Let us denote by $\boldsymbol{P}_1', \boldsymbol{P}_2', \ldots$ the projections on the plane Π of the acting forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ From the definition of a moment with respect to an axis (p. 233) it follows that the moment of the force $\boldsymbol{P}_i'$ with respect to O_1 is equal to the moment of the force $\boldsymbol{P}_i$ with respect to l. Consequently the total moment of the forces $\{\boldsymbol{P}_i'\}$ with respect to the point O_1 is equal to the total moment of the forces $\{\boldsymbol{P}_i\}$ with respect to the axis l.

Therefore: *the necessary and sufficient condition for the equilibrium of a system of forces $\{\boldsymbol{P}_i\}$ is that the total moment of the forces $\{\boldsymbol{P}_i'\}$ with respect to O_1 be zero.*

This condition is such as if the projection of a body on the plane Π had a fixed point O_1 and the projections of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ acted on the projection of the body.

In order to see whether a system of forces acting on a rigid body having a fixed axis is in equilibrium, it is therefore sufficient to know only the projections of the acting forces on a plane perpendicular to the axis and the point of intersection of the axis with this plane.

Plane motion of a body. Let it be possible for a rigid body to move only in such a way that the path of each of its points is plane and lies in a plane parallel to a certain fixed plane Π.

We then say that the body can execute only a *plane motion* and we call the plane Π a *directional plane.*

An example of a body executing a plane motion is a cylinder whose bases lie in two parallel planes Π and Π' (Fig. 198). If there is no friction the reactions of the planes Π and Π' are perpendicular to these planes.

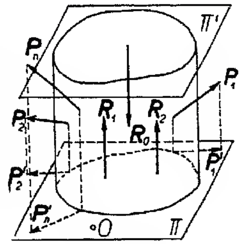


Fig. 198.

In general, let us assume that *whenever there is no friction the reactions which cause the body to execute only a plane motion are perpendicular to the directional plane Π.*

It is obvious, therefore, that a system of forces perpendicular to the directional plane is in equilibrium; this means that forces perpendicular to Π cannot move a body which can execute only a plane motion if the body is at rest.

Let a body which can execute only a plane motion be in equilibrium under the action of a system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ The acting forces therefore balance the reactions $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots$, i. e. they form a system equipollent to zero. It follows from this that the projections of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ and those of the reactions $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots$ on the directional plane Π also form a system equipollent to zero. Since the projections of the reactions are zero (because the reactions are perpendicular to Π), the projections $\boldsymbol{P}_1'$, $\boldsymbol{P}_2', \ldots$ of the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ on the directional plane themselves also form a system equipollent to zero.

Let O be an arbitrary point of the plane Π. If the forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ are in equilibrium, we obtain:

$$\Sigma \boldsymbol{P}_i' = 0, \quad \Sigma \mathrm{Mom}_O \boldsymbol{P}_i' = 0. \tag{3}$$

Condition (3) is therefore a necessary condition for the equilibrium the system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ We shall prove that it is also a sufficient condition.

With this in view, let us assume that the system of forces $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ satisfies equations (3). In virtue of the theorem on reduction this system is equipollent to a system consisting of the force $\boldsymbol{P} = \Sigma \boldsymbol{P}_i$ and a certain couple $\boldsymbol{U}, -\boldsymbol{U}$. Let us denote by $\boldsymbol{P}', \boldsymbol{U}'$, and $-\boldsymbol{U}'$, the projections of these forces on the plane Π. By (3) we obtain:

$$\boldsymbol{P}' = 0, \quad \mathrm{Mom}(\boldsymbol{U}', -\boldsymbol{U}') = 0.$$

Consequently $\boldsymbol{P}$ is perpendicular to Π; and the couple $\boldsymbol{U}, -\boldsymbol{U}$ lies in the plane perpendicular to Π. Hence we can rotate the couple $\boldsymbol{U}, -\boldsymbol{U}$ in

its plane (leaving the moment unchanged) so that in the new position the forces are perpendicular to Π. Denoting by $\mathbf{V}$, $-\mathbf{V}$ the new couple equipollent to the former, we see that the system of forces $\mathbf{P}_1$, $\mathbf{P}_2$, ... is equipollent to the system of forces $\mathbf{P}$, $\mathbf{V}$, and $-\mathbf{V}$, perpendicular to Π. It follows that the system $\mathbf{P}_1$, $\mathbf{P}_2$, ... is in equilibrium.

Therefore: *a necessary and sufficient condition that a system of forces acting on a rigid body which can execute only a plane motion be in equilibrium is that the projections of these forces on the directional plane form a system equipollent to zero.*

Hence, in order to find out whether a system of forces acting on a body which can execute only a plane motion is in equilibrium, it is sufficient to know the projections of the acting forces on the directional plane.

Example 1. Lever. A beam having a fixed horizontal axis perpendicular to it is called a *lever*.

We assume that the forces acting on a lever lie in one plane Π perpendicular to the axis of rotation and passing through the centre of gravity.

Let us denote by $\mathbf{Q}_1$, $\mathbf{Q}_2$, ... the forces acting on the beam at the points $A_1, A_2, \ldots$, by $\mathbf{Q}$ the weight of the beam acting at its centre of gravity S, by $Q_1, Q_2, \ldots, Q$ the absolute values, and by $q_1, q_2, \ldots, q$ the arms of these forces with respect to the point of intersection O of the plane Π with the axis of rotation (Fig. 199).

The moment of the system of forces $\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}$, with respect to the axis of rotation in this case is equal to the moment of this system with respect to O. The acting forces will therefore be in equilibrium if the sum of their moments with respect to O is zero. Hence the condition of equilibrium can be written in the form

$$\pm Q_1 q_1 \pm Q_2 q_2 \pm \cdots \pm Qq = 0, \tag{4}$$

where the signs + and — are taken according to the rule given on p. 233.

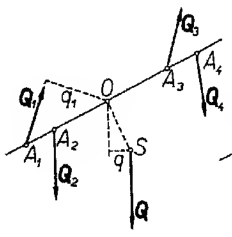


Fig. 199.

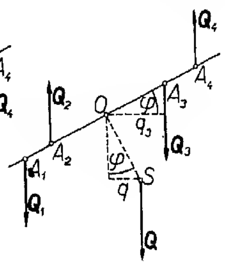


Fig. 200.

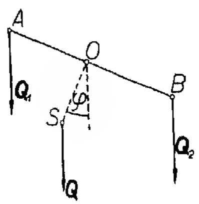


Fig. 201.

Let us assume that the center of gravity lies under the axis of rotation when the beam has a horizontal position. It follows from this, obviously, that for a beam on which no forces act (except gravity) a horizontal position is a position of equilibrium. Let us assume, in addition, that the acting forces have a vertical direction (Fig. 200).

Let φ denote the angle which the beam makes with the horizontal. Since OS is perpendicular to the beam, OS also makes an angle φ with the vertical.

Hence we have:

$$q_1 = OA_1 \cos\varphi, \quad q_2 = OA_2 \cos\varphi, \quad \ldots, \quad q = OS \sin\varphi,$$

whence by substituting in (4)

$$(\pm Q_1 \cdot OA_1 \pm Q_2 \cdot OA_2 \pm \ldots) \cos\varphi \pm Q \cdot OS \sin\varphi = 0,$$

or, dividing by $\cos\varphi$,

$$\pm Q_1 \cdot OA_1 \pm Q_2 \cdot OA_2 \pm \ldots \pm Q \cdot OS \tan\varphi = 0.$$

Knowing the forces $\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}$, we can calculate from equation (5) the angle φ which the beam makes with the horizontal in the position of equilibrium.

In particular, if the beam is acted upon by two forces $\mathbf{Q}_1$ and $\mathbf{Q}_2$, directed downwards and applied on opposite sides of the beam (as in Fig. 201), we obtain from (5) $-Q_1 \cdot OA + Q_2 \cdot OB - Q \cdot OS \tan\varphi = 0$, whence

$$\tan\varphi = (Q_2 \cdot OB - Q_1 \cdot OA) / Q \cdot OS. \tag{6}$$

If $\varphi = 0$ (i. e. if the beam is in equilibrium in a horizontal position), we obtain

$$Q_1 \cdot OA = Q_2 \cdot OB.$$

In particular, therefore, if $OA = OB$, then $Q_1 = Q_2$.

An instrument called a *balance*, which serves to compare the weights of two bodies and indirectly their masses, depends on this principle.

Example 2. A rigid body, having a fixed axis l, is in equilibrium under the action of the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, whose points of application are $A_1, A_2, \ldots, A_n$ (Fig. 202). Give the necessary and sufficient conditions which these forces must satisfy in order that the body continue to be in equilibrium, if it is

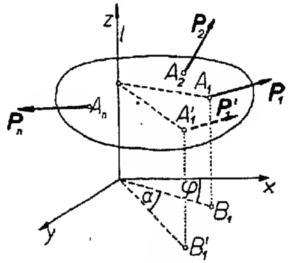


Fig. 202.

turned about the axis l through an arbitrary angle α and during this rotation the directions, senses, magnitudes, and the points of application (in the body) of these forces remain unchanged.

Let us take the axis l as the z-axis of the coordinate system; denote by $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ the coordinates of the points of application $A_1, A_2, \ldots$, and by $x_1', y_1', z_1', x_2', y_2', z_2', \ldots$ the coordinates of the points $A_1', A_2', \ldots$, into which the points $A_1, A_2, \ldots$ went when the body turned through an angle α about the axis l. Let B_1 and B_1' be the projections of the points A_1 and A_1' on the xy-plane, and φ the angle between OB and the x-axis. Putting $r_1 = OB_1 = OB_1'$, we have:

$$x_1 = r_1 \cos\varphi, \quad y_1 = r_1 \sin\varphi, \tag{7}$$

$$x_1' = r_1 \cos(\varphi + \alpha), \quad y_1' = r_1 \sin(\varphi + \alpha), \quad z_1 = z_1'. \tag{8}$$

Consequently $x_1' = r_1 \cos\varphi \cos\alpha - r_1 \sin\varphi \sin\alpha$, whence by (7)

$$x_1' = x_1 \cos\alpha - y_1 \sin\alpha, \text{ and similarly } y_1' = y_1 \cos\alpha + x_1 \sin\alpha. \tag{9}$$

Analogous formulae are obtained for the remaining points $A_2', A_3', \ldots$

Since the body has to maintain equilibrium after turning through an angle α, the moment of the forces with respect to the z-axis must be zero, i. e.

$$\Sigma(P_{i_x} y_i' - P_{i_y} x_i') = 0.$$

Substituting for x_i', y_i' the expressions from formulae (9), we obtain

$$\cos\alpha\, \Sigma(P_{i_x} y_i - P_{i_y} x_i) + \sin\alpha\, \Sigma(P_{i_x} x_i + P_{i_y} y_i) = 0. \tag{10}$$

Since equilibrium occurs for $\alpha = 0$, substituting $\alpha = 0$ in formula (10), we obtain

$$\Sigma(P_{i_x} y_i - P_{i_y} x_i) = 0. \tag{11}$$

From (10) we have for $\alpha = \frac{1}{2}\pi$

$$\Sigma(P_{i_x} x_i + P_{i_y} y_i) = 0. \tag{12}$$

Conversely, if conditions (11) and (12) hold, then obviously condition (10) holds for every α. Equations (11) and (12) are therefore the sought for necessary and sufficient conditions.

Determination of the reactions acting on a fixed axis. Let a rigid body have a line l fixed at the two points O and O'. We can then assume that the forces of reaction act at the points O and O'.

Let us assume that a system $\{\boldsymbol{P}_i\}$ of forces acting on the body is in equilibrium.

Let the point O be the origin of a system of coordinates and the axis l the z-axis (Fig. 203). Let us denote by x_i, y_i, z_i, the coordinates of the points of application of the forces $\{\boldsymbol{P}_i\}$, by $\boldsymbol{R}$ and $\boldsymbol{N}$ the reactions at the points O and O', and by d the length of the segment OO'.

Since the system of forces $\{\boldsymbol{P}_i\}$ together with the reactions $\boldsymbol{R}$ and $\boldsymbol{N}$ is in equilibrium, forming the projections of the sum and total moment with respect to O on the axes of the coordinate system, we obtain six equations:

$$\Sigma P_{i_x} + R_x + N_x = 0, \tag{I}$$

$$\Sigma P_{i_y} + R_y + N_y = 0, \tag{II}$$

$$\Sigma P_{i_z} + R_z + N_z = 0, \tag{III}$$

$$\Sigma(P_{i_y} z_i - P_{i_z} y_i) + N_y d = 0, \tag{IV}$$

$$\Sigma(P_{i_z} x_i - P_{i_x} z_i) - N_x d = 0, \tag{V}$$

$$\Sigma(P_{i_x} y_i - P_{i_y} x_i) = 0. \tag{VI}$$

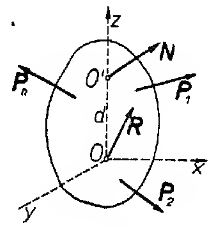


Fig. 203.

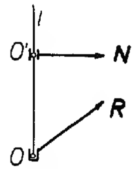


Fig. 204.

From equations (IV) and (V) we can determine N_x and N_y. Next we determine R_x and R_y from equations (I) and (II). Finally we calculate $R_z + N_z$ from equation (III).

We see, therefore, that the above equations do not permit us to determine the reactions. It is true that the number of unknowns is six (R_x, R_y, R_z and N_x, N_y, N_z), i. e. as many as there are equations, however, they appear only in five equations. Equation (VI) expresses the condition for the equilibrium of the acting forces. From equations (I)—(V) we can determine only the components of reaction perpendicular to the axis l and the sum of the components parallel to the axis l.

Problems in which the forces of reaction cannot be determined from the conditions of equilibrium are called *statically indeterminate*.

Therefore the calculation of the reactions of a rigid body which is fixed at two points is statically indeterminate.

If we assume that the given body is not a rigid body, but one that can be deformed, then the forces of reaction could be calculated by appealing to the theory of elasticity.

Our problem can be made statically determinate by assuming that the point O' is fixed in a smooth bearing (Fig. 204).

The reaction $\mathbf{N}$ is then perpendicular to the axis l. In this case we have $N_z = 0$, and therefore we can determine R_z from equation (III).

Example 3. A heavy rectangular plate has a horizontal axis l fixed at the point O and at the bearing O' (Fig. 205). The centre of the side parallel to the axis is the point of application of the force $\mathbf{P}$ perpendicular to the plate. Given are: the weight $\mathbf{Q}$ whose origin is at the centre S of the rectangle, sides a and b of the rectangle, and the length $d = OO'$. Determine in the position of equilibrium the reactions $\mathbf{R}$ and $\mathbf{N}$ (at O and $O)'$ and the angle φ which the plate makes with the horizontal.

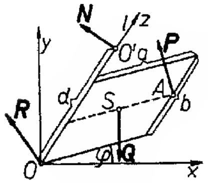


Fig. 205.

Let O be the origin and the axis l the z-axis of the coordinate system; let us give the x-axis a horizontal direction and the y-axis a direction vertically upwards. We shall be able to apply equations (I)—(VI), p. 277.

The point S has the coordinates $\frac{1}{2}a \cos \varphi, \frac{1}{2}a \sin \varphi, \frac{1}{2}b$; and the point A: $a \cos \varphi,\ a \sin \varphi,\ \frac{1}{2}b$. We have:

$$Q_x = 0, \quad Q_y = -Q, \quad Q_z = 0, \quad \text{where } Q = |\mathbf{Q}|;$$
$$P_x = -P \sin\varphi, \quad P_y = P \cos\varphi, \quad P_z = 0, \quad \text{where } P = |\mathbf{P}|.$$

Since $N_z = 0$, we get by formulae (I)—(VI), p. 277:

$$-P \sin\varphi + R_x + N_x = 0, \quad P \cos\varphi - Q + R_y + N_y = 0, \quad R_z = 0,$$
$$\tfrac{1}{2}bP \cos\varphi - \tfrac{1}{2}Qb + N_y d = 0, \quad \tfrac{1}{2}bP \sin\varphi - N_x d = 0,$$
$$-Pa + \tfrac{1}{2}Qa \cos\varphi = 0.$$

From the last equation we obtain $\cos \varphi = 2P \,/\, Q$, and from the remaining equations we determine R_x, R_y and N_x, N_y. We have $R_z = N_z = 0$.

Equilibrium is obviously possible if $2P \,/\, Q \leq 1$, i. e. if $P \leq Q \,/\, 2$.

§ 12. Equilibrium of heavy supported bodies. If a rigid body which is not acted upon by any forces other than the force of gravity rests on a horizontal plane Π and is in equilibrium, then the forces of reaction which the plane exerts on the body (at the points of support) balance the weights of the individual points of the body.

Let us assume that the supporting plane Π is smooth. The reactions are then perpendicular to the plane; hence they have a resultant $\mathbf{F}$ acting at a certain point O of the plane Π. The point O was called the *centre of pressure* (p. 260). The weights of the individual points of the body have a resultant $\mathbf{Q}$ whose point of application is at the centre of gravity S.

If the body is in equilibrium, the forces $\mathbf{F}$ and $\mathbf{Q}$ balance each other. Consequently $\mathbf{F} + \mathbf{Q} = 0$, and moreover $\mathbf{F}$ and $\mathbf{Q}$ lie on one line. Because of this the centre of pressure lies at the point of intersection of the direction of the force $\mathbf{Q}$ with the plane Π. The centre of pressure O is therefore the projection of the centre of gravity S on the supporting plane Π.

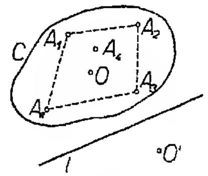


Fig. 206.

If a body rests on a plane Π only at one point O and is in equilibrium, then the reaction acts at O. Consequently the centre of gravity lies above the point of support.

Let the body now rest on the plane Π at the points $A_1, A_2, \ldots$ and let C be an arbitrary closed convex polygon enclosing all the points of support $A_1, A_2, \ldots$ (Fig. 206). We shall prove that the centre of pressure O in this case also lies either within or on the polygon C.

For let us assume that the centre of pressure lies outside the polygon C at the point O'. Let us draw an arbitrary line l in the plane Π such that the point O' and the line C lie on opposite sides of the line l. The moments of the forces of reaction with respect to l would therefore be directed opposite to the moment of the force $\mathbf{F}$. This is impossible, however, because the total moment of the forces of reaction is equal to the moment of the force $\mathbf{F}$. Hence the centre of pressure must lie within the convex polygon C. If K is the smallest convex polygon (in the figure the polygon A_1, A_2, A_3, A_n) within which the points of support lie,[1]) then the centre of pressure also lies within this polygon. Since we have assumed that the body is in equilibrium, the direction of the force of gravity passes through the centre of pressure and therefore also falls within the polygon K.

We shall now prove that if the weight falls within the polygon K, then reactions will appear which balance the weight.

We shall consider two cases:

1° A body is supported at two points A and B. In this case the polygon K is the line segment AB. If the direction of the force of gravity

[1]) In geometry it is proved that such a polygon always exists and lies within every convex polygon containing the points of support.

passes through the point G of the segment AB, then there exist two forces of reaction $\mathbf{R}_1$ and $\mathbf{R}_2$ directed vertically upwards and having their origins at A and B. These forces can be determined graphically as on p. 253, or calculated as in example 4, p. 263. It follows from this by the principle given on p. 260 that the reactions balance the weight.

2° A body is supported at $n > 2$ points. If the points are collinear, then denoting the extreme points of support by A and B, we can proceed as in case 1°. Suppose, then, that not all the points of support are collinear. If the force of gravity falls within the polygon K at the point O, then we can find three points of support such that the point O will lie within a triangle of which these points are the vertices (the points A_1, A_3, A_n, in the figure). As we shall show (*vide* example 4), we can then determine the reactions acting at the vertices of this triangle and balancing the force of gravity.

We therefore have the following theorem:

If a heavy rigid body rests on a smooth horizontal plane, then the necessary and sufficient condition that the reactions of the plane balance the weight of the body is that the force of gravity fall within the smallest convex polygon K containing all the points of support.

More generally: let a rigid body rest on a smooth horizontal plane and in addition to the force of gravity let other forces act on it. If the body is in equilibrium, then the resultant $\mathbf{F}$ of the reactions balances the acting forces. It follows from this that the forces acting on the body have a vertical resultant $-\mathbf{F}$ whose direction passes through the centre of pressure O and falls within the polygon K. Conversely, if the acting forces have a vertical resultant, directed downwards and falling within the polygon K, then reactions will appear which balance the forces acting on the body. The proof is carried out as before.

Example 4. A three-legged stool rests on the floor Π. Determine the reactions at the points of support A_1, A_2, A_3, under the assumption that there is no friction.

Let us denote by S the centre of gravity of the stool, by S' the projection of S on Π, by $\mathbf{Q}$ the weight of the stool, and by $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$ the reactions (Fig. 207).

The problem can be solved most simply by calculating the total moments of the forces with respect to the lines A_1A_2, A_2A_3, A_3A_1; these moments are obviously zero.

Let h_3 denote the distance of A_3 from A_1A_2, and w_3 the distance of S' from A_1A_2. Taking the moment with respect to the axis A_1A_2, we obtain:

$$R_3h_3 - Qw_3 = 0, \text{ where } R_3 = |\mathbf{R}_3| \text{ and } Q = |\mathbf{Q}|,$$

because the moments of the forces $\mathbf{R}_1$ and $\mathbf{R}_2$ are zero. Consequently

$$R_3 = Qw_3 \,/\, h_3.$$

Analogously we obtain

$$R_1 = Qw_1 \,/\, h_1 \quad \text{and} \quad R_2 = Qw_2 \,/\, h_2.$$

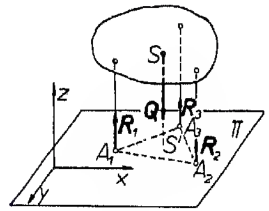


Fig. 207.

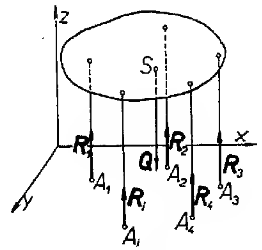


Fig. 208.

Example 5. Let a rigid body rest on a horizontal plane Π at the n points of support $A_1, A_2, \ldots, A_n$. Let us take the plane Π as the xy-plane of the coordinate system (x, y, z); denote by $x_1, y_1, 0, x_2, y_2, 0, \ldots, x_n, y_n, 0$, the coordinates of the points of support $A_1, A_2, \ldots, A_n$, and by x_0, y_0, z_0, the coordinates of the centre of gravity S (Fig. 208).

Let $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n$ denote the reactions, $\mathbf{Q}$ the weight, and $R_1, R_2, \ldots, R_n, Q$, the absolute values of these forces.

Forming the projections on the coordinate axes, we obtain for the projection on the z-axis

$$R_1 + R_2 + \ldots + R_n - Q = 0 \tag{13}$$

The remaining two equations drop out because they become identically zero.

Forming next the moments with respect to the origin of the coordinate system, we obtain only two equations:

$$-R_1y_1 - R_2y_2 - \ldots + Qy_0 = 0, \quad R_1x_1 + R_2x_2 + \ldots - Qx_0 = 0, \tag{14}$$

because the moment of the forces with respect to the z-axis is zero.

We thus have only three equations for the determination of the reactions. Hence if $n > 3$, then we shall not be able to determine the reactions. The problem of determining the reactions in the case, for example, of

a table standing on four legs is therefore statically indeterminate (*vide* p. 277).

However, if a body supported at $n > 3$ points is not rigid, then the reactions can be determined by appealing to the theory of elasticity. We shall show this in the next example.

Example 6. A rectangular table rests on four legs at the points A_1, A_2, A_3, A_4, of a smooth horizontal plane Π (Fig. 209).

Let us denote the reactions by $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4$, the weight by $\mathbf{Q}$, and the absolute values of these forces by R_1, R_2, R_3, R_4, Q. Let us assume that the points A_1, A_2, A_3, A_4, are the vertices of the rectangle and let us put:

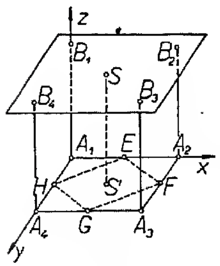


Fig. 209.

$$A_1A_2 = A_3A_4 = a, \quad A_1A_4 = A_2A_3 = b. \tag{15}$$

Let us take A_1 as the origin of the coordinate system, the lines A_1A_2 and A_1A_4 as the x and y axes, and the sense of the z-axis vertically upwards. Finally, let us denote by x_0, y_0, z_0, the coordinates of the centre of gravity S of the table top.

Forming the projections of the forces on the coordinate axes and taking the moments of the forces with respect to these axes, we obtain (cf. equations (13) and (14) of example 5):

$$R_1 + R_2 + R_3 + R_4 - Q = 0, \tag{16}$$

$$-(R_3 + R_4)\,b + Qy_0 = 0, \quad (R_2 + R_3)\,a - Qx_0 = 0. \tag{17}$$

The reactions cannot be determined from these equations.

Let us assume, however, that the table top and the plane Π (on which the legs of the table rest) are rigid, that the legs of the table are not rigid, but can be compressed, and that the reactions are proportional (in magnitude) to the contraction of the respective legs.

Therefore, if we denote the original length of the legs by l, and their lengths after compression by z_1, z_2, z_3, z_4, then the contractions are $l - z_1$, $l - z_2$, $l - z_3$, $l - z_4$, whence according to the assumption

$$R_1 = m(l - z_1), \; R_2 = m(l - z_2), \; R_3 = m(l - z_3), \; R_4 = m(l - z_4), \tag{18}$$

where m is the factor of proportionality.

Equations (16), (17) and (18) constitute a system of seven equations with eight unknowns R_1, R_2, R_3, R_4, and z_1, z_2, z_3, z_4. We obtain the eighth equation by stipulating that the points B_1, B_2, B_3, B_4, at which the table

top rests on the legs, lie in one plane; for we have assumed that the table top is rigid.

The points B_1, B_2, B_3, and B_4, have the coordinates $0, 0, z_1$, $a, 0, z_2$, a, b, z_3, and $0, b, z_4$. As is known from analytic geometry, the condition that these points lie in one plane is expressed by the formula

$$\begin{vmatrix} 0 & 0 & z_1 & 1 \\ a & 0 & z_2 & 1 \\ a & b & z_3 & 1 \\ 0 & b & z_4 & 1 \end{vmatrix} = 0.$$

Evaluating the determinant, we obtain:

$$z_1 - z_2 + z_3 - z_4 = 0. \tag{19}$$

From equations (16)—(19) we can determine the unknown reactions. We obtain:

$$\begin{aligned} z_1 &= l + \frac{1}{4m} Q\left[-3 + 2\left(\frac{x_0}{a} + \frac{y_0}{b}\right)\right], \quad z_2 = l + \frac{1}{4m} Q\left[-1 - 2\left(\frac{x_0}{a} - \frac{y_0}{b}\right)\right], \\ z_3 &= l + \frac{1}{4m} Q\left[1 - 2\left(\frac{x_0}{a} + \frac{y_0}{b}\right)\right], \quad z_4 = l + \frac{1}{4m} Q\left[-1 + 2\left(\frac{x_0}{a} - \frac{y_0}{b}\right)\right], \end{aligned} \tag{20}$$

$$\begin{aligned} R_1 &= \tfrac{1}{4}Q\left[3 - 2\left(\frac{x_0}{a} + \frac{y_0}{b}\right)\right], \quad R_2 = \tfrac{1}{4}Q\left[1 + 2\left(\frac{x_0}{a} - \frac{y_0}{b}\right)\right], \\ R_3 &= \tfrac{1}{4}Q\left[-1 + 2\left(\frac{x_0}{a} + \frac{y_0}{b}\right)\right], \quad R_4 = \tfrac{1}{4}Q\left[1 - 2\left(\frac{x_0}{a} - \frac{y_0}{b}\right)\right]. \end{aligned} \tag{21}$$

In order that formulae (21) give non-negative values for the reactions $R_1, \ldots, R_4$ the following relations must hold:

$$\tfrac{1}{2} \leqq \frac{x_0}{a} + \frac{y_0}{b} \leqq \tfrac{3}{2}, \quad -\tfrac{1}{2} \leqq \frac{x_0}{a} - \frac{y_0}{b} \leqq \tfrac{1}{2}. \tag{22}$$

It follows from this that the projection of the centre of gravity on the horizontal plane, i. e. the point $S'(x_0, y_0, 0)$, must lie within the parallelogram $EFGH$ whose vertices are the midpoints of the sides of the rectangle $A_1A_2A_3A_4$.

Let us suppose that the point S' falls within the triangle A_1EH (beyond the side EH). Then we would have

$$\frac{x_0}{a} + \frac{y_0}{b} < \tfrac{1}{2}, \;\text{ i. e. }\; 1 - 2\left(\frac{x_0}{a} + \frac{y_0}{b}\right) > 0,$$

whence by (20) $z_3 > l$; this means that the leg A_3B_3 becomes elongated, which is obviously impossible.

We must therefore assume that *the table rests on only three legs*, namely, at the points A_1, A_2, A_4. Putting $R_3 = 0$, we then obtain from equations (17):

$$R_4 = Qy_0 / b, \quad R_2 = Qx_0 / a,$$

and from equation (16)

$$R_1 = Q\left[1 - \left(\frac{x_0}{a} + \frac{y_0}{b}\right)\right].$$

Example 7. A heavy cylinder rests on a smooth horizontal plane. The cylinder is acted upon by a force couple $\mathbf{P}$ and $-\mathbf{P}$ lying in a vertical plane passing through the centre of gravity S. What can be the maximum magnitude of the moment of the couple if the cylinder is in equilibrium?

Let us denote by $\mathbf{M}$ the moment of the couple, by $\mathbf{Q}$ the weight of the cylinder, by $\mathbf{F}$ the resultant of the forces of reaction acting at O, and by r the radius of the base of the cylinder (Fig. 210).

If the cylinder is in equilibrium, the sum of the forces is zero; hence

$$\mathbf{F} + \mathbf{Q} = 0 \quad \text{or} \quad \mathbf{F} = -\mathbf{Q}.$$

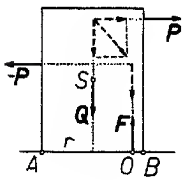


Fig. 210.

Assuming that S lies on the axis of the cylinder and putting $d = AO$, we get from the calculation of the moment with respect to A

$$|\mathbf{M}| + |\mathbf{Q}|r - |\mathbf{F}|d = 0.$$

Since $|\mathbf{Q}| = |\mathbf{F}|$,

$$|\mathbf{M}| = Q(d - r),$$

where $Q = |\mathbf{Q}|$. Since the maximum value of d is $2r$, the maximum value of $|\mathbf{M}|$ is Qr.

Hence if $|\mathbf{M}| > Qr$, equilibrium is impossible. However, if $|\mathbf{M}| \leqq Qr$, then, as is easily verified, the resultant of the forces $\mathbf{Q}$, $\mathbf{P}$ and $-\mathbf{P}$ is equal to $\mathbf{Q}$ and intersects the horizontal plane within the base of the cylinder. The cylinder can therefore remain in equilibrium.

§ 13. Internal forces. Through an arbitrary point O of an axis chosen in a given rigid body, e. g. in a beam, let us pass a plane Π perpendicular to this axis. The plane will divide the body into two parts I and II. Let us assume that the body is acted upon by certain forces and that it is in equilibrium. In many problems of engineering mechanics it is convenient to consider the parts I and II as separate rigid bodies tangent along the intersecting plane Π (Fig. 211).

From such a conception it follows that part II acts on part I with certain forces. These forces are called *stresses*.

Taking the point O as the centre of reduction, we can replace the stresses by one force $\mathbf{R}$ with its origin at O and a force couple of moment $\mathbf{M}$.

The component $\overline{OA}$ of the force $\mathbf{R}$, perpendicular to the section, is called the *compressive* or *tensile* resultant at O, depending on whether the component is directed towards part I or away from it.

The component $\overline{OB}$ of the force $\mathbf{R}$, tangent to the section, is called the resultant *bending* or *shearing* force at O.

The component $\overline{OC}$ of the moment $\mathbf{M}$, perpendicular to the intersecting plane Π, is called the *twisting moment* at O, and the component $\overline{OD}$ tangent to the surface is called the *bending moment* at O.

The twisting moment can be considered as the moment of a certain force couple lying in the intersecting plane, and the bending moment as the moment of a force couple lying in the plane tangent to the axis at the point O. The action of these couples, of which the first tends to twist and the second to bend, explain to us the names of the moments.

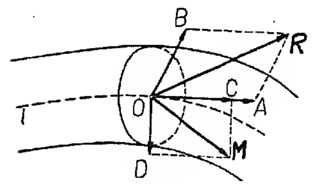


Fig. 211.

If the body is in equilibrium, then the external forces acting on part I balance the stresses. Consequently $-\mathbf{R}$ and $-\mathbf{M}$ are equal, respectively, to the sum and total moment with respect to O of the external forces acting on part I. Knowing the external forces, we can therefore determine $\mathbf{R}$ and $\mathbf{M}$.

A knowledge of the forces $\mathbf{R}$ and of the moment $\mathbf{M}$ is of great importance in the subject of strength of materials. In general, the larger the forces $\mathbf{R}$ and $\mathbf{M}$ are, the greater is the possibility that the body will be ruptured.

According to the law of action and reaction, the stresses with which part I acts on II can be replaced by the sum $-\mathbf{R}$ with its origin at O and by a couple of moment $-\mathbf{M}$.

Example 1. A beam supported at the points A and B carries the loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5$, directed vertically downwards and situated at the distances $x_1, x_2, \ldots, x_5$, from A (Fig. 174). Assuming that the supports are smooth, we obtain (cf. formulae (13) and (14), p. 263):

$$R_2 = (P_1x_1 + P_2x_2 + \ldots + P_5x_5) / d,$$
$$R_1 = [P_1(d - x_1) + \ldots + P_5(d - x_5)] / d,$$

where $AB = d$, and P_i, R_1, R_2, denote the absolute values of the forces and reactions.

Let us cut the beam by a plane perpendicular to the axis at the point C at a distance x from A. Let us denote by $\boldsymbol{R}$ the sum, and by $\boldsymbol{M}$ the moment with respect to C, of the stresses of part CB on part AC. Assuming that the cut occurs between the forces P_3 and P_4, we obtain:

$$-\boldsymbol{R} = \boldsymbol{R}_1 + \boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3,$$
$$-\boldsymbol{M} = \mathrm{Mom}_C\boldsymbol{R}_1 + \mathrm{Mom}_C\boldsymbol{P}_1 + \mathrm{Mom}_C\boldsymbol{P}_2 + \mathrm{Mom}_C\boldsymbol{P}_3.$$

Giving the y-axis a vertical direction with an upward sense, we obtain:

$$R_y = R_1 - P_1 - P_2 - P_3,$$
$$M = xR_1 - (x - x_1)P_1 - (x - x_2)P_2 - (x - x_3)P_3.$$

Since $\boldsymbol{R}$ and $\boldsymbol{M}$ lie in the intersecting plane, $\boldsymbol{R}$ is the bending force and $\boldsymbol{M}$ the bending moment. The compressive (or tensile) force and the twisting moment are zero. The force $\boldsymbol{R}$ and the moment $\boldsymbol{M}$ can also be determined by means of a string polygon as on pp. 252 and 253.

Example 2. A beam, built-in as in the Fig. 212, is loaded at A by the force $\boldsymbol{P}$. Let us form a section at the point C. Denote by $\boldsymbol{R}$ the sum, and by $\boldsymbol{M}$ the moment with respect to C, of the stresses of the left part on the right part. Consequently we have:

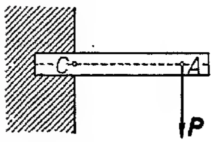


Fig. 212.

$$-\boldsymbol{R} = \boldsymbol{P}, \quad -\boldsymbol{M} = \mathrm{Mom}_C\boldsymbol{P}.$$

Hence the right part acts on the left part (built-in) with stresses of sum $-\boldsymbol{R}$ and moment $-\boldsymbol{M}$. The reactions of the wall, balancing these stresses, therefore have the sum $\boldsymbol{R} = -\boldsymbol{P}$ and the moment $\boldsymbol{M} = -\mathrm{Mom}_C\boldsymbol{P}$.

III. SYSTEMS OF BODIES

§ 14. Conditions of equilibrium. A necessary and sufficient condition for the equilibrium of a system of rigid bodies (free or not) is that each body of the system be in equilibrium. It follows from this that *the necessary and sufficient condition for the equilibrium of a system of rigid bodies is that, for each body separately, the forces acting on this body balance the reactions.*

The forces with which two bodies of a system act on each other are

called the *internal* forces of the system. The remaining forces are called *external.*

For example, if two bodies of a system touch each other, then the reactions at the points of contact are internal forces. On the other hand, those acting forces and reactions which arise from bodies not belonging to the system (e. g. from supports) are external forces.

Internal forces occur in pairs and are subject to the law of action and reaction; consequently the sum and total moment of the external forces are zero.

If a system of bodies is in equilibrium, then, for each body, the external forces acting on the body balance the internal forces. It follows from this that the external forces acting on the entire system balance the internal forces of the system. Since (as we have mentioned above) the internal forces have a sum and total moment equal to zero, *it follows that if a system of rigid bodies is in equilibrium, the sum and total moment of the external forces are zero.*

This condition is only sufficient, but not necessary for the equilibrium of a system of rigid bodies.

Each part of a system of rigid bodies which is in equilibrium is obviously itself in equilibrium. The external forces with respect to a certain part of the system are:

a) those external forces of the whole system which act on its given part,

b) the reactions exerted on this part by the remaining bodies of the system.

It follows from this that *the necessary and sufficient condition for the equilibrium of a system of rigid bodies is that the external forces acting on any part of the system balance the reactions exerted on this part by the remaining bodies of the system.*

For we can choose individual bodies of the system as the parts of the system.

Example. Two heavy rods AC and BC, lying in a vertical plane and touching at the point C, lean against vertical walls at A and B, and against horizontal plane at C (Fig. 213). Given are: the weights of the rods $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ acting at the centres of gravity, as well as the lengths $l_1 = AC$, $l_2 = BC$, $a_1 = S_1C$, $a_2 = S_2C$ and the distance d between the vertical walls. Determine in the posi-

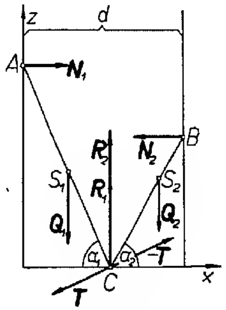


Fig. 213.

tion of equilibrium the angles α_1 and α_2 which the rods make with the horizontal under the assumption that there is no friction.

Let us denote the reactions of the vertical walls by $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ (these reactions therefore have a horizontal direction), the reactions of the horizontal wall by $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ (hence having a vertical direction), finally, the force with which the rod CB acts on the rod AC at the point C by $\boldsymbol{T}$. Then by the law of action and reaction the rod AC acts on the rod CB with a force $-\boldsymbol{T}$. Nothing can be said beforehand about the direction of the force $\boldsymbol{T}$.

Let us select the x and z axes in the plane of the rods, giving the x-axis a horizontal direction and the z-axis a direction vertically upwards. If the rod AC is in equilibrium, the forces acting on this rod balance one another. Therefore, forming their projections on the x and z axes and calculating the moment with respect to C, we obtain:

$$N_1 + T_x = 0, \quad -Q_1 + R_1 + T_y = 0, \tag{1}$$

$$N_1 l_1 \sin\alpha_1 - Q_1 a_1 \cos\alpha_1 = 0, \tag{2}$$

where N_1, R_1, and Q_1, denote the absolute values of the corresponding forces. Similarly, for the rod CB we get:

$$-N_2 - T_x = 0, \quad -Q_2 + R_2 - T_y = 0, \tag{3}$$

$$-N_2 l_2 \sin\alpha_2 + Q_2 a_2 \cos\alpha_2 = 0. \tag{4}$$

From equations (2) and (4) we have:

$$N_1 = Q_1 a_1 \cot\alpha_1 / l_1, \quad N_2 = Q_2 a_2 \cot\alpha_2 / l_2, \tag{5}$$

and from the first of the equations (1) and (3) $N_1 = N_2$, whence by (5)

$$Q_1 a_1 \cot\alpha_1 / l_1 = Q_2 a_2 \cot\alpha_2 / l_2. \tag{6}$$

Moreover, as is seen from the drawing,

$$l_1 \cos\alpha_1 + l_2 \cos\alpha_2 = d. \tag{7}$$

From equations (6) and (7) we can determine the angles α_1 and α_2.

Remark. We cannot determine the forces $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, and $\boldsymbol{T}$, from equations (1)—(4). However, we can obtain the forces $\boldsymbol{T}' = \boldsymbol{R}_1 + \boldsymbol{T}$ and $\boldsymbol{T}'' = \boldsymbol{R}_2 - \boldsymbol{T}$. They are the resultants of the reactions acting on the rods at C. We get:

$$T'_x = T_x = -N_1, \quad T'_y = R_1 + T_y = Q_1,$$
$$T''_x = -T_x = N_2, \quad T''_y = R_2 - T_y = Q_2.$$

§ 15. Systems of bars. If two forces act at the ends A and B of a rigid bar and the bar is in equilibrium, then these forces (because their sum and

total moment are equal to zero) act along the bar, are equal in magnitude and oppositely directed. Let us denote these forces by $\boldsymbol{P}$ and $-\boldsymbol{P}$.

Stresses in bars. Let us cut a bar at some point C and remove its right part CB (Fig. 214). Now, in order that the left part of the bar remain in equilibrium, it would be necessary to add the force $-\boldsymbol{P}$ with its initial point at C. Consequently we can assume that the right part CB acts on the left part AC with a force $-\boldsymbol{P}$. This force is called the *stress* in the bar.

If the forces $\boldsymbol{P}$ and $-\boldsymbol{P}$ are directed towards each other, the stress is called a *compression* (Fig. 215), in the opposite case a *tension* (Fig. 214).

At every point of the bar the stress has the same magnitude and direction; the sense of the stress, however, depends on whether we are considering the reaction of the right part on the left, or conversely. With reference to the end of the bar, we can only talk about the reaction of the whole bar on the end. Therefore, if we give the magnitude of the stress and its kind (i. e. whether it is a tension or a compression), then the stresses at the ends of the bar will be completely defined.

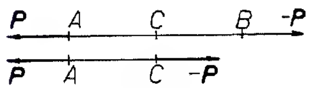


Fig. 214.

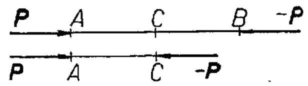


Fig. 215.

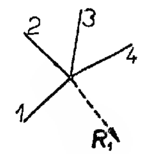


Fig. 216.

Pin-connections. Let us imagine that several rigid bars are so connected that they must constantly be in contact with each other at certain points, e. g. at the ends. If, in addition to this, the connection does not cause other limitations of the motion of the bars, we say that the bars are *pin-connected* and the points at which the bars are pinned are called *joints*.

A pin-connection can be obtained approximately by joining, for example, the ends of the bars by a very short inextensible string.

According to the theorem on reduction, the reactions which one bar exerts on the other can be replaced by one force $\boldsymbol{R}$ acting at the point of contact and a force couple of moment $\boldsymbol{M}$.

For simplicity's sake we shall assume that $\boldsymbol{M} = 0$. We then say that *the joint is smooth*.

It should be noted that not always can we assume that a joint is smooth; examples of this will be given later (p. 294).

In the case of a smooth joint, the reaction with which one bar acts on the other is a force acting at the point of contact, i. e. at the joint. In particular, if several bars come together at a smooth joint, then the

reaction exerted on a certain bar by those remaining will be a force acting at the joint (e. g. the reaction $\boldsymbol{R}_1$ of the bars 2, 3, 4, on the bar 1, shown in Fig. 216).

Systems of bars. Let us consider a system of bars connected at their ends. If some external force acts at a joint, then it is necessary to specify clearly on which bar this force acts.

If a system of bars is in equilibrium, the forces acting on each bar must obviously balance the reactions exerted on this bar.

In Fig. 217 is shown a system of bars in which the external forces $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, and $\boldsymbol{P}_3$, are acting on the bar AB. The force $\boldsymbol{P}_1$ is acting at the end A of the bar. These forces balance the reactions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ at the ends A and B.

It is often convenient to consider a joint as a separate material point (as a separate body) connected with the ends of the bars coming together at this joint. In other words, it is assumed that the ends of the bars are not connected together directly, but by means of a joint. Under this assumption the reactions of the pinned bars are replaced by the reactions of the joint on these bars and the forces applied at a joint are considered as forces acting on the joint, and not on the bars. The only internal forces of a system of bars will then be the reactions of the joints on the bars and those of the bars on the joints.

In the case of the equilibrium of a system each bar and joint is in equilibrium; consequently (p. 286):

1° *external forces acting on an arbitrary bar* (not attached at a joint) *balance the reactions which the joints exert on this bar*;

2° *external forces acting at any joint whatsoever balance the reactions of the bars at this joint.*

In Fig. 218 the force $\boldsymbol{P}_2$ with its origin at A balances the reactions $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, $\boldsymbol{S}_3$, of the bars 1, 2, 3, on the joint A. On the other hand, the external forces $\boldsymbol{P}_1$, $\boldsymbol{P}_3$, acting on the bar AB balance the reactions $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ of the joints A and B on this bar.

In general, nothing can be said in advance about the directions of the reactions of the joints on the bars. The situation is different, however, when the external forces are applied only at the joints.

Let us consider just such a system of bars which remain in equilibrium (Fig. 219). Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ denote the reactions of the joints A and B on the bar connecting these joints. Since no external forces act on the bar, the reactions $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ must balance (this follows from condition 1°). Therefore the reactions act along the bar and we have $\boldsymbol{T}_1 = -\boldsymbol{T}_2$.

Hence: *if a system of pin-connected bars is in equilibrium, and the external forces are applied only at the joints, then the reactions of the joints on the bars are forces directed along the bars; the reactions which the joints exert on the bar connecting them are equal in magnitude and direction, but opposite in sense.*

The reactions of the joints A and B cause a stress in the bar which may be a tension or a compression (in our case the stress is a tension). By

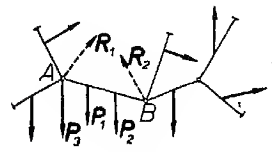


Fig. 217.

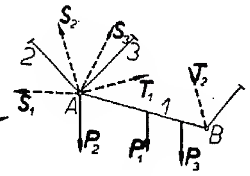


Fig. 218.

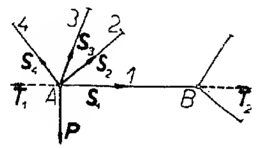


Fig. 219.

the law of action and reaction the bar AB acts on the joint A with a force $\boldsymbol{S}_1 = -\boldsymbol{T}_1$. The force $\boldsymbol{S}_1$ is therefore a stress in the bar at the end A. The reactions of the bars on the joints are then equal to the stresses in these bars. From condition 2°, therefore, we obtain the following theorem:

The external forces applied at a joint balance the stresses in the bars (coming together at this joint).

In Fig. 219 the stresses $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, $\boldsymbol{S}_3$, $\boldsymbol{S}_4$ in the bars 1, 2, 3, 4, balance the force $\boldsymbol{P}$ acting at the joint A; therefore $\boldsymbol{P} + \boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3 + \boldsymbol{S}_4 = 0$.

Example 1. Three bars AB, BC, and CD, pin-connected at the points B and C, and fixed by means of the joints at the points A and D, remain in equilibrium in a vertical plane under the action of the vertical forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ whose origins are E and F. Given are: the force $\boldsymbol{P}$ and the points of application E and F. Determine the force $\boldsymbol{Q}$ (Fig. 220).

The reactions $\boldsymbol{R}_1$ and $\boldsymbol{T}_1$ act at the points A and B of the bars AB and BC, respectively. We have $\boldsymbol{R}_1 + \boldsymbol{T}_1 = 0$; the reactions $\boldsymbol{R}_1$ and $\boldsymbol{T}_1$ therefore act along the bar AB.

The reaction $-\boldsymbol{T}_1$, the force $\boldsymbol{P}$, and the reaction $\boldsymbol{T}_2$ of the bar CD at the point C, act on the bar BC. Since the bar BC is in equilibrium, the directions of these forces intersect at one point G, which we find as the intersection of the line AB and of the direction of the force $\boldsymbol{P}$. Having the point G, we can obtain the direction CG of the reaction $\boldsymbol{T}_2$. Since $-\boldsymbol{T}_1 + \boldsymbol{P} + \boldsymbol{T}_2 = 0$, knowing the

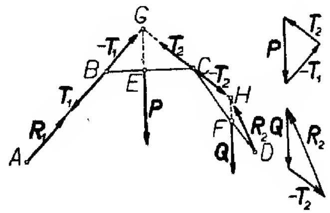


Fig. 220.

directions of the forces $\mathbf{T}_1, \mathbf{T}_2$, and the force $\mathbf{P}$, we can determine the forces $\mathbf{T}_1, \mathbf{T}_2$ from the triangle of forces.

The reactions $-\mathbf{T}_2$, $\mathbf{R}_2$, and the force $\mathbf{Q}$, act on the bar CD. These forces intersect at one point H which we obtain as the point of intersection of the directions of the forces $-\mathbf{T}_2$ and $\mathbf{Q}$. Having the point H, we obtain the direction of the force $\mathbf{R}_2$. Since $-\mathbf{T}_2 + \mathbf{Q} + \mathbf{R}_2 = 0$, knowing $\mathbf{T}_2$, we obtain the forces $\mathbf{Q}$ and $\mathbf{R}_2$ from the triangle of forces.

Example 2. Four bars 1, 2, 3, 4, are pin-connected at A, B, and C, and fixed at the joints E and F. The bars are inclined to the horizontal at the angles α, β, γ, and δ. Vertical forces $\mathbf{P}_1, \mathbf{P}_2$, and $\mathbf{P}_3$, act at the joints A, B, and C. The force $\mathbf{P}_1$ is given. Determine the forces $\mathbf{P}_2$ and $\mathbf{P}_3$ as well as the reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ at E and F (Fig. 221).

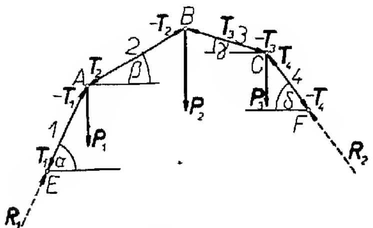


Fig. 221.

Since the external forces act only at the joints, the stresses in the bars at each joint balance the external forces acting at this joint.

At the joint E the stress $\mathbf{T}_1$ in the bar 1 balances the reaction $\mathbf{R}_1 : \mathbf{T}_1 + \mathbf{R}_1 = 0$.

At the joint A the stresses $-\mathbf{T}_1$ in bar 1 and $\mathbf{T}_2$ in bar 2 balance the force $\mathbf{P}_1$: $-\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{P}_1 = 0$. Denoting the absolute values of these forces by T_1, T_2, and P_1, and forming their projections on the horizontal and vertical directions, we obtain:

$$T_1 \cos\alpha - T_2 \cos\beta = 0, \quad T_1 \sin\alpha - T_2 \sin\beta - P_1 = 0.$$

From these equations we calculate T_1 and T_2.

At the joint B for the stress $\mathbf{T}_3$ in bar 3 we get the relation $-\mathbf{T}_2 + \mathbf{T}_3 + \mathbf{P}_2 = 0$. Forming the projections on the horizontal and vertical directions and putting $T_3 = |\mathbf{T}_3|$ and $P_2 = |\mathbf{P}_2|$, we obtain:

$$T_2 \cos\beta - T_3 \cos\gamma = 0, \quad T_2 \sin\beta + T_3 \sin\gamma - P_2 = 0.$$

Using an analogous notation, we get at the joint C:

$$T_3 \cos\gamma - T_4 \cos\delta = 0, \quad -T_3 \sin\gamma + T_4 \sin\delta - P_3 = 0,$$

from which we calculate T_4 and P_3.

At the joint F we finally obtain $-\mathbf{T}_4 + \mathbf{R}_2 = 0$ or $\mathbf{R}_2 = \mathbf{T}_4$.

Example 3. Decimal balance. A beam AC, supported at O, is connected at the points B and C with the beams DF and GK by means of the

rods BD and CG. The beam DF, supported at the point F, rests at the point H on the beam GK, supported at K. At the points B, C, G and D there are pin-connections (Fig. 222).

A weight $\mathbf{Q}$, which is to be weighed, is put on the beam DF and balanced by the weight $\mathbf{P}$ placed on a pan hanging from A. The weights of the beams and bars are neglected. Determine the relation between the weights $\mathbf{P}$ and $\mathbf{Q}$.

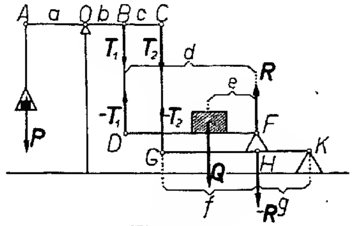


Fig. 222.

Let us denote by $\mathbf{T}_1$, $\mathbf{T}_2$, the stresses in the bars BD and CG at the points B and C.

If the beam AC is in equilibrium, the sum of the moments of the forces acting on it with respect to O is equal to zero:

$$-Pa + bT_1 + (b + c)\,T_2 = 0, \tag{8}$$

where $P = |\mathbf{P}|$, $T_1 = |\mathbf{T}_1|$, $T_2 = |\mathbf{T}_2|$, and the lengths a, b, and c, are those shown in the figure.

The forces acting on the beam DF are: the stress $-\mathbf{T}_1$ in the bar BD at the point D, the weight $\mathbf{Q}$, and the reaction $\mathbf{R}$ at the point F. Forming the projections on a vertical direction and taking the moment with respect to F, we obtain in the position of equilibrium for the beam DF:

$$T_1 + R - Q = 0, \quad T_1 d - Qe = 0, \tag{9}$$

where $R = |\mathbf{R}|$, $Q = |\mathbf{Q}|$, and the lengths d and e are those given in the figure.

The forces acting on the beam GK are: the stress $-\mathbf{T}_2$ in the bar CG at the point G and the reaction $-\mathbf{R}$ of the beam DF at the point H. Forming the moment of these forces with respect to the point of support K, we obtain

$$T_2(f + g) - Rg = 0. \tag{10}$$

From equations (9) we obtain:

$$T_1 = Q \cdot \frac{e}{d}, \quad R = Q \cdot \frac{d - e}{d},$$

whence by equation (10)

$$T_2 = Q \cdot \frac{d - e}{d} \cdot \frac{g}{f + g}.$$

Substituting the values obtained in equation (8) for T_1 and T_2, we obtain

$$P = Q\frac{(bf - cg)\,e + (b + c)\,gd}{ad(f + g)}. \tag{11}$$

If we assume that $bf - cg = 0$, whence

$$b / c = g / f, \tag{12}$$

then P will be independent of e, i. e. of the position of the weight $\mathbf{Q}$ on the beam DF. Hence in virtue of (11) and (12)

$$P = Q \cdot \frac{b}{a}.$$

For $b / a = \frac{1}{10}$ we have a *decimal balance*.

Example 4. Two bars fixed at the ends A and B and pinned at C are collinear. A force $\mathbf{P}$ whose origin is at D acts on the bar AC in a direction perpendicular to AC (Fig. 223). The system of bars is in equilibrium, since the point C cannot change its position.

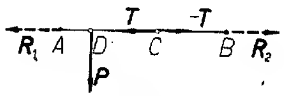


Fig. 223.

Let us suppose for a moment that the bar AC acts on CB with a force $\mathbf{T}$ whose initial point is at C. Consequently the bar CB would act on the bar AC with a force $-\mathbf{T}$ also acting at C.

Let us denote by $\mathbf{R}_1$ and $\mathbf{R}_2$ the reactions at A and B.

Since the bar CB is in equilibrium, the forces $\mathbf{R}_2$ and $\mathbf{T}$ act along the bar AB. It follows from this that the bar AC cannot be in equilibrium because the forces $-\mathbf{T}$, $\mathbf{R}_1$, and $\mathbf{P}$, acting on this bar do not balance one another, for their total moment with respect to A is equal to the moment of the force $\mathbf{P}$ with respect to A, which is different from zero. We have thus arrived at a contradiction.

We must therefore assume that the bar AC acts on the bar CB with a force equipollent to one force and a force couple of moment different from zero.

§ 16. Frames.

§ 16. Frames. A system of rigid bars, pin-connected and forming as a whole a rigid body, is called a *space frame* (or *truss*).

Examples of space frames are 1. a system of three bars, pin-connected and forming a triangle (Fig. 224a), 2. a system of six bars forming the edges of a tetrahedron (Fig. 224b). On the other hand, a system of bars forming the edges of a rectangular parallelepiped and pin-connected at the vertices is not a frame because the bars can change their relative positions.

Joints of a frame are also called *nodes*.

Plane frame. If a system of pin-connected rigid bars is coplanar and the bars cannot change their mutual positions in this plane, then such a system is called a *plane frame*.

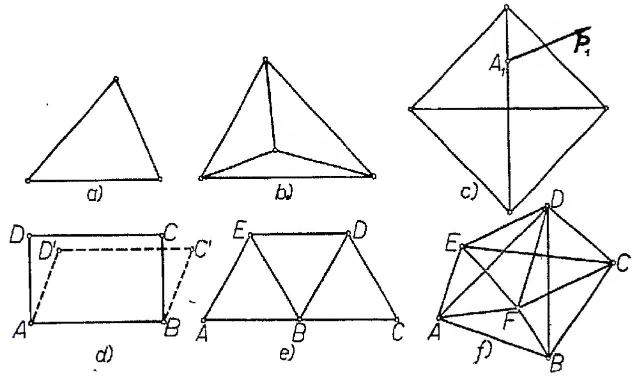


Fig. 224.

Examples of plane frames are represented in Fig. 224a, b, c, e, and f.

The system of bars in Fig. 224d does not form a frame, even a plane frame, because the bars can change their relative positions; they can assume e. g. the position indicated by the dotted lines.

A plane frame does not form a rigid system if we admit motions of the bars in space. For example, if we fix the joints B, C, D, and E, of the frame in Fig. 224e, then we can rotate the bars AE and AB in space about EB. In the plane of the frame, the bars AE and AB cannot move.

If we remove the bar AB in the frame shown in Fig. 224f, then in its plane the system of bars continues to remain a rigid system, i. e. a frame. Such a bar is called a *redundant* bar.

The frame shown in Fig. 224e does not have any redundant bars.

Analytical method of determining stresses in a frame. A plane frame has p bars and w joints $A_1, A_2, \ldots, A_w$, at which the external forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_w$ are applied. When the joint A_i is connected by a bar with the joint A_j (Fig. 225), we denote the length of this bar by d_{ij} and the number whose ab-

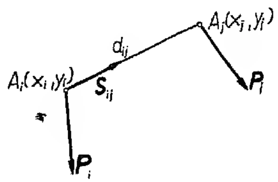


Fig. 225.

solute value is equal to the magnitude of the stress in this rod by S_{ij}; the sign is $+$ or $-$ depending on whether the stress is a tension or a compression (p. 289).

Let us choose an arbitrary coordinate system and denote the coordinates of the joints $A_1, A_2, \ldots, A_w$, by $x_1, y_1, x_2, y_2, \ldots, x_w, y_w$. From the definition of the number S_{ij} it follows that for $i = 1, 2, \ldots, w$, the stress in the bar at the joint A_i has the projections

$$\frac{x_j - x_i}{d_{ij}} S_{ij}, \quad \frac{y_j - y_i}{d_{ij}} S_{ij},$$

on the coordinate axes.

Since the external force $\mathbf{P}_i$ at the joint A_i balances the stresses in the bars pinned at this joint (p. 291),

$$P_{i_x} + \sum_j \frac{x_j - x_i}{d_{ij}} S_{ij} = 0, \quad P_{i_y} + \sum_j \frac{y_j - y_i}{d_{ij}} S_{ij} = 0,$$

i. e.

$$P_{i_x} = \sum_j \frac{x_i - x_j}{d_{ij}} S_{ij}, \quad P_{i_y} = \sum_j \frac{y_i - y_j}{d_{ij}} S_{ij}, \tag{1}$$

where the summation extends over all indices j for which the joint A_j is connected by a bar with the joint A_i. Since, by hypothesis, there are w joints, system (1) consists of $2w$ equations.

Equations (1) serve to determine the stresses S_{ij} when the forces $\mathbf{P}_i$ are given. It may happen, however, that system (1) does not possess a solution or there are too few equations to determine the unknowns S_{ij}.

For instance, in the frame shown in Fig. 224c, p. 295, the bars in contact at the joint A_1 are collinear and the external force $\mathbf{P}_1$ acting at this joint does not lie on this line. The stresses in these bars cannot balance the force $\mathbf{P}_1$. The system of equations (1) for this frame does not therefore have a solution (p. 294).

In the frame of Fig. 224f, p. 295, we have 13 bars and 6 vertices. The number of unknown stresses is consequently 13 and the number of equations in system (1) is only $2 \cdot 6 = 12$. Therefore there are too few equations.

In equations (1) let us denote the right sides of the first equations by E_i, and those of the second by F_i. Equations (1) then assume the form:

$$P_{i_x} = E_i, \quad P_{i_y} = F_i. \tag{2}$$

It can be easily verified by calculation that:

$$\sum_{i=1}^{w} E_i = 0, \quad \sum_{i=1}^{w} F_i = 0, \quad \sum_{i=1}^{w} (E_i y_i - F_i x_i) = 0. \tag{3}$$

The equalities (3) are identities, i. e. they hold for all values of S_{ij}.

The identities (3) can also be derived without calculation in the following manner:

Let us choose S_{ij} entirely arbitrarily and determine P_{i_x} and P_{i_y} from equations (1). Since these equations express the fact that the stresses at every joint of the bars balance the external forces, the forces so determined will be in equilibrium (p. 290). Consequently the external forces will be in equilibrium, i. e. the following equalities will hold:

$$\sum_{i=1}^{w} P_{i_x} = 0, \quad \sum_{i=1}^{w} P_{i_y} = 0, \quad \sum_{i=1}^{w} (P_{i_x} y_i - P_{i_y} x_i) = 0. \tag{4}$$

It follows from this, in virtue of (2), that relations (3) must be satisfied identically for all values of S_{ij}.

Let us now assume that for a certain frame equations (1) have a solution for every system of forces $\{\mathbf{P}_i\}$ equipollent to zero. Let us further assume that the right sides of equations (1) satisfy identically some linear relation of the form

$$a_1 E_1 + a_2 E_2 + \ldots + b_1 F_1 + b_2 F_2 + \ldots = 0, \tag{5}$$

where $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are certain constants.

Let the system of forces $\{\mathbf{P}_i\}$ be equipollent to zero; equations (1) therefore have a solution. By (2) and (5) the forces $\{\mathbf{P}_i\}$ must consequently satisfy the relation

$$a_1 P_{1_x} + a_2 P_{2_x} + \ldots + a_w P_{w_x} + b_1 P_{1_y} + b_2 P_{2_y} + \ldots + b_w P_{w_y} = 0. \tag{6}$$

Hence, if the forces $\{\mathbf{P}_i\}$ satisfy equations (4), then they also satisfy equation (6). Relation (6) is therefore dependent on relations (4). It follows from this that relation (5) depends on relations (3).

The right sides of equations (1) consequently satisfy only three independent relations (3). Hence the system of equations (1) has $2w - 3$ independent equations (while three equations depend on the remaining ones). The unknowns must be at least as many as there are independent equations, i. e. $\geq 2w - 3$. Since there are as many unknowns S_{ij} as there are bars, namely p,

$$p \geq 2w - 3. \tag{7}$$

When $p > 2w - 3$, the number of independent equations is less than the number of unknowns; hence there exist infinitely many solutions. When $p = 2w - 3$, there are as many unknowns as linearly independent equations; consequently the stresses S_{ij} are uniquely determined.

A frame is said to be *statically determinate* if equations (1) determine

uniquely the stresses S_{ij} of the bars for every system of forces $\{\mathbf{P}_i\}$ which is in equilibrium. Therefore we have proved the theorem

I. *If a frame is statically determinate, then* $p = 2w - 3$ (where p denotes the number of bars and w the number of joints).

One can prove the theorem

II. *A statically determinate frame does not possess redundant bars.*

The conditions expressed in theorems I and II are necessary, but not sufficient, in order that a frame be statically determinate (*vide* Fig. 224c, p. 295).

Determination of stresses in a frame (by means of force diagrams). We are to determine the stresses in the bars of the frame shown in the Fig. 226. Joints are denoted by the letters A, B, C, D, and bars by the numbers *1*, *2*, *3*, *4*, *5*. The frame is loaded by a vertical force $\mathbf{P}$ at the joint C and rests on smooth supports A and B.

Let us first determine the reactions at A and B. On account of symmetry each of the reactions is equal to $-\frac{1}{2}\mathbf{P}$.

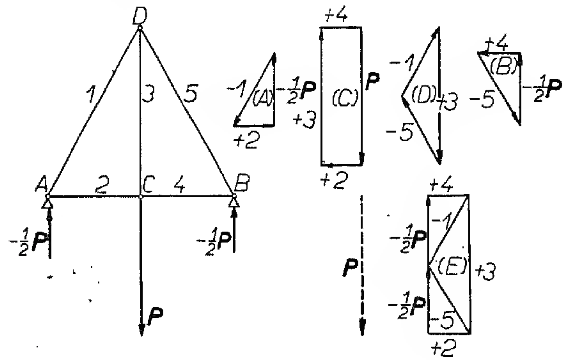


Fig. 226.

The joint A is acted upon by the external force $-\frac{1}{2}\mathbf{P}$ and the stresses in the bars *1* and *2*. Since these forces balance one another, they form a closed polygon which we can draw because we know the force $-\frac{1}{2}\mathbf{P}$ and the directions of the stresses. This polygon is shown in Fig. 226 (A); the stresses are denoted by the numbers *1* and *2*, and the signs before the numbers denote whether the stress is a tension (+) or a compression (—).

The joint C is acted upon by the stresses in the bars *2*, *3*, and *4*, which balance the force $\mathbf{P}$. Since only two forces are unknown, namely, the stresses in the bars *3* and *4*, we can draw a force polygon, remembering

that the stress in bar *2* at the joint C has a sense opposite to that at the joint A. This polygon is shown in Fig. 226 (C).

No external forces act at the joint D; consequently the stresses balance one another. Therefore they form a closed polygon which can be drawn (Fig. 226 (D)), remembering that the stresses in bars *1* and *3* at the joint D have senses opposite to those at A and C.

We have determined the stresses in every bar. In order to verify our reasoning we can form another polygon for the joint B (Fig. 226 (B)).

Proceeding in this manner we have drawn each force twice. However we can simplify matters by combining all the polygons (A), (B), (C), and (D), as in Fig. 226 (E).

In Fig. 228 (E) each force appears only once; such a drawing is called a *Cremona force diagram* for the given frame.

We shall give certain directions for obtaining a Cremona force diagram in the following example:

Fig. 227 represents a frame and Fig. 228 its Cremona force diagram. The frame is loaded at the joints G and F by forces $\mathbf{P}$ and $2\mathbf{P}$. At the joints A and E it rests on smooth supports. Bars *2*, *6*, and *10*, are horizontal and equal in length. Calculating the moment of the external forces with respect to E and A, we find that the reactions at A and E are $-\frac{4}{3}\mathbf{P}$ and $-\frac{5}{3}\mathbf{P}$, respectively.

We now draw the polygon of external forces in the order in which they appear on the perimeter of the frame. For instance, going clockwise we draw in turn $-\frac{4}{3}\mathbf{P}$, $-\frac{5}{3}\mathbf{P}$, $2\mathbf{P}$, and $\mathbf{P}$.

We next construct a polygon for the joint A. Let us note that bar *2* connects the joints A and G at which the external forces $-\frac{4}{3}\mathbf{P}$ and $\mathbf{P}$ act. In the force diagram the stress in bar *2* is drawn from the origin of the force $-\frac{4}{3}\mathbf{P}$ and from the terminus of the force $\mathbf{P}$. The force $-\frac{4}{3}\mathbf{P}$ defines the sense of the forces in the polygon for the joint A. We obtain the

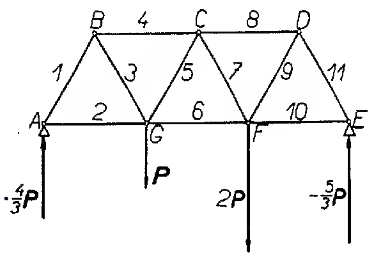


Fig. 227.

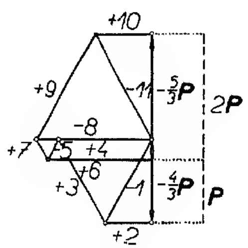


Fig. 228.

stresses in the bars *1* and *2* and indicate that bar *1* is in compression (—) while *2* is in tension (+).

Let us now proceed to consider a joint where only two stresses are unknown. Such a joint is B. We determine the polygon of stresses in the bars *1*, *3*, and *4*. We obtain the sense of the forces because we know from the preceding polygon that bar *1* is in compression.

Let us next consider the joint G at which only two stresses are unknown, namely, the stresses in bars *5* and *6*. Proceeding as before, we consider in turn the joints C, F, D, and E.

In preparing a Cremona force diagram it is necessary to adhere to the following rules:

1. The external forces in the polygon of forces are drawn on the diagram in the order in which they appear on the perimeter of the frame.

2. If a bar on the perimeter of the frame connects joints which are the origins of the external forces, then the stress in the bar is drawn on the diagram of forces from the point at which the terminus of one external force meets with the origin of the other.

Let us note that the Cremona force diagram represented in Fig. 228, p. 299, has, besides, the following two properties:

a) the forces acting at a joint form a closed polygon in the diagram,

b) if three bars form a triangle, then in the diagram their stresses have origin at one point.

A Cremona force diagram having the above two properties is called a *reciprocal force diagram.*

This name has reference to the so-called theory of reciprocal figures.

Let us note that it is not possible to construct a Cremona force diagram for every statically determinate frame.

Determination of stresses by means of sections. Let us suppose that a frame is such that it is possible to cut three of its bars whose origins are not at one joint, in such a manner that the frame is divided into two parts. If at least two of the bars cut are not parallel, then it is possible to determine the stresses in the bars cut without calculating the stresses in the remaining bars.

Let us denote the bars cut by *1, 2, 3*. If one part of the frame is removed, e. g. the right part, then the left part will remain in equilibrium after the addition of the stresses $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$.

Let the bars *1* and *2* intersect at the point O (Fig. 229). Since the left part of the frame is in equilibrium, the external forces acting on this part balance the stresses $\mathbf{S}_1, \mathbf{S}_2$, and $\mathbf{S}_3$. Denoting by $\mathbf{M}$ the moment of the

external forces (acting on the left part of the frame) with respect to O, and by d the distance of O from the bar *3*, we get $|\mathbf{M}| = |\mathbf{S}_3|d$, i. e. $|\mathbf{S}_3| = |\mathbf{M}|/d$.

We choose the sense of the force $\mathbf{S}_3$ such that $\mathbf{M}$ and the moment of the force $\mathbf{S}_3$ with respect to O have opposite signs. The moment $\mathbf{M}$ can be obtained by determining at first the resultant $\mathbf{R}$ (or possibly a resultant couple) of the external forces acting on the left part, and next calculating the moment of the resultant $\mathbf{R}$ (or a resultant couple) with respect to O. We calculate $\mathbf{S}_2$ and $\mathbf{S}_1$ similarly by forming the moment with respect to the point of intersection of bars *1, 3* and *2, 3*.

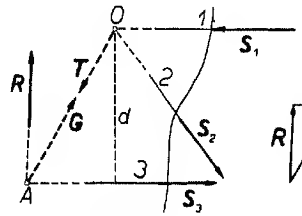


Fig. 229.

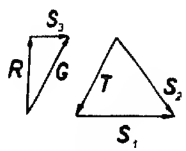


Fig. 230.

If bars *1* and *3* were parallel, we would obtain the force $\mathbf{S}_2$ by forming the projections of the forces on a line perpendicular to the bars *1* and *3* (for the projections of the forces $\mathbf{S}_1$ and $\mathbf{S}_3$ will be zero).

The method described above of calculating stresses was given by J. W. Ritter. The stresses $\mathbf{S}_1, \mathbf{S}_2$, and $\mathbf{S}_3$, can also be determined graphically by means of a method given by K. Culmann.

Let the bars *1* and *2* intersect at O, let the external forces have a resultant $\mathbf{R}$, and let $\mathbf{R}$ and $\mathbf{S}_3$ intersect at the point A. Let us denote the resultant of the forces $\mathbf{S}_1$ and $\mathbf{S}_2$ by $\mathbf{T}$, and the resultant of the forces $\mathbf{R}$ and $\mathbf{S}_3$ by $\mathbf{G}$.

Since the forces $\mathbf{R}, \mathbf{S}_1, \mathbf{S}_2$, and $\mathbf{S}_3$, are in equilibrium, the forces $\mathbf{T}$ and $\mathbf{G}$ are also in equilibrium. It follows from this that $\mathbf{T} = -\mathbf{G}$ and that the forces $\mathbf{T}$ and $\mathbf{G}$ are collinear. Since $\mathbf{T}$ has its origin at O, and $\mathbf{G}$ at A, the forces $\mathbf{T}$ and $\mathbf{G}$ lie on the line OA. Knowing already the direction of the forces $\mathbf{T}$ and $\mathbf{G}$, we determine the triangle of forces $\mathbf{R}, \mathbf{S}_3$, and $\mathbf{G}$, from which we obtain the forces $\mathbf{S}_3$ and $\mathbf{G}$. Since $\mathbf{T} = -\mathbf{G}$, we can construct the triangle of forces $\mathbf{S}_1, \mathbf{S}_2$, and $\mathbf{T}$, from which we can get $\mathbf{S}_1$ and $\mathbf{S}_2$ (Fig. 230).

If the resultant $\mathbf{R}$ were parallel to the bar *3*, the force $\mathbf{G}$ would also be parallel to *3* and would pass through O. Since $\mathbf{R}$ is the resultant of the forces $\mathbf{G}$ and $-\mathbf{S}_3$, the problem would then be reduced to the resolution of

the force $\mathbf{R}$ into two forces $\mathbf{G}$ and $-\mathbf{S}_3$ whose positions are given. Such a problem was solved by means of a string polygon on p. 253.

Finally, if the external forces were reduced to a couple $\mathbf{R}_1$ and $\mathbf{R}_2$, we should have $\mathbf{R}_1 = -\mathbf{R}_2$ and the resultant $\mathbf{G}$ of the forces $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{S}_3$, would be equal to $\mathbf{S}_3$ and would have its origin at O. The problem would then be reduced to the resolution of the system of forces $\mathbf{R}_1$ and $\mathbf{R}_2$ into two forces $\mathbf{G}$ and $-\mathbf{S}_3$ whose positions are given (cf. p. 253).

§ 17. Equilibrium of heavy cables. Chain. A system of rigid pin-connected rods is called a *chain* if only two rods are pinned at each joint. The rods of a chain are also called *links*.

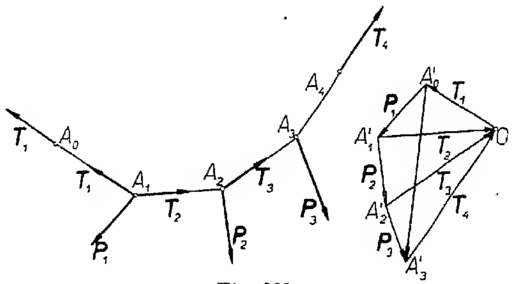


Fig. 231.

Let us assume that a chain consisting of the links A_0A_1, A_1A_2, A_2A_3, and A_3A_4, pin-connected at the joints A_1, A_2, and A_3, remains in equilibrium under the action of the forces $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, applied at the joints, and the forces $\mathbf{T}_1$ and $\mathbf{T}_4$ applied at A_0 and A_4 (Fig. 291). The forces $\mathbf{T}_1$ and $\mathbf{T}_4$ obviously have the directions of the rods A_0A_1 and A_3A_4 (p. 291).

It is easy to show that *a chain in equilibrium assumes the form of a string polygon* of the system of forces $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$.

To show this let us construct a polygon of forces $A_0'A_1'A_2'A_3'$ for the system $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$. For the pole O let us take the point of intersection of the lines drawn from the points A_0', A_3' and parallel to the extreme rods of the chain.

Since the chain is in equilibrium, the sum of the forces is zero:

$$\mathbf{T}_1 + \mathbf{T}_4 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = 0.$$

Since $\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = \overline{A_0'A_3'}$, and $\overline{OA_0'}$, $\overline{A_3'O}$ are parallel to $\mathbf{T}_1$, $\mathbf{T}_3$, from triangle $A_0'A_3'O$ we obtain:

$$\mathbf{T}_1 = \overline{OA_0'},\quad \mathbf{T}_3 = \overline{A_3'O}. \tag{1}$$

Let us now consider the joint A_1. The stress of the link A_0A_1 at the joint A_1 is $\mathbf{T}_1$; let us denote by $\mathbf{T}_2$ the stress of the link A_1A_2 at the joint A_1. We obviously have

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{P}_1 = 0. \tag{2}$$

From the string polygon we obtain $\overline{OA_0'} + \mathbf{P}_1 + \overline{A_1'O} = 0$, whence by (1) $\mathbf{T}_1 + \mathbf{P}_1 + \overline{A_1'O} = 0$, and from this by (2) $\overline{A_1'O} = \mathbf{T}_2$. The segment $A_1'O$ is therefore parallel to the rod A_1A_2.

Similarly, we ascertain that the segments $A_2'O$ and $A_3'O$ are parallel to the rods A_2A_3 and A_3A_4.

It follows from this that the string polygon drawn from the point A_1 will assume the form of a chain.

Cable. A *rope* or a *cable* (*flexible* and *inextensible*) is defined as a material line which can be bent arbitrarily without changing its length or that of any of its parts.

A rope can therefore assume the form of an arbitrary curve of the same length. A cable can be considered approximately as a chain consisting of very many small links.

Let a heavy cable (flexible and inextensible) be suspended from two points A and B. Let us assume that the density of the cable ϱ = constant. The weight of a portion of the cable of length s cm is therefore

$$Q = s\varrho g = s\delta, \tag{3}$$

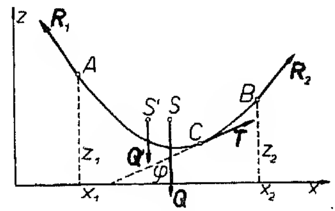


Fig. 232.

where $\delta = \varrho g$.

Let us determine the form that the cable will assume under the action of its own weight.

Let us choose a system of coordinates (x, y, z), giving the z-axis a vertical direction and an upward sense; let the xz-plane be taken vertically and passing through the points A and B (Fig. 232).

The external forces acting on the cable are: the weight, acting at the centre of gravity S of the cable, and the reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ at the points A and B. If the cable is in equilibrium, these forces balance each other. It follows from this that they lie in one vertical plane, namely, the xz-plane. Consequently:

$$R_{1_y} = 0,\quad R_{2_y} = 0. \tag{4}$$

Let us cut the cable at an arbitrary point C and remove the part CB. In order that the portion AC remain in equilibrium, it is necessary to add at the point C the force $\mathbf{T}$ which the portion CB exerts on the portion AC. The force $\mathbf{T}$ is the *tension of the cable.*

Since the cable is considered approximately as a chain consisting of small links, the force $\mathbf{T}$ is tangent to CB.

Let us denote the length of the arc AC by s. The external forces acting on the part AC are: the weight $\mathbf{Q}'$ of magnitude $s\delta$, acting at the centre of gravity S' of the part AC, the reaction $\mathbf{R}_1$, and the tension $\mathbf{T}$. Since the sum of these forces is zero, because the part AC remains in equilibrium, forming the projections on the coordinate axes, we obtain:

$$T_x + R_{1_x} = 0, \quad T_z + R_{1_z} - s\delta = 0, \quad T_y + R_{1_y} = 0. \tag{5}$$

From (4) $R_{1_y} = 0$; hence $T_y = 0$. Since the tension $\mathbf{T}$ is tangent to the curve, and C is an arbitrary point of this curve, the tangent at each point is parallel to the vertical xz-plane. It follows from this that the curve lies in a vertical plane, namely, the xz-plane, because it has in common with it the two points A and B. The first of the equations (5) gives

$$T_x = -R_{1_x} = \text{const.} \tag{6}$$

Therefore: *the horizontal component of the tension of the cable is the same at each point of the cable.*

Let us denote by φ the angle which $\mathbf{T}$ makes with the x-axis. We therefore have $\tan\varphi = T_z / T_x$, whence by the second of the equations (5)

$$\tan\varphi = (-\delta s / R_{1_x}) + (R_{1_z} / R_{1_x}). \tag{7}$$

Let us put:

$$a = -\delta / R_{1_x}, \quad a' = R_{1_z} / R_{1_x}. \tag{8}$$

If $z = f(x)$ is the equation of the curve, then $z' = \tan\varphi$. Hence in virtue of (7) and (8)

$$z' = as + a'. \tag{9}$$

Equation (9) is the differential equation of the curve whose form the cable assumes. Differentiating it, we obtain $z'' = a\,\mathrm{d}s / \mathrm{d}x$, and since $\mathrm{d}s = \sqrt{1 + z'^2}\,\mathrm{d}x$, it follows that $z'' = a\sqrt{1 + z'^2}$. Let us substitute $z' = w$. Hence $z'' = \mathrm{d}w / \mathrm{d}x$, whence $\mathrm{d}w / \mathrm{d}x = a\sqrt{1 + w^2}$, i. e. $\mathrm{d}w / \sqrt{1 + w^2} = a\,\mathrm{d}x$. Integrating, we obtain $\int \mathrm{d}w / \sqrt{1 + w^2} = \int a\,\mathrm{d}x$; therefore $\ln(\sqrt{1 + w^2} + w) = ax + c$, where c is the constant of integration. Consequently

$$\sqrt{1 + w^2} + w = \sqrt{1 + z'^2} + z' = e^{ax+c}. \tag{10}$$

We have

$$1 / (\sqrt{1 + z'^2} + z') = \sqrt{1 + z'^2} - z' = e^{-ax-c}. \tag{11}$$

From equations (10) and (11) we obtain

$$z' = \tfrac{1}{2}(e^{ax+c} - e^{-ax-c}), \tag{12}$$

$$\mathrm{d}s / \mathrm{d}x = \sqrt{1 + z'^2} = \tfrac{1}{2}(e^{ax+c} + e^{-ax-c}). \tag{13}$$

Integrating equations (12) and (13), we obtain

$$z = \frac{1}{2a}(e^{ax+c} + e^{-ax-c}) + c', \tag{14}$$

$$s = \frac{1}{2a}(e^{ax+c} - e^{-ax-c}) + c'', \tag{15}$$

where c' and c'' are certain constants.

The curve defined by equation (14) is called the *catenary*.

Therefore: *a cable hangs in the form of a catenary.*

Equations (14) and (15) depend on four constants a, c, c', and c''. These constants can be determined if we know, for instance, the coordinates x_1, z_1, x_2, z_2, of the points A, B and the length l of the cable, because from the conditions that $z = z_1$, for $x = x_1$, and $z = z_2$, for $x = x_2$, we obtain by (14):

$$z_1 = \frac{1}{2a}(e^{ax_1+c} + e^{-ax_1-c}) + c', \quad z_2 = \frac{1}{2a}(e^{ax_2+c} + e^{-ax_2-c}) + c'. \tag{16}$$

While from the conditions that $s = 0$, for $x = x_1$, and $s = l$, for $x = x_2$, we obtain by (15):

$$0 = \frac{1}{2a}(e^{ax_1+c} - e^{-ax_1-c}) + c'', \quad l = \frac{1}{2a}(e^{ax_2+c} + e^{-ax_2-c}) + c''. \tag{17}$$

It can be shown that equations (16) and (17) define the constants a, c, c', and c'', uniquely.

Let us still compute the tension $\mathbf{T}$ of the cable at an arbitrary point C whose coordinates are x, z. From equations (6) and (8) we get

$$T_x = \delta / a. \tag{18}$$

Since $T_z / T_x = \tan\varphi = z'$, $T_z = T_x z'$; consequently

$$T_z = \delta z' / a. \tag{19}$$

From equations (18) and (19) we obtain

$$T = \sqrt{T_x^2 + T_z^2} = \frac{\delta}{a}\sqrt{1 + z'^2} = \frac{\delta}{a}\cdot\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{\delta}{2a}(e^{ax+c} + e^{-ax-c}),$$

and hence by (14)

$$T = \delta(z - c') \tag{20}$$

Loaded cable. Let a force $\boldsymbol{P}$ directed vertically downwards be applied at a point C of a cable. As we already know, parts CB and AC of the cable are catenaries. Let us denote the constants for the curves BC and CA in equations (14) and (15) by a_1, c_1, c_1', c_1'' and a_2, c_2, c_2', c_2'', respectively, and the tensions in the parts BC and CA at the point C by $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ (Fig. 233).

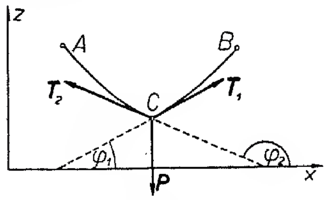


Fig. 233.

Considering the cable as a chain consisting of many small links, and the point C as a joint, we have in the position of equilibrium $\boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{P} = 0$. Forming projections on the x and z axes and putting $P = |\boldsymbol{P}|$, we obtain:

$$T_{1_x} + T_{2_x} = 0, \quad T_{1_z} + T_{2_z} - P = 0. \tag{21}$$

Denoting the right-hand and the left-hand derivatives at C by z_1' and z_2', we obtain by (18) and (19):

$$T_{1_x} = \delta / a_1, \; T_{1_z} = \delta z_1' / a_1, \; T_{2_x} = -\delta / a_2, \; T_{2_z} = -\delta z_2' / a_2,$$

whence by (21)

$$\frac{\delta}{a_1} - \frac{\delta}{a_2} = 0, \qquad \frac{\delta z_1'}{a_1} - \frac{\delta z_2'}{a_2} - P = 0.$$

Hence we get:

$$a_1 = a_2, \qquad z_1' - z_2' = \frac{Pa_1}{\delta} = \frac{Pa_2}{\delta}. \tag{22}$$

Knowing the lengths l_1 and l_2 of the arcs BC and CA we can obtain the equations of the curves CB and AC. In this case it is necessary to determine ten constants $a_1, c_1, c_1', c_1'', a_2, c_2, c_2', c_2''$ and x_0, z_0, where x_0 and z_0 are the coordinates of the point C. To determine these constants for CB and AC we have two sets of four equations analogous to (16) and (17), and in addition two equations (22), i. e. ten altogether.

CHAPTER VII

KINEMATICS OF A RIGID BODY

§ 1. Displacement and rotation of a body about an axis. According to the definition of a rigid body (p. 231), its points do not change their mutual distances during motion. When the point A moved to the point B, the vector $\overline{AB}$ was called the *displacement* of the point (p. 34). During a change of position of a rigid body, the points of this body undergo, in general, various displacements.

We shall first become acquainted with certain theorems from geometry which give the resolution of the displacements of the points of a body. These theorems will be helpful to us in determining the velocities of these points.

Parallel displacement or translation. A body is said to undergo a *parallel displacement* or a *translation* if the displacements of all the points of the body during a change of its position are equal.

The displacement common to all points of the body is called the *displacement vector* or the *displacement of the body.*

The position of the body after a displacement is therefore determined by the initial position and the displacement vector.

Let us assume that the points A_1, B_1 moved to the points A_2, B_2 after a translation. Since the displacements of both points are equal, $\overline{A_1A_2} = \overline{B_1B_2}$. It follows from this that $\overline{A_1B_1} = \overline{A_2B_2}$.

Therefore: *the vectors attached to a body do not change either their sense or direction during a translation.*

Conversely, it is easy to prove that *if the vectors in a body maintain their sense and direction during a displacement of the body, then the displacement is a translation.*

For let us assume that two arbitrary points A_1, B_1 moved to the points A_2, B_2 (Fig. 234). By hypothesis, $\overline{A_1B_1} = \overline{A_2B_2}$; hence $\overline{A_1A_2} = \overline{B_1B_2}$.

Loaded cable. Let a force $\boldsymbol{P}$ directed vertically downwards be applied at a point C of a cable. As we already know, parts CB and AC of the cable are catenaries. Let us denote the constants for the curves BC and CA in equations (14) and (15) by a_1, c_1, c_1', c_1'' and a_2, c_2, c_2', c_2'', respectively, and the tensions in the parts BC and CA at the point C by $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ (Fig. 233).

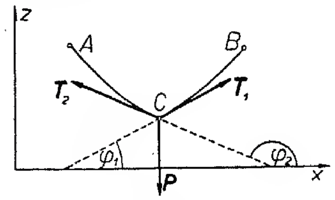


Fig. 233.

Considering the cable as a chain consisting of many small links, and the point C as a joint, we have in the position of equilibrium $\boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{P} = 0$. Forming projections on the x and z axes and putting $P = |\boldsymbol{P}|$, we obtain:

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whence by (21)

$$\frac{\delta}{a_1} - \frac{\delta}{a_2} = 0, \qquad \frac{\delta z_1'}{a_1} - \frac{\delta z_2'}{a_2} - P = 0.$$

Hence we get:

$$a_1 = a_2, \qquad z_1' - z_2' = \frac{Pa_1}{\delta} = \frac{Pa_2}{\delta}. \tag{22}$$

Knowing the lengths l_1 and l_2 of the arcs BC and CA we can obtain the equations of the curves CB and AC. In this case it is necessary to determine ten constants $a_1, c_1, c_1', c_1'', a_2, c_2, c_2', c_2''$ and x_0, z_0, where x_0 and z_0 are the coordinates of the point C. To determine these constants for CB and AC we have two sets of four equations analogous to (16) and (17), and in addition two equations (22), i. e. ten altogether.

CHAPTER VII

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The displacement common to all points of the body is called the *displacement vector* or the *displacement of the body.*

The position of the body after a displacement is therefore determined by the initial position and the displacement vector.

Let us assume that the points A_1, B_1 moved to the points A_2, B_2 after a translation. Since the displacements of both points are equal, $\overline{A_1A_2} = \overline{B_1B_2}$. It follows from this that $\overline{A_1B_1} = \overline{A_2B_2}$.

Therefore: *the vectors attached to a body do not change either their sense or direction during a translation.*

Conversely, it is easy to prove that *if the vectors in a body maintain their sense and direction during a displacement of the body, then the displacement is a translation.*

For let us assume that two arbitrary points A_1, B_1 moved to the points A_2, B_2 (Fig. 234). By hypothesis, $\overline{A_1B_1} = \overline{A_2B_2}$; hence $\overline{A_1A_2} = \overline{B_1B_2}$.

The points A_1, B_1 therefore have equal displacements, i. e. the change of position of the body is a translation.

It is easy to see that lines and planes in a body remain parallel to one another after a translation.

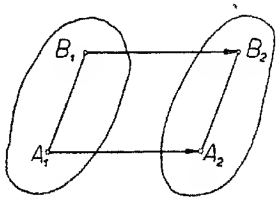

Fig. 234.

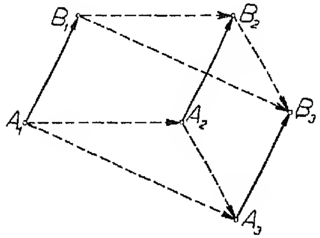

Fig. 235.

Let us suppose that a body has made two successive translations: first, from position I to position II, and next, from position II to position III. Let A_1, B_1 be two arbitrary points in position I, and A_2, B_2 and A_3, B_3 their corresponding points in positions II and III (Fig. 235). By hypothesis, $\overline{A_1A_2} = \overline{B_1B_2}$ and $\overline{A_2A_3} = \overline{B_2B_3}$. Since $\overline{A_1A_3} = \overline{A_1A_2} + \overline{A_2A_3}$ and $\overline{B_1B_3} = \overline{B_1B_2} + \overline{B_2B_3}$, it follows that $\overline{A_1A_3} = \overline{B_1B_3}$. Consequently we can go directly from position I to position III by means of one translation. Denoting the displacements in passing from I to II, from II to III, and from I to III, by $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}$, we obviously obtain

$$\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2.$$

Therefore: *if a body has made several successive translations, then the final position can be obtained from the initial position by means of one translation; the displacement of the body from the initial position to the final position is equal to the sum of the displacements of the separate translations.*

This theorem can be called the *law of composition of displacements.*

Since the resultant displacement is the sum of the component displacements, then (in virtue of the commutativity of the sum of vectors) the resultant displacement does not depend on the order in which the body made the component displacements.

Rotation about an axis. If two points of a body, e. g. K and M, remained fixed during a change of position of the body, then, obviously, all the points of the line l passing through K and M will also remain fixed. We then say that the body *rotated about the line l*; this line is called the *axis of rotation.*

If some plane Π_1 in the initial position of the body passes through the axis of rotation, then the corresponding plane Π_2 in the final position will also pass through this axis.

Let us give the axis l an arbitrary sense. The angle φ through which it is necessary to rotate the plane Π_1 (counterclockwise with respect to axis l) in order that it fall on Π_2 and in order that the corresponding points coincide is called the *angle of rotation.*

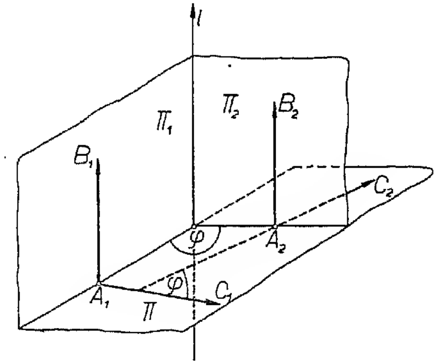

Fig. 236.

The rotation of a body about an axis is determined by giving the axis and the angle of rotation. During a rotation the points of a body remain in planes perpendicular to the axis of rotation.

During a rotation every vector $\overline{A_1B_1}$ parallel to the axis of rotation falls on a vector $\overline{A_2B_2}$ parallel to the vector $\overline{A_1B_1}$. It is easy to prove that only vectors parallel to the axis do not change either direction or sense during a rotation.

Let us note that if a vector $\overline{A_1C_1}$ lies in a plane Π perpendicular to the axis of rotation, then the angle which this vector makes with the corresponding vector $\overline{A_2C_2}$ is (relative to the chosen sense of the axis of rotation) equal to the angle of rotation (Fig. 236).

If the body makes several rotations about this same axis l through the angles $\varphi_1, \varphi_2, \ldots$, then the final displacement is obviously also a rotation about the axis l through the angle $\varphi = \varphi_1 + \varphi_2 + \ldots$ It follows from this, in virtue of the commutativity of the sum, that the final position does not depend on the order in which the partial rotations of the body were made.

The situation is quite different when the body makes successive rotations about various axes as the example on p. 314 shows.

Example. A rigid body made two successive rotations about two parallel lines l and m which are rigidly attached to the body. The rotations had opposite senses, but the angles of rotation were equal. Prove that the body can be displaced from its initial position to its final position by means of a translation.

Let C be an arbitrary point of the body. Through C let us pass a plane Π perpendicular to the given axes of rotation l and m. Let L and M denote the corresponding points of intersection and φ the angle of rotation (Fig. 237).

During a rotation about the axis l through an angle φ the axis m will assume the position of the line m'; let us denote the point of intersection of this line with the plane Π by M'. Next, after a rotation about the axis m' through an angle φ the axis l will assume the position of the line l'; let us denote its point of intersection with the plane Π by L'.

Finally, let C_1 be the position of the point C after a rotation about the line l, and C' the position of the point C_1 after a rotation about the line m'.

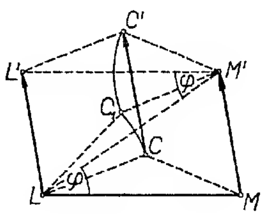

Fig. 237.

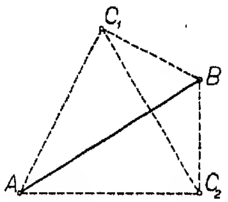

Fig. 238.

The triangle LMC assumed ultimately the position $L'M'C'$, while $\overline{LL'} = \overline{MM'} = \overline{CC'}$ (as in Fig. 237). Since the displacements of the points situated on the axis l (or m) are equal, the above relation indicates that the displacements of all the points are equal. It is easy to verify that

$$LL' = 2LM \sin \tfrac{1}{2}\varphi. \tag{1}$$

§ 2. Displacements of points of a body in plane motion. The motion of a plane figure moving in a plane is called a *plane motion*.

The position of a figure in plane motion is determined by the position of two of its arbitrary points.

For suppose that there are two possible positions of the figure at which the two points A and B would occupy the same positions. Let us consider an arbitrary point C of the figure which in one position is at C_1, and in the another at C_2 (Fig. 238). The triangles ABC_1 and ABC_2 are congruent and are situated symmetrically with respect to AB. Therefore

they cannot be made to coincide without taking them out of the plane. This, however, is contrary to the hypothesis that in the plane motion the triangle ABC once occupied the position ABC_1, and the second time the position ABC_2.

Rotation about a point. If a figure lying in the plane Π is rotated about a line l perpendicular to this plane, then the figure will continue to remain in the plane Π. Such a rotation is called a *rotation* of the figure *about a point* (the point of intersection of the line l with the plane Π).

Theorem. *Every figure in plane motion can be displaced from one arbitrary position to another by means of one translation and one rotation.*

For let A_1, B_1 be two points of this figure in the first position, and A_2, B_2 the corresponding points in the second position. Let us first translate the figure so that the point A_1 falls on A_2. After this displacement the point B_1 will fall on a certain point B_1'. Let us now rotate the figure about A_2 so that B_1' falls on B_2. Since the two points A_1, B_1 coincided with the corresponding points A_2, B_2 after a translation and a rotation, the remaining points will also coincide. Thus, we have displaced the figure from one position to another by means of one translation and one rotation, q. e. d.

We shall now prove that the most general displacement of a figure in plane motion is either a translation or a rotation.

I Theorem of Euler. *A figure can be displaced from one position to another* (which it occupies in plane motion) *either by means of a translation or a rotation.*

Proof. Let A_1B_1 be an arbitrary segment of the figure in the initial position I, and A_2B_2 the same segment in the final position II.

In the case when the vectors $\overline{A_1B_1}$ and $\overline{A_2B_2}$ are equal, we have $\overline{A_1A_2} = \overline{B_1B_2}$. Therefore, by a parallel displacement of the figure so that the point A_1 falls on A_2, the point B_1 will fall on B_2. Since the figure will have the points A_2 and B_2 in common with position II after this displacement, it will have all points in common. In this case, therefore, it is possible to displace the figure from position I to position II by means of a translation.

In the case when the vectors $\overline{A_1B_1}$ and $\overline{A_2B_2}$ are not equal, let us draw the perpendicular bisectors l_1 and l_2 of the segments A_1A_2 and B_1B_2.

Let us assume at first that these bisectors are distinct and intersect at the point O (Fig. 239). The triangles OA_1B_1 and OA_2B_2 are congruent. Denote by B_1' the point symmetrical to B_1 with respect to the bisector l_1.

The triangles OA_1B_1 and OA_2B_1' are obviously situated symmetrically with respect to l_1. They are consequently congruent and not superposable without taking them out of the plane. Hence if the figure is rotated about O so that the point A_1 falls on A_2, then, since the point B_1 cannot fall on B_1', the point B_1 will fall on B_2. After this rotation the figure will therefore have the points A_2 and B_2 in common with position II, and consequently all the other points in common.

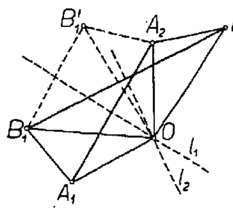

Fig. 239.

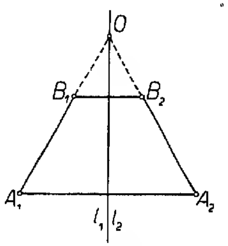

Fig. 240.

Next, let us assume that the bisectors of the segments A_1A_2 and B_1B_2 are identical (Fig. 240). In this case the segments A_1B_1 and A_2B_2 are situated symmetrically with respect to l_1 (or l_2); the centre of rotation will be the point of intersection of the line A_1B_1 with l_1 (or A_2B_2 with l_2). In this case, therefore, the displacement of the figure from position I to position II can be made by means of a rotation, q. e. d.

Plane motion of a body. If a body can move only in such a way that its points remain constantly in planes parallel to a certain fixed plane Π, then the body is said to be in *plane motion*, and the plane Π is called the *directional plane* (cf. the definition and example on p. 272).

Let us cut a body in plane motion by a plane Π' parallel to the directional plane Π. Let C be the plane section. The position of the plane section C obviously determines the position of the entire body. Since the plane section C must remain constantly in one and the same plane Π', by I Theorem of Euler we can displace this figure from the arbitrary position it occupies to another arbitrary position, either by means of a translation, or by means of a rotation about a point lying in Π'.

It follows from this that *a body in plane motion can be displaced from one position to another, either by means of a translation, or by means of a rotation about an axis perpendicular to the directional plane.*

§ 3. Displacements of the points of a body. If a rigid body has one fixed point, then it can rotate about this point, and if it has two fixed

points, it can rotate about an axis passing through these points. Giving the position of one or two points of a body is, therefore, not sufficient to determine the positions of all the other points of the body. But we have the following

Theorem I. *The position of all the points of a rigid body is determined by the position of three of its points, provided the points are not collinear.*

Proof. Let us suppose that there exist two distinct positions of the body at which three non-collinear points A, B, C, would occupy the same positions. Let us consider an arbitrary point D of this body which in the first position is at D_1, and in the second at D_2. The tetrahedrons $ABCD_1$ and $ABCD_2$ have a common base and correspondingly equal edges (Fig. 241). It follows from this that they are symmetrically placed with respect to the plane ABC. Therefore they cannot be brought into complete coincidence. This is, however, contrary to the assumption that the first position of the tetrahedron $ABCD$ is $ABCD_1$ and the other one $ABCD_2$.

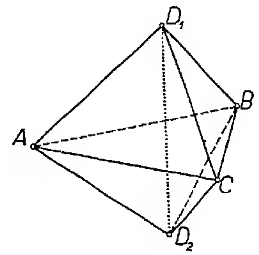

Fig. 241.

Rotation about a point of a body. If one point of a body remains fixed during a displacement of the body, then the body is said to have been *rotated* about this point.

II Theorem of Euler. *A rotation about a point is equivalent to a rotation about a line passing through this point.*

Proof. Let us suppose that a body has been rotated about the point O. In the initial position I let us select in the body an arbitrary segment A_1B_1 (not passing through O) and let A_2B_2 be the corresponding segment in the final position II. Let us draw the planes of symmetry Π_1 and Π_2 of the segments A_1A_2 and B_1B_2.

Let us assume at first that Π_1 and Π_2 are distinct and that they intersect in the line l (Fig. 242). The line l passes through O because l is the locus of points equidistant from A_1 and A_2, as well as from B_1 and B_2, while $OA_1 = OA_2$ and $OB_1 = OB_2$. Let C be an arbitrary point (different from O) on the line l.

The tetrahedrons OCA_1B_1 and OCA_2B_2 are equal. It is easy to show that they are also superposable. The vertices O, C, A_1 and O, C, A_2 are placed symmetrically with respect to Π_1; hence, if the tetrahedrons OCA_1B_1 and OCA_2B_2 were not superposable, the vertices B_1 and B_2

would have to be placed symmetrically with respect to Π_1, which is impossible, since B_1 and B_2 are placed symmetrically with respect to Π_2, and $\Pi_1 \neq \Pi_2$. We have proved, therefore, that the tetrahedrons OCA_1B_1 and OCA_2B_2 are equal and superposable.

If we now rotate the body about the axis l so that A_1 falls on A_2, then the tetrahedron OCA_1B_1 will fall on the tetrahedron OCA_2B_2. After this rotation the body will therefore have three points in common (namely O, A_2 and B_2) with the body in position II, and consequently all other points. We have thus displaced the body from position I to position II by means of a rotation about the line l.

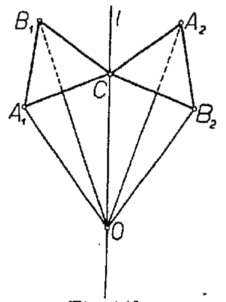

Fig. 242.

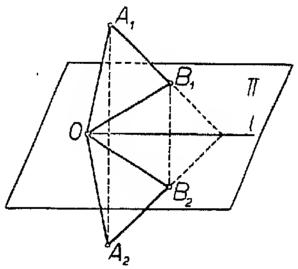

Fig. 243.

Let us now assume that the segments A_1A_2 and B_1B_2 have a common plane of symmetry Π (Fig. 243). Then the triangles OA_1B_1 and OA_2B_2 are situated symmetrically with respect to Π. The axis of rotation in this case will be the line of intersection of the plane OA_1B_1 (or OA_2B_2) with the plane Π.

Remark 1. From II Theorem of Euler it follows that during a rotation of a body about a point, there exists in the body a certain line having the property that its points do not change their position.

Remark 2. If a body makes two successive rotations about two axes passing through one point O, then the body can be displaced from its initial position to its final position by means of one rotation about an axis passing through O, because the point O did not change its position. Therefore the composition of two rotations about an axis passing through one point is a rotation about an axis passing through the same point.

Example. A body made two successive rotations about the axes of a fixed coordinate system: first about the z-axis and then about the x-axis, both rotations counterclockwise through a right angle.

Since the origin O of the system remained fixed during both rotations, we can displace the body from its initial position to its final position by means of one rotation about a certain axis l passing through O (Fig. 244).

Let us consider the point $A(0, 0, 1)$ on the z-axis in the initial position. After a rotation about this axis the point A did not change its position, and after a rotation about the x-axis it assumed the position $A'(0, 1, 0)$.

Let us next consider the point $B(0,1, 0)$ on the y-axis in the initial position. After a rotation about the z-axis the point B occupied the position $B'(1, 0, 0)$ on the x-axis, and then during a rotation about this axis the point B' did not change its position any more. The sought for axis l will therefore be the intersection of the planes of symmetry of the segments AA' and BB'.

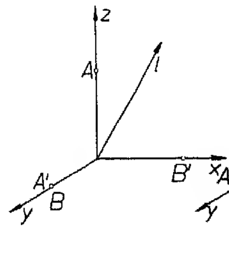

Fig. 244.

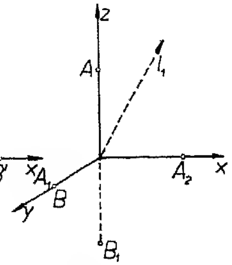

Fig. 245.

The plane of symmetry of the segment AA' has the equation $y = z$, and the plane of symmetry of the segment BB' has the equation $x = y$. The axis l, being the intersection of both planes, consequently has the equation

$$x = y = z.$$

Let us now suppose that the body had made the same rotations in the reverse order, i. e. first about the x-axis, and then about the z-axis (Fig. 245). The points $A(0, 0, 1)$ and $B(0, 1, 0)$ after a rotation about the x-axis will occupy the positions $A_1(0, 1, 0)$ and $B_1(0, 0, -1)$, and then, after a rotation about the z-axis, they will assume the positions $A_2(1, 0, 0)$ and $B_2(0, 0, -1)$. The planes of symmetry of the segments AA_2 and BB_2 have the equations $x = z$ and $y = -z$. The axis l_1 about which it is necessary to rotate the body in order that it go from its initial position to its final position will therefore have the equation

$$x = -y = z.$$

It follows from this that the final position depends on the order in which the rotations were made.

Chasles' theorem. *A body can be displaced from one arbitrary position to another by means of one translation and one rotation about an axis.*

In general, this can be done in infinitely many ways, but the axes

of rotation will always be parallel and the angles of rotation equal (if the axes have the same sense).

Proof. Let O_1 be an arbitrary point of the body in the initial position I, and O_2 the corresponding point in the final position II. Let us first translate the body to the position I′ so that O_1 falls on O_2. If the position I′ is identical with II, then the body has been displaced from the position I to the position II by means of one translation, conformably to the requirements of the theorem.

Let us assume, therefore, that the position I′ is different from II. Since the positions II and I′, of the body have the point O_2 in common (Fig. 246), it follows that, by II Theorem of Euler, we can displace it from the position I′ to the position II by means of a rotation about a certain axis l passing through the point O_2.

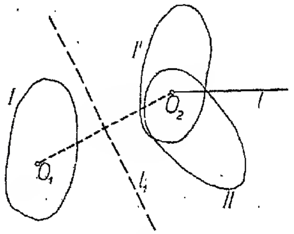

Fig. 246.

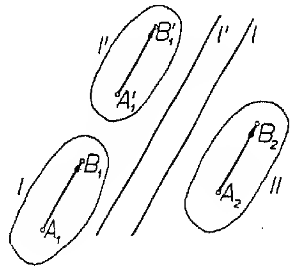

Fig. 247.

In each case we have therefore displaced the body from the position I to the position II by means of one translation and one rotation; thus we have proved the first part of the theorem.

Had we chosen a different point O_1 in the beginning, then in general we would have obtained a different translation and a different rotation about a different axis. We shall show, however, that in every case the axes of rotation would be parallel.

In the position I let us consider in the body an arbitrary vector $\overline{A_1B_1}$ parallel to the axis of rotation l (Fig. 247). The corresponding vector $\overline{A_2B_2}$ in the position II will have the same direction and sense as the vector $\overline{A_1B_1}$. For neither a translation nor a rotation (about a parallel axis) changes the direction or sense of a vector.

Let us now assume that the body has been displaced from the position I to the position II by means of a different translation and rotation about a different axis l'. After this displacement let the vector $\overline{A_1B_1}$ fall on the vector $\overline{A_1'B_1'}$ (Fig. 247). Obviously, after a rotation about the new axis l', the vector $\overline{A_1'B_1'}$ will fall on the vector $\overline{A_2B_2}$. Since the displacement changes neither the direction nor sense of the vector, $\overline{A_1'B_1'}$ has the same sense and direction as the vector $\overline{A_1B_1}$. It follows from this that the vector $\overline{A_1'B_1'}$ will also have the same sense and direction as the vector $\overline{A_2B_2}$. Hence the axis of rotation l' must be parallel to $\overline{A_2B_2}$, i. e. to the axis l.

Finally, we shall show that the angles of rotation about the axes l and l' are equal, provided that the axes l and l' are given the same senses. In the position I let us select in the body an arbitrary vector $\boldsymbol{a}_1$ perpendicular to l and obviously to l' at the same time. The angle which vector $\boldsymbol{a}_1$ makes with the corresponding vector $\boldsymbol{a}_2$ in the position II relative to the chosen sense of the axis is the angle of rotation (p. 309). The angle of rotation is therefore in both cases the same, q. e. d.

Theorem 2. *A body can be displaced from one arbitrary position to another by means of two successive rotations.*

Proof. Let O_1 be an arbitrary point of the body in the position I, and O_2 the corresponding point of the body in the position II. Let us rotate the body through 180° about the axis l_1, which is the axis of symmetry of the segment O_1O_2. By means of this rotation the point O_1 will fall on the point O_2. The body will assume the position II′ which has the point O_2 in common with the position II. Consequently we can go from position II′ to position II by means of a rotation about a certain axis l passing through O_2. In this manner we have displaced a body from position I to position II by means of two rotations about the lines l_1 and l, q. e. d.

Twist. If a body makes a translation and then a rotation about an axis parallel to the translation, then the body is said to have made a *twist*.

In particular, a translation or a rotation is also called a twist.

Theorem 3. *A body can always be displaced from one arbitrary position to another by means of a twist and this can be done in only one way.*

Proof. Let us consider in the body in position I an arbitrary triangle $A_1B_1C_1$ lying in the plane Π_1 perpendicular to the possible axes of rotation. Let us denote by $A_2B_2C_2$ the corresponding triangle and by Π_2 the corresponding plane in the position II. The planes Π_1 and Π_2 are therefore parallel. Let us now displace the body from position I to position I′ by giving the body a displacement perpendicular to Π_1 so that the plane Π_1 assumes the position of the plane Π_2 (Fig. 248). By means of this displa-

cement the triangle $A_1B_1C_1$ coincides with the triangle $A_1'B_1'C_1'$ lying in the plane Π_2 in which the triangle $A_2B_2C_2$ also lies.

We can displace the body from position I′ to position II first by means of a translation so that A_1' falls on A_2, and next by means of a rotation about a line perpendicular to Π_2 and passing through A_2. It follows from this that the triangle $A_1'B_1'C_1'$, remaining constantly in the plane Π_2, will fall on $A_2B_2C_2$. Hence by II Theorem of Euler we can displace the triangle $A_1'B_1'C_1'$ to $A_2B_2C_2$ by means of a translation or of a rotation (about a certain point O lying in Π_2, i. e. by means of a rotation about an axis l perpendicular to Π_2 at the point O). By this translation (or rotation) the body is displaced from position I′ to position II.

It follows from this that a body is displaced from position I to position II either by means of a translation (if the displacement from I′ to II is a translation), or by means of a translation and of a rotation about the axis l parallel to the translation, q. e. d.

The axis l is called the *axis of twist.*

During a twist the axis of twist slides along itself. Every other line changes its position during a twist. It follows from this that a body cannot be displaced from position I to position II by means of a twist along any other axis l'.

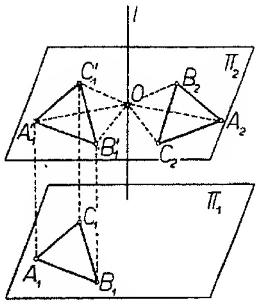

Fig. 248.

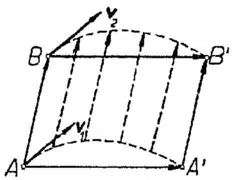

Fig. 249.

§ 4. Advancing motion and rotation about an axis. Advancing motion. If every position that a body assumes during motion can be obtained from the initial position by means of a translation, then the body is said to move with an *advancing motion.*

From the definition of an advancing motion it follows that every two positions of a body in an advancing motion can be obtained from each other by means of a translation.

Let A and B be two arbitrary points of a body. In an advancing motion the vector $\overline{AB}$ changes neither its direction nor its sense. Therefore, if the path of the point A is translated so that the displacement of all its points is equal to the vector $\overline{AB}$, then the path of the point A will coincide with the path of the point B (Fig. 249).

Therefore: *in an advancing motion the paths of all points are congruent and they can cover one another by means of a translation.*

The advancing motion of a body is therefore determined by the motion of one of its points and the position of the body at the initial moment. For if we know the motion of the point A, for example, then the displacement vector of the body from its initial position to its position at the time t will also be known, because it is equal to the known displacement of the point A.

Conversely: *if the vectors attached to a rigid body do not change their direction, then the body moves with an advancing motion.*

Indeed, let us consider a fixed system of coordinates (x, y, z) and two points in the body $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. By hypothesis, the vector $\overline{AB}$ does not change its direction (that it does not change its length is obvious); we shall show that it also does not change its sense. During the motion the projections of the vector $\overline{AB}$ on the coordinate axes are constant in absolute value. Hence $|x_2 - x_1|$, $|y_2 - y_1|$, and $|z_2 - z_1|$, are certain constants. Since $x_2 - x_1$ is a continuous function of the time t, therefore from the fact that $|x_2 - x_1| = \text{const.}$ it follows that $x_2 - x_1$ is also a constant. Similarly $y_2 - y_1 = \text{const.}$ and $z_2 - z_1 = \text{const.}$ Consequently the vector $\overline{AB}$ does not change its sense.

Therefore, denoting by A, B and A', B' the positions of the given points at two arbitrary moments t and t', we obtain $\overline{AB} = \overline{A'B'}$, whence $\overline{AA'} = \overline{BB'}$. The displacements of two arbitrary points are consequently equal, and we can displace the body from the position at the moment t to the position at the moment t' by means of a translation. Hence the motion of the body is an advancing motion.

In a body moving with an advancing motion let us consider two arbitrary points A_1, A_2 at the moment t and the positions A_1', A_2' of these points at the moment $t + \Delta t$. Denoting by $\mathbf{v}_1$ and $\mathbf{v}_2$ the velocities of the points A_1 and A_2, we obtain (p. 35):

$$\mathbf{v}_1 = \lim_{\Delta t \to 0} \frac{\overline{A_1A_1'}}{\Delta t}, \quad \mathbf{v}_2 = \lim_{\Delta t \to 0} \frac{\overline{A_2A_2'}}{\Delta t}.$$

Since the body moves with an advancing motion, $\overline{A_1A_1'} = \overline{A_2A_2'}$, whence $\mathbf{v}_1 = \mathbf{v}_2$.

Therefore: *in an advancing motion the velocities of all the points of a body at an arbitrary moment t are equal to one another.*

The velocity of an advancing motion at the moment t is called the common velocity of all the points of a body at this moment.

Let the body now move in such a way that at each moment t the velocities of all the points of the body are equal to one another. Let us select two arbitrary points of the body $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

Since the velocity of the point A at each moment is always equal to the velocity of the point B,

$$x_1^{\cdot} = x_2^{\cdot}, \quad y_1^{\cdot} = y_2^{\cdot}, \quad z_1^{\cdot} = z_2^{\cdot}, \quad \text{whence} \quad x_2^{\cdot} - x_1^{\cdot} = 0.$$

Consequently $x_2 - x_1 = c_1$ and similarly $y_2 - y_1 = c_2$, $z_2 - z_1 = c_3$, where c_1, c_2, and c_3 are certain constants. It follows from this that the vector $\overline{AB}$ has constant projections on the coordinate axes, and hence does not change either its direction or sense during motion. The motion is consequently an advancing motion.

Therefore: *if all the points of a body have equal velocities at each moment, then the body moves with an advancing motion.*

Rotation about an axis. If a body moves so that all the points on a certain line l remain at rest, then we say that the body *rotates about the axis* l (p. 308).

During a rotation all the points move in circles lying in planes perpendicular to the axis of rotation; the centres of these circles lie on the axis of rotation.

The radii joining the points of the body with the centres of the circles along which these points move sweep out equal angles in equal times. It follows from this that during a rotation about an axis all the points have equal angular velocities at each moment. Their common angular velocity is called the *angular velocity* of the rotation about the axis.

From the definition of the angular velocity vector (p. 45), it follows that during a rotation of a body about an axis all the points have the same angular velocity vector lying on the axis of rotation. This vector is called the *angular velocity vector* of the rotating body.

Example 1. If the vertices A and D of a parallelogram $ABCD$ are fixed, then the sides AB and DC can rotate about the vertices A and D. During these rotations the side BC remains constantly parallel to AD. Consequently BC moves with an advancing motion.

Example 2. A circle with centre O' and radius r moves with an advancing motion and remains constantly tangent to the circle K with centre O and radius R. Determine the path of an arbitrary point A (*vide* Fig. 284 on p. 368).

The centre O' obviously moves along a circle K' with centre O and radius $a = R - r$. The path of the point A (dotted in Fig. 284) will consequently be a circle of radius a. The centre O_1 of this circle is obtained by giving the centre O a displacement $\overline{OO_1}$ equal to the vector $\overline{O'A}$.

§ 5. Distribution of velocities in a rigid body. When a rigid body moves, its points can in general have various velocities at a given moment.

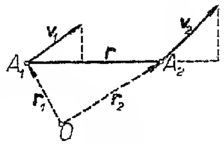

Fig. 250.

Relations among the velocities of the points of a body. Let us consider in the body two points A_1 and A_2 whose velocities are $\mathbf{v}_1$ and $\mathbf{v}_2$. Let O be the origin of the coordinate system (Fig. 250). Let us put:

$$\mathbf{r}_1 = \overline{OA}_1, \quad \mathbf{r}_2 = \overline{OA}_2, \quad \mathbf{r} = \overline{A_1A_2} = \mathbf{r}_2 - \mathbf{r}_1.$$

Consequently (p. 35, (III)):

$$\mathbf{r}_1^{\cdot} = \mathbf{v}_1, \quad \mathbf{r}_2^{\cdot} = \mathbf{v}_2, \quad \mathbf{r}^{\cdot} = \mathbf{r}_2^{\cdot} - \mathbf{r}_1^{\cdot} = \mathbf{v}_2 - \mathbf{v}_1. \tag{1}$$

We have $\mathbf{r}^2 = |\mathbf{r}|^2$. Since $|\mathbf{r}| = \text{const.}$, forming the derivative, we obtain $2\mathbf{r}\mathbf{r}^{\cdot} = 0$, i. e. $\mathbf{r}^{\cdot}\mathbf{r} = 0$; hence by (1) $\mathbf{r}(\mathbf{v}_2 - \mathbf{v}_1) = 0$, whence

$$\mathbf{r}\mathbf{v}_2 = \mathbf{r}\mathbf{v}_1. \tag{2}$$

From the definition of a scalar product it follows that

$$\mathbf{r}\mathbf{v}_1 = |\mathbf{r}| \operatorname{Proj}_{\overline{A_1A_2}}\mathbf{v}_1 \quad \text{and} \quad \mathbf{r}\mathbf{v}_2 = |\mathbf{r}| \operatorname{Proj}_{\overline{A_1A_2}}\mathbf{v}_2.$$

Hence by (2) we obtain, after dividing by $|\mathbf{r}|$,

$$\operatorname{Proj}_{\overline{A_1A_2}}\mathbf{v}_1 = \operatorname{Proj}_{\overline{A_1A_2}}\mathbf{v}_2. \tag{3}$$

We have proved, therefore, that *in a rigid body the projections of the velocities of two points on the segment joining these points are equal.*

We can also say that the components of the velocities of two points with respect to the segment joining these points are equal.

Example 1. Let the velocities $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of three non-collinear points A_1, A_2, A_3 of a body be given. Let us choose an arbitrary point C, not lying in the plane $A_1A_2A_3$, and denote its velocity by $\mathbf{v}$ (Fig. 251). From the point C let us draw the vectors $\overline{CB}_1$, $\overline{CB}_2$, $\overline{CB}_3$, equal to the projections of the velocities $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ on the lines A_1C, A_2C, A_3C.

According to the theorem proved, these vectors will also be the projections of the vector **v** with its origin C on the lines A_1C, A_2C, and A_3C. If planes perpendicular to these lines are passed through the points B_1, B_2, and B_3, then the point of intersection of these planes will be the terminus of the vector **v**.

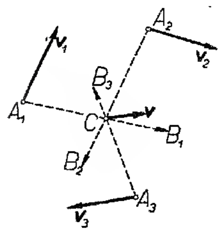

Fig. 251.

In order to determine the velocity of a point D lying in the plane $A_1A_2A_3$, we first determine the velocity of an arbitrary point C not lying in the plane $A_1A_2A_3$, and then the velocity of the point C as before (by taking from among the points A_1, A_2, A_3, and C, three points which are not coplanar with D).

Therefore: *the velocities of all the points of a rigid body are determined by the velocities of three of its non-collinear points.*

Example 2. Velocities of points of a straight line and a plane. Let us give an arbitrary sense to a moving line l and take on it a point O whose coordinates are x_0, y_0, z_0. Denote the angles which the axis l makes with the coordinate axes by α, β, γ, and put:

$$a = \cos\alpha, \quad b = \cos\beta, \quad c = \cos\gamma.$$

For an arbitrary point $A(x, y, z)$ of the axis l, having the coordinate r on this axis, we have:

$$x = x_0 + ar,\ y = y_0 + br,\ z = z_0 + cr. \tag{4}$$

Denoting the velocity of an arbitrary point A by **v** and calculating the derivative of (4) with respect to time, we obtain (because $r =$ const.):

$$v_x = x^\cdot = x_0^\cdot + a^\cdot r, \quad v_y = y^\cdot = y_0^\cdot + b^\cdot r, \quad v_z = z^\cdot = z_0^\cdot + c^\cdot r. \tag{5}$$

From the point A let us draw the vector $\overline{AA'}$ equal to the velocity **v** of the point A, and denote by ξ, η, ζ, the coordinates of the point A'. Since $\xi = x + v_x$ etc., we get in virtue of (4) and (5) the equations

$$\xi = x_0 + x_0^\cdot + (a + a^\cdot)\, r, \quad \eta = y_0 + y_0^\cdot + (b + b^\cdot)\, r,$$
$$\zeta = z_0 + z_0^\cdot + (c + c^\cdot)\, r.$$

Being equations of the first degree of the parameter r, they are the equations of a line.

Therefore: *if velocity vectors are drawn from points lying on a straight line, then the ends of these vectors lie on a straight line.*

By making use of this theorem similar theorem for a plane can easily be proved.

Namely: *if velocity vectors are drawn from points lying in a plane, then the ends of these vectors lie in one plane.*

Knowing the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of two points A_1 and A_2 of the line l, the velocity **v** of an arbitrary point A of this line can be determined in the following manner (*vide* Fig. 252):

We pass a line m through the ends of the velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ drawn from the points A_1 and A_2. The end of the velocity vector **v** drawn from A lies on this line. On the line l we determine a point B so that the vector $\overline{AB}$ is equal to the projection of $\mathbf{v}_1$ (or $\mathbf{v}_2$) on the axis l. According to the theorem proved on p. 321, $\overline{AB}$ is the projection of **v** on l. If we pass a plane through B and perpendicular to l, then the point A' in which this plane cuts m will be the end of the vector **v** drawn from A.

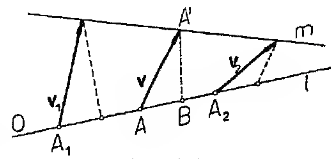

Fig. 252.

Instantaneous motion of a rigid body. Let us consider in a rigid body an arbitrary point A with coordinates x, y, z at a certain time t and denote by **v** its velocity. Since **v** depends on the point A, **v** is a function of the coordinates x, y, z at the time t. We can therefore assume that

$$\mathbf{v} = \mathbf{F}(x, y, z). \tag{6}$$

The vector function (6) defines the velocity at the time t at every point of the body.

The distribution of the velocities in a body at a certain instant is called the *instantaneous motion* of the body.

The instantaneous motion of a body is consequently determined by giving the vector function (6).

In an advancing motion all the points have the same velocity. Function (6) will therefore assume the form $\mathbf{v} =$ const.

Denoting by ω the angular velocity vector, and by O an arbitrary point on the axis l we have for the point A, in a rotating motion about the axis l (cf. formulae (2) and (III), p. 46):

$$\mathbf{v} = \mathrm{Mom}_A\,\boldsymbol{\omega} \quad \text{or} \quad \mathbf{v} = \overline{OA} \times \boldsymbol{\omega}. \tag{7}$$

The instantaneous motion of a body at the moment t is called the *instantaneous advancing motion* if all the points of the body have the same velocity. This velocity is called the *instantaneous velocity of the advancing motion.*

The instantaneous motion of a body at the moment t is called an *instantaneous rotation* about an axis l if the velocities of the points of the body are expressed by formula (7), i. e. if the velocities of the points of a body at the moment t are such as if the body were rotating about the axis l. The velocity $\boldsymbol{\omega}$ is called the *instantaneous angular velocity vector*, and the axis l the *instantaneous axis of rotation*.

If the instantaneous motion at each instant of time is a rotation about a fixed line l, then it is a rotation about the axis l.

For let us choose a fixed coordinate system $O(x, y, z)$, taking the axis l as the z-axis. Let A be an arbitrary point of the body whose coordinates are x, y, z, and $\boldsymbol{\omega}$ the instantaneous velocity vector. Then $\omega_x = 0$ and $\omega_y = 0$. According to formula (7) the projections of the velocity $\mathbf{v}$ of the point A are:

$$v_x = x^{\cdot} = y\omega_z, \quad v_y = y^{\cdot} = -x\omega_z, \quad v_z = z^{\cdot} = 0. \tag{8}$$

As $z^{\cdot} = 0$, $z = \text{const}$. The points therefore move in planes perpendicular to the axis l. From equations (8) we get $xx^{\cdot} + yy^{\cdot} = 0$, whence after integration $\frac{1}{2}x^2 + \frac{1}{2}y^2 = \text{const}$. The points of the body therefore move in planes perpendicular to the axis l at a constant distance from l, and hence the body rotates about l.

§ 6. Instantaneous plane motion. Let a plane figure move in the plane Π. Let us select two arbitrary coordinate systems in this plane: the one (x, y) fixed, and the other (ξ, η) moving with an advancing motion and having its origin at an arbitrarily chosen point M of the given figure (Fig. 253).

The instantaneous motion of the figure with respect to the system (ξ, η) will then be a rotation about the point M (i. e. about the axis l perpendicular to Π at the point M). The relative velocities of the points at the moment t will therefore be such as if the figure were rotating about the point M with a certain angular velocity ω. Consequently we can say that the instantaneous relative motion is a rotation about the point M. Since the system (ξ, η) moves with an advancing motion, the velocity of transport is the same for all points of the figure (p. 58) and equal to the velocity of the point M. Denoting by $\mathbf{v}$ the absolute velocity of an arbitrary point A, by $\mathbf{v}_r$ the relative velocity of the point A, and by $\mathbf{v}_t$ the velocity of trausport (i. e. the velocity of the point M), we therefore obtain (p. 58).

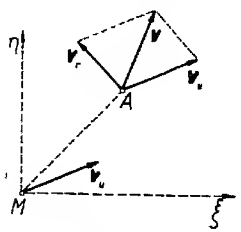

Fig. 253.

$$\mathbf{v} = \mathbf{v}_t + \mathbf{v}_r. \tag{1}$$

In view of this we can say that the velocities of the points of the figure are such as if the figure were executing two motions simultaneously: an advancing motion with the velocity $\mathbf{v}_t$ of an arbitrary point M of the figure, and a rotation about this point M.

Therefore: *the instantaneous motion of a figure in plane motion is composed of an instantaneous advancing motion and an instantaneous rotation; the advancing motion has the velocity of an arbitrary point of the figure, and the rotation is about this point.*

In general, the instantaneous motion of a figure can be represented in infinitely many ways as the composition of an instantaneous advancing motion and an instantaneous rotation, for this depends on the choice of the point M.

In all these representations, however, the instantaneous angular velocities are equal. For let us choose in the figure an arbitrary axis k at the time t and let k' denote the position of this axis at the time $t + \Delta t$. The angle $\Delta\varphi$ which the axis k' makes with the axis k is equal to the angle through which the figure turned about M in the relative motion. It follows from this that $\Delta\varphi$ does not depend on the choice of the point M, and the same is true of ω, because $\omega = \lim\limits_{\Delta t \to 0} \Delta\varphi / \Delta t$.

In particular, an instantaneous motion can be an instantaneous advancing motion (if the instantaneous angular velocity ω is zero) or an instantaneous rotation (if the point M has the velocity $\mathbf{v}_t = 0$).

For each point A of the figure the velocity $\mathbf{v}_r$ (of the instantaneous rotation) is perpendicular to the segment MA, where $|\mathbf{v}_r| = MA \cdot \omega$. Knowing the sense of the instantaneous rotation, we can therefore determine the sense of $\mathbf{v}_r$ and then obtain the velocity $\mathbf{v}$ of the point A from formula (1).

Now let $\mathbf{v}_t \neq 0$ and $\omega \neq 0$. On a line l perpendicular to $\mathbf{v}_t$ and passing through M let us consider two points O and O' at a distance r from M, where $r = |\mathbf{v}_t| / \omega$ (Fig. 254).

The velocities of the instantaneous rotation $\mathbf{v}_r$ and $\mathbf{v}_r'$ of the points O and O' are perpendicular to l and therefore parallel to $\mathbf{v}_t$. In addition we have $|\mathbf{v}_r| = MO \cdot \omega = r\omega = |\mathbf{v}_t|$ and similarly $|\mathbf{v}_r'| = |\mathbf{v}_t|$. Since $\mathbf{v}_r$ and $\mathbf{v}_r'$ have opposite senses, it follows that for one of the points O and O', e. g. for O, we have $\mathbf{v}_r = -\mathbf{v}_t$. Consequently the velocity of the point O is $\mathbf{v} = \mathbf{v}_r + \mathbf{v}_t = 0$. Therefore, if the origin of the system (ξ, η) is placed at the point O, then the velocity of transport will be zero and hence the instantaneous motion will be an instantaneous rotation about O.

The point O is called the *instantaneous centre of rotation.*

The velocity $\mathbf{v}$ of an arbitrary point A is perpendicular to the segment OA and has a sense which depends on the sense of the instantaneous rotation. Moreover

$$|\mathbf{v}| = OA \cdot \omega. \tag{2}$$

Therefore: *an instantaneous plane motion is either an instantaneous advancing motion or an instantaneous rotation about the instantaneous centre of rotation.*

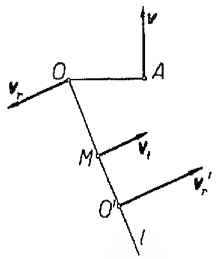

Fig. 254.

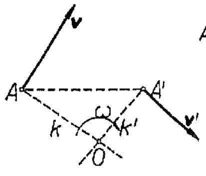

Fig. 255.

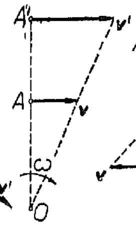

Fig. 256.

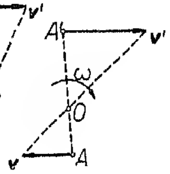

Fig. 257.

Determination of the instantaneous centre of rotation. The instantaneous centre of rotation obviously has a zero velocity. According to (2) every other point has a velocity $\mathbf{v} \neq 0$ (if $\omega \neq 0$). Consequently there exists only one instantaneous centre of rotation (when $\omega \neq 0$).

If at an arbitrary point A we draw a perpendicular to the velocity $\mathbf{v}$ of this point, then the instantaneous centre of rotation will lie on this line (Fig. 255).

The instantaneous centre of rotation can in general be determined if the velocity $\mathbf{v}$ of one point, e. g. A, is known, as well as the direction of the velocity $\mathbf{v}'$ of another point A'. Let us draw k and k' perpendicular to $\mathbf{v}$ and $\mathbf{v}'$ at the points A and A'. The point O of intersection of these lines is the instantaneous centre of rotation. From formula (2) we obtain $\omega = |\mathbf{v}| \,/\, OA$. The sense of the instantaneous rotation is obtained from the sense of the velocity $\mathbf{v}$.

When the lines k and k' are parallel, the instantaneous motion is an instantaneous advancing motion.

When the lines k and k' coincide, in order to determine the instantaneous motion, we must know in addition the velocity $\mathbf{v}'$ of the point A'. Knowing only the direction of the velocity $\mathbf{v}'$, then, is not sufficient.

When $\mathbf{v} = \mathbf{v}'$, the instantaneous motion is an advancing motion.

However, when $\mathbf{v} \neq \mathbf{v}'$, the instantaneous motion is an instantaneous rotation. Denoting, then, by O the instantaneous centre of rotation (obviously lying on the lines k and k'), we have by (2) $|\mathbf{v}| = OA \cdot \omega$ and $|\mathbf{v}'| = OA' \cdot \omega$. Consequently

$$\frac{OA}{OA'} = \frac{|\mathbf{v}|}{|\mathbf{v}'|}. \tag{3}$$

If $\mathbf{v}$ and $\mathbf{v}'$ have the same senses (Fig. 256) and $|\mathbf{v}| < |\mathbf{v}'|$, then the point O lies on the prolongation of the segment $A'A$ beyond the point A. We then have $OA' - OA = A'A$, whence by (3)

$$OA = A'A \cdot \frac{|\mathbf{v}|}{|\mathbf{v}'| - |\mathbf{v}|}. \tag{4}$$

On the other hand, when $\mathbf{v}$ and $\mathbf{v}'$ have opposite senses (Fig. 257), the point O lies on the segment AA'. We then have $OA + OA' = AA'$, whence by (3)

$$OA = AA' \cdot \frac{|\mathbf{v}|}{|\mathbf{v}| + |\mathbf{v}'|}. \tag{5}$$

Example 1. A rod AB moves in a plane in such a way that both of its ends move constantly along the curves C and C'. The velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the points A and B are tangent to the curves C and C' (Fig. 258).

Drawing perpendiculars to the tangents at the points A and B, we obtain the instantaneous centre of rotation O of the rod AB as the point of intersection of these perpendiculars. Denoting the instantaneous angular velocity by ω, we obtain:

$$|\mathbf{v}_1| = OA \cdot \omega, \quad |\mathbf{v}_2| = OB \cdot \omega \quad \text{and} \quad |\mathbf{v}_1| \,/\, |\mathbf{v}_2| = OA \,/\, OB.$$

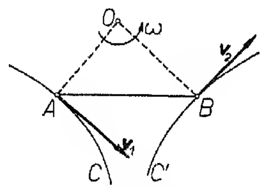

Fig. 258.

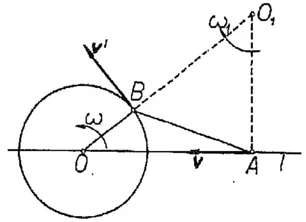

Fig. 259.

Example 2. In a crank-mechanism the rod AB moves in such a way that its end B is pin-connected with the rod OB fixed at O, and the other end A moves along a line l passing through O. The rod OB revolves about the point O with an angular velocity ω (Fig. 259). What is the velocity of the point A?

The velocity $\mathbf{v}_1'$ of the point B is perpendicular to OB, and the velocity $\mathbf{v}$ of the point A has the direction of the line l. Drawing the perpendiculars to $\mathbf{v}'$ and $\mathbf{v}$, we obtain the instantaneous centre of rotation O_1 of the rod AB. Now, $OB \cdot \omega = |\mathbf{v}'| = O_1B \cdot \omega_1$; consequently $\omega_1 = OB \cdot \omega \,/\, O_1B$, whence

$$|\mathbf{v}| = O_1A \cdot \omega_1 = O_1A \cdot OB \cdot \omega \,/\, O_1B.$$

Example 3. A system of pin-connected rods AO and OB moves in a plane (Fig. 260). The velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the points A and B are given. Determine the velocity of the point O.

Let us denote by α and β the angles which the two velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ make with the rods OA and OB, by δ the angle which the velocity $\mathbf{v}$ of the point O makes with the rod OA, and by φ the angle AOB.

Since the projections of the velocity $\mathbf{v}$ on the rods OA and OB are equal to the corresponding projections of $\mathbf{v}_1$ and $\mathbf{v}_2$ on these rods, denoting the absolute values of these velocities by v, v_1 and v_2, we obtain

$$v_1 \cos\alpha = v\cos\delta, \quad v_2\cos\beta = v\cos(\varphi - \delta), \tag{6}$$

whence

$$\cos(\varphi - \delta) \,/\, \cos\delta = v_2\cos\beta \,/\, v_1\cos\alpha;$$

therefore $\cos\varphi + \tan\delta\sin\varphi = v_2\cos\beta \,/\, v_1\cos\alpha$, whence

$$\tan\delta = (v_2\cos\beta - v_1\cos\alpha\cos\varphi) \,/\, v_1\cos\alpha\sin\varphi.$$

Knowing δ, we determine v from equation (6).

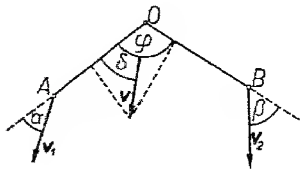

Fig. 260.

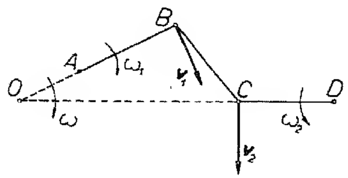

Fig. 261.

Example 4. A system of three rods AB, BC, and CD, pin-connected and fixed at A and D, lies in a plane (Fig. 261). The rod AB rotates about A with an angular velocity ω_1. Determine the angular velocity ω_2 of the rod CD about D.

The points B and C move in circle with centres at A and D; their velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are therefore perpendicular to AB and CD. The instantaneous centre of rotation of the rod BC is obtained by drawing perpendiculars to $\mathbf{v}_1$ and $\mathbf{v}_2$, or by prolonging the segments AB and DC to their

point of intersection O. The point O is the instantaneous centre of rotation of the rod BC.

Let us denote by ω the instantaneous angular velocity of the rod BC. Putting $v_1 = |\mathbf{v}_1|$ and $v_2 = |\mathbf{v}_2|$, we therefore obtain:

$$v_1 = OB \cdot \omega \quad \text{and} \quad v_2 = OC \cdot \omega. \tag{7}$$

The rod AB rotates about A with an angular velocity ω_1; consequently

$$v_1 = AB \cdot \omega_1 \quad \text{and similarly} \quad v_2 = CD \cdot \omega_2. \tag{8}$$

From (7) and (8) we obtain $AB \cdot \omega_1 = OB \cdot \omega$, i.e. $\omega = AB \cdot \omega_1 \,/\, OB$, whence by (7) $v_2 = AB \cdot OC \cdot \omega_1 \,/\, OB$, and since by (8) $\omega_2 = v_2 \,/\, CD$,

$$\omega_2 = \frac{AB \cdot OC}{OB \cdot CD}\,\omega_1.$$

Example 5. A system of rods, pin-connected at the joints B, C, D, and E, is given (Fig. 262). The rods AB, FD, and EG, can rotate about the points A, F, and G, which are fixed. The rod GE rotates about G with an angular velocity ω. Determine the angular velocities ω' and ω'' of the rods FD and AB about F and A.

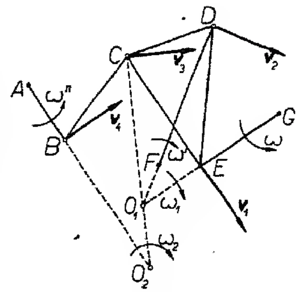

Fig. 262.

Let us denote the velocities of the points E, D, C, B by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, and their absolute values by v_1, v_2, v_3, v_4. The points E and D move along the circumferences of circles with centres at G and F. The velocities of these points are perpendicular to EG and FD.

Let O_1 be the point of intersection of the perpendiculars to $\mathbf{v}_1$ and $\mathbf{v}_2$ (i. e. of the prolongations of the rods GE and DF). The point O_1 will be the instantaneous centre of rotation of the rod ED and at the same time of the triangle EDC, because the rods ED, DC, and CE, form a rigid system.

Denoting by ω_1 the instantaneous velocity of rotation about O_1, whose sense is determined from the sense of $\mathbf{v}_1$, we have:

$$v_1 = O_1E \cdot \omega_1, \quad v_2 = O_1D \cdot \omega_1. \tag{9}$$

The sense of $\mathbf{v}_2$ is determined from the sense of the angular velocity ω_1.

Since $v_1 = GE \cdot \omega$, in virtue of (9):

$$\omega_1 = \frac{GE}{O_1E}\,\omega, \quad v_2 = \frac{O_1D \cdot GE}{O_1E}\,\omega_1 \tag{10}$$

The rod FD rotates about F with an angular velocity ω'; hence $v_2 = FD \cdot \omega'$, whence $\omega' = v_2 / FD$. Consequently by (10)

$$\omega' = \frac{O_1D \cdot GE}{O_1E \cdot FD}\omega. \tag{11}$$

The sense of ω' is obtained from the sense of $\mathbf{v}_2$.

The velocity $\mathbf{v}_3$ of the point C is perpendicular to O_1C; we therefore have

$$v_3 = O_1C \cdot \omega_1. \tag{12}$$

The sense of $\mathbf{v}_3$ is determined from the sense of the rotation about O_1.

In order to determine the centre O_2 of the instantaneous rotation of the rod BC, let us note that the velocity $\mathbf{v}_4$ of the point B is perpendicular to the rod AB. We therefore draw perpendiculars to the velocities $\mathbf{v}_4$ and $\mathbf{v}_3$, i. e. we prolong AB and CO_1 to their intersection at O_2.

Denoting by ω_2 the angular velocity of the instantaneous rotation about O_2, we have:

$$v_3 = O_2C \cdot \omega_2, \quad v_4 = O_2B \cdot \omega_2. \tag{13}$$

The sense of the instantaneous rotation is obtained from the sense of $\mathbf{v}_3$.

From (13) we get:

$$\omega_2 = v_3 / O_2C, \quad v_4 = O_2B \cdot v_3 / O_2C. \tag{14}$$

The sense of $\mathbf{v}_4$ is obtained from the sense of the rotation about O_2.

Since the rod AB rotates about A with an angular velocity ω'', $v_4 = AB \cdot \omega''$, whence $\omega'' = v_4 / AB$. Hence by (14), (12) and (10) we get

$$\omega'' = \frac{O_2B \cdot O_1C \cdot GE}{AB \cdot O_2C \cdot O_1E}\omega.$$

The sense of the angular velocity ω'' is obtained from the sense of $\mathbf{v}_4$.

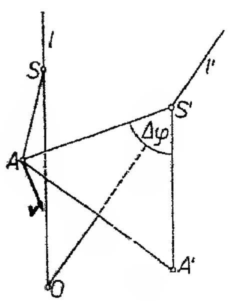

Fig. 263.

§ 7. Instantaneous space motion. We shall first consider a particular case.

Rotation about a point. Let us suppose that a body rotating about a fixed point O occupied position I at time t and position II at time $t + \Delta t$. The body can therefore be displaced from position I to position II by means of a rotation about a certain line l' through an angle $\Delta\varphi$ (Fig. 263). Let us assume that the line l' tends to a certain line l when Δt tends to zero.

Let us set

$$\lim_{\Delta t \to 0} \Delta\varphi / \Delta t = \omega. \tag{1}$$

Let us denote by A an arbitrary point of the body in position I, and by A' the corresponding point in position II. Let S' be the centre of the circle along which the point A moves during its rotation about the line l', and S the limiting position of the point S' as $\Delta t \to 0$. Finally, let us denote by $\mathbf{v}$ the velocity of the point A. We have

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\overline{AA'}}{\Delta t};$$

consequently

$$|\mathbf{v}| = \lim_{\Delta t \to 0} \left|\frac{\overline{AA'}}{\Delta t}\right| = \lim_{\Delta t \to 0} \frac{2AS' \sin\frac{1}{2}\Delta\varphi}{\Delta t} = \lim_{\Delta t \to 0} \frac{AS' \sin\frac{1}{2}\Delta\varphi}{\frac{1}{2}\Delta\varphi} \cdot \frac{\Delta\varphi}{\Delta t},$$

and since $\lim_{\Delta\varphi \to 0}(\sin\frac{1}{2}\Delta\varphi / \frac{1}{2}\Delta\varphi) = 1$, it follows by (1) that

$$|\mathbf{v}| = AS \cdot \omega. \tag{2}$$

Let us note that the vector $\overline{AA'} / \Delta t$ is perpendicular to l' and makes an angle of $90° - \frac{1}{2}\Delta\varphi$ with the segment $S'A$; the vector $\mathbf{v}$ is then in the limit perpendicular to l and AS. The velocity of the point A at the time t is therefore such as if the body were rotating about the axis l with an angular velocity ω. Thus we have proved the following theorem:

The instantaneous motion of a body rotating about a certain point O is an instantaneous rotation about an axis passing through O.

The velocity $\mathbf{v}$ of the point A is perpendicular to the plane Π passing through l and A, whence $\mathbf{v} \perp OA$, since OA lies in Π.

Hence: *during the rotation of a body about the point O, the velocities of the points of the body are perpendicular to the lines connecting these points with the point O.*

The axis l of the instantaneous rotation lies in a plane passing through the point A and perpendicular to the velocity of the point A. Therefore, knowing the directions of the velocities of two points of the body, we obtain the instantaneous axis of rotation as the line of intersection of the planes passing through these points and perpendicular to the directions of the velocities. The instantaneous angular velocity is obtained from equation (2).

Example 1. The sphere $x^2 + y^2 + z^2 = 1$ rotates about the centre O. The velocity $\mathbf{v}$ of the point $A(1, 0, 0)$ at a certain instant t and the direction of the velocity $\mathbf{w}$ of the point $B(0, 1, 0)$ at the same instant are given.

Determine the instantaneous axis of rotation, the angular velocity, and the velocity $\mathbf{w}$ (at the moment t).

Since the velocity $\mathbf{v}$ is perpendicular to OA, i. e. to the x-axis, $v_x = 0$. Let us denote the cosines of the angles which the velocity $\mathbf{w}$ makes with the coordinate axes by a, b, c. We obviously have $b = 0$, because $\mathbf{w}$ is perpendicular to OB, i. e. to the y-axis.

The instantaneous axis of rotation is the intersection of the planes passing through O as well as through A and B, and perpendicular to the velocities $\mathbf{v}$ and $\mathbf{w}$. The equations of these planes are the following:

$$yv_y + zv_z = 0, \quad ax + cz = 0, \tag{3}$$

from which

$$\frac{x}{c\,/\,a} = \frac{y}{v_z\,/\,v_y} = \frac{z}{-1}. \tag{4}$$

Equations (4) are the equations of the instantaneous axis of rotation.

Let $\boldsymbol{\omega}$ denote the instantaneous angular velocity. The projections of $\boldsymbol{\omega}$ on the coordinate axes are proportional to the direction numbers of the axis of rotation, i. e. to the numbers $c\,/\,a$, $v_z\,/\,v_y$, and -1. Denoting by λ the factor of proportionality we get:

$$\omega_x = \lambda c\,/\,a, \quad \omega_y = \lambda v_z\,/\,v_y, \quad \omega_z = -\lambda. \tag{5}$$

In order to determine λ we calculate the velocity $\mathbf{v}$ by making use of the formula $\mathbf{v} = \overline{OA} \times \boldsymbol{\omega}$. We obtain $v_x = 0$, $v_y = \lambda$, and $v_z = \lambda v_z\,/\,v_y$, whence $\lambda = v_y$, and hence by (5):

$$\omega_x = cv_y\,/\,a, \quad \omega_y = v_z, \quad \omega_z = -v_y. \tag{6}$$

Since $\mathbf{w} = \overline{OB} \times \boldsymbol{\omega}$, we get by (6):

$$w_x = -v_y, \quad w_y = 0, \quad w_z = -cv_y\,/\,a.$$

Instantaneous motion in the general case. Let us place the origin of the coordinate system (ξ, η, ζ), moving with an advancing motion, at an arbitrary point M of a given body. Since the point M is fixed relative to the system (ξ, η, ζ), the motion of the body relative to the system (ξ, η, ζ) is a rotation about the point M. The instantaneous relative motion will therefore be an instantaneous rotation about the axis l passing through M (Fig. 264).

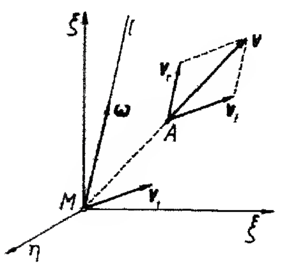

Fig. 264.

Let A be an arbitrary point of the body. Let us denote by $\mathbf{v}$ the absolute velocity of

the point A, by $\mathbf{v}_t$ the velocity of transport, and by $\mathbf{v}_r$ the relative velocity. Consequently

$$\mathbf{v} = \mathbf{v}_t + \mathbf{v}_r. \tag{7}$$

The velocity of transport $\mathbf{v}_t$ is equal to the velocity of the point M, because the system (ξ, η, ζ) moves with an advancing motion. The velocities of the points of the body are therefore such as if the body were executing two motions simultaneously: an advancing motion with the velocity $\mathbf{v}_t$ of an arbitrary point M of the body, and a rotation about a certain axis passing through M.

Hence: *an instantaneous motion of a body is composed of an instantaneous advancing motion with the velocity of an arbitrary point M of this body, and an instantaneous rotation about the instantaneous axis of rotation passing through the point M.*

The instantaneous motion of a body can in general be represented in infinitely many ways as the composition of an instantaneous advancing motion and an instantaneous rotation, for this depends on the choice of the point M.

We shall show that *the instantaneous angular velocity vectors are equal for all possible representations* (i. e. *that the instantaneous axes of rotation are parallel and the instantaneous angular velocities are equal*).

Let us suppose that a point M of the body at the time t coincided with the point M' at the time $t + \Delta t$. Relative to the fixed system, the change of position of the body in the time Δt is the composition of the displacement $\overline{MM'}$ and the rotation through an angle $\Delta\varphi$ about a certain axis l'. Relative to the system (ξ, η, ζ) the change of position is only a rotation about the axis l' through the angle $\Delta\varphi$, because the system (ξ, η, ζ) made the displacement $\overline{MM'}$ in the time Δt. Consequently the limiting position of the axis l' is the instantaneous axis of rotation l, and the instantaneous angular velocity is $\omega = \lim\limits_{\Delta t \to 0} \Delta\varphi\,/\,\Delta t$.

From the theorem given on p. 315 it follows that had we chosen another point M_1 in the body, then with similar notations the axis l_1' would be parallel to l', while $\Delta\varphi_1 = \Delta\varphi$. Consequently the instantaneous axis of rotation l_1 passing through M_1 is parallel to l, and the instantaneous angular velocity ω_1 is equal to ω. Therefore the angular velocity vectors for M and for M_1 will be equal.

Remark. Denoting the instantaneous angular velocity vector by $\boldsymbol{\omega}$, we obtain by (III), p. 46, $\mathbf{v}_r = \overline{MA} \times \boldsymbol{\omega}$ for an arbitrary point A. Hence by (7)

$$\mathbf{v} = \mathbf{v}_t + \overline{MA} \times \boldsymbol{\omega}. \tag{I}$$

Velocity of transport. Let a system of coordinates (ξ, η, ζ) with origin M move in space relative to a fixed coordinate system (x, y, z) (Fig. 265). The system (ξ, η, ζ) (together with the points rigidly attached to it) can be considered as a rigid body. The instantaneous motion of the system (ξ, η, ζ) will therefore be the composition of the instantaneous advancing motion with a velocity $\mathbf{v}_0$ of the origin M of the system, and the rotation with an instantaneous angular velocity $\boldsymbol{\omega}$ about an instantaneous axis of rotation passing through M. The velocity of transport $\mathbf{v}_t$ of an arbitrary point A in motion relative to the system (ξ, η, ζ) is the velocity the point A would possess were it rigidly attached to the system (ξ, η, ζ). Consequently $\mathbf{v}_t$ is the sum of the velocity $\mathbf{v}_0$ and the velocity $\mathbf{w}$ of the instantaneous rotation. In view of the preceding we have by (I)

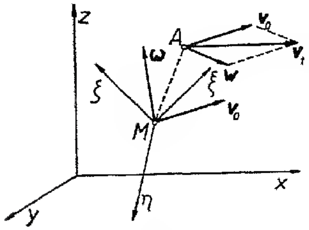

Fig. 265.

$$\mathbf{v}_t = \mathbf{v}_0 + \overline{MA} \times \boldsymbol{\omega}. \tag{8}$$

Therefore: *the velocity of transport of an arbitrary point is such as if this point were rigidly attached to the moving coordinate system, and this system executed two simultaneous motions: an advancing motion with a velocity of the origin of the system, and a rotation about an axis passing through the origin of the system.*

This theorem was given without proof on p. 62.

Instantaneous twist. An instantaneous motion of a body is called an *instantaneous twist* or an *instantaneous screw motion* if the velocity of the instantaneous advancing motion is parallel to the instantaneous axis of rotation.

In particular, an instantaneous advancing motion or an instantaneous rotation is also called an instantaneous twist.

The instantaneous axis of twist is called the *central axis* of the instantaneous rotation.

By means of the theorem given on p. 317, we shall prove the following theorem:

An instantaneous motion of a rigid body can be represented as an instantaneous twist and this can be done in only one way.

Proof. Let us assume that we have displaced the body from position I at the instant t to the position II at the instant $t + \Delta t$ by means of a twist about the axis l' (p. 317). Let l be the limiting position of the line

l' as $\Delta t \to 0$ (Fig. 266). Let us consider an arbitrary point O on l at the instant t and denote the position of the point O at the instant $t + \Delta t$ by O'. During the displacement of the body from position I to position II by means of a twist, the point O will occupy the position O_1 after the displacement, and then it will go to the position O' after a rotation about l' through an angle $\Delta\varphi$. Denoting by $\mathbf{u}$ the velocity of the point O, we therefore have

$$\mathbf{u} = \lim_{\Delta t \to 0} \frac{\overline{OO'}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{OO_1} + \overline{O_1O'}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{OO_1}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\overline{O_1O'}}{\Delta t}. \tag{9}$$

Let S be the centre of the circle along which the point O moved during the rotation about the axis l' through the angle $\Delta\varphi$. Consequently $|\overline{O_1O'}| = 2SO_1 \cdot \sin\frac{1}{2}\Delta\varphi$, whence

$$\lim_{\Delta t \to 0} \left| \frac{\overline{O_1O'}}{\Delta t} \right| = \lim_{\Delta t \to 0} 2SO_1 \cdot \left| \frac{\sin(\frac{1}{2}\Delta\varphi)}{\Delta t} \right| = \lim_{\Delta t \to 0} SO_1 \cdot \left| \frac{\sin(\frac{1}{2}\Delta\varphi)}{\frac{1}{2}\Delta\varphi} \right| \cdot \left| \frac{\Delta\varphi}{\Delta t} \right|.$$

Since $\lim\limits_{\Delta t \to 0} SO_1 = 0$, and

$$\lim_{\Delta\varphi \to 0} \left| \frac{\sin(\frac{1}{2}\Delta\varphi)}{\frac{1}{2}\Delta\varphi} \right| = 1, \qquad \lim_{\Delta t \to 0} \left| \frac{\overline{O_1O'}}{\Delta t} \right| = 0.$$

Therefore, by (9), $\mathbf{u} = \lim\limits_{\Delta t \to 0} \overline{OO_1} / \Delta t$. But $\overline{OO_1} \parallel l'$; hence because l' tends to l as $\Delta t \to 0$, the velocity $\mathbf{u}$ has the direction of the axis l, which is the instantaneous axis of rotation passing through O. The instantaneous motion is therefore a twist, because the velocity $\mathbf{u}$ of the instantaneous advancing motion has the direction of the instantaneous axis of rotation l.

The points lying on the instantaneous axis of twist have velocities equal to $\mathbf{u}$, and hence parallel to the axis of twist. The points situated outside the axis of twist have velocities which are not parallel to the axis of twist, for the velocity of such a point is the sum of the velocity $\mathbf{u}$ and the velocity of rotation $\mathbf{w}$ perpendicular to $\mathbf{u}$. Consequently the sum $\mathbf{u} + \mathbf{w}$ is never parallel to $\mathbf{u}$ (and hence also to l), except when $\mathbf{w} = 0$, i. e. when the point lies on the axis of twist.

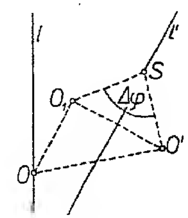

Fig. 266.

It follows from this that an instantaneous motion can be represented as an instantaneous twist in only one way. For, if we had represented this instantaneous motion as a twist about another axis l_1, then by the theorem on p. 333, the lines l and l_1 would be parallel and in that case the velocities

of the points lying on l_1 would be parallel to l_1 and l, which is impossible, since — as we have just proved — only points situated on the axis l have velocities parallel to l.

Example 2. Let a body move in such a way that its instantaneous motion is a twist of constant advancing velocity $\mathbf{u}$ and angular velocity $\boldsymbol{\omega}$ about a fixed axis l.

Let us choose the z-axis as the instantaneous axis of rotation. Let us denote by ω and u the components of $\boldsymbol{\omega}$ and $\mathbf{u}$ with respect to the z-axis. The velocity $\mathbf{v}$ of a point $A(x, y, z)$ of the body is expressed, in virtue of (I), p. 333, by the formula

$$\mathbf{v} = \mathbf{u} + \overline{OA} \times \boldsymbol{\omega},$$

where O is the origin of the system. Since $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$, and $u_x = 0$, $u_y = 0$, $u_z = u$:

$$x^{\cdot} = \omega y, \quad y^{\cdot} = -\omega x, \quad z^{\cdot} = u. \tag{10}$$

The last of the equations (10) gives after integration,

$$z = ut + c, \tag{11}$$

where c is an arbitrary constant. Differentiating the first of the equations (10), we obtain $x^{\cdot\cdot} = \omega y^{\cdot}$. Substituting for $y^{\cdot}$ the value from the second equation, we get the equation $x^{\cdot\cdot} + \omega^2 x = 0$, whose general solution has the form

$$x = a \sin \omega t + b \cos \omega t, \tag{12}$$

where a and b are arbitrary constants. Since $y = x^{\cdot} / \omega$,

$$y = a \cos \omega t - b \sin \omega t. \tag{13}$$

Let us assume that the point A had the coordinates $x_0 = r$, $y_0 = 0$, and $z_0 = 0$, at $t = 0$. Substituting $t = 0$ in (11)—(13), we get $a = 0$, $b = r$, and $c = 0$, whence:

$$x = r \cos \omega t, \quad y = -r \sin \omega t, \quad z = ut.$$

The point will therefore move with a screw motion (p. 55) on a cylindrical surface whose axis is the z-axis (because $x^2 + y^2 = r^2$), describing the so-called *helix*. If we develop the lateral surface of the cylinder, the helix will appear as a straight line. The helix consequently cuts all the generatrices at the same angle α. The distance of two neighbouring points of a helix on the same generatrix is called the *lead of the helix* and we denote it by h. We therefore have

$$\tan \alpha = 2r\pi / h, \tag{14}$$

where r is the radius of the base of the cylinder. Since the time for one revolution is $2\pi / |\omega|$ or $h / |\mathbf{u}|$,

$$h = 2\pi|\mathbf{u}| / |\boldsymbol{\omega}|, \quad \tan\alpha = r|\boldsymbol{\omega}| / |\mathbf{u}|. \tag{15}$$

Determination of the motion of a body. Let us select in the body an arbitrary point O whose coordinates are x_0, y_0, z_0 with respect to a certain fixed coordinate system (x, y, z). Let us denote the instantaneous angular velocity by $\boldsymbol{\omega}$ and the velocity of the point O by $\mathbf{u}$. A point A of the body whose coordinates are x, y, z, has the velocity ((I), p. 333)

$$\mathbf{v} = \mathbf{u} + \overline{OA} \times \boldsymbol{\omega}. \tag{16}$$

Since: $v_x = x^{\cdot}$, $v_y = y^{\cdot}$, $v_z = z^{\cdot}$, and $u_x = x_0^{\cdot}$, $u_y = y_0^{\cdot}$, $u_z = z_0^{\cdot}$, we get from formula (16):

$$\begin{aligned} x^{\cdot} &= x_0^{\cdot} + (y - y_0)\,\omega_z - (z - z_0)\,\omega_y, \\ y^{\cdot} &= y_0^{\cdot} + (z - z_0)\,\omega_x - (x - x_0)\,\omega_z, \\ z^{\cdot} &= z_0^{\cdot} + (x - x_0)\,\omega_y - (y - y_0)\,\omega_x. \end{aligned} \tag{17}$$

If the motion of the point O and the angular velocity $\boldsymbol{\omega}$ are given by the functions:

$$x_0 = f(t), \; y_0 = \varphi(t), \; z_0 = \psi(t); \quad \omega_x = \alpha(t), \; \omega_y = \beta(t), \; \omega_z = \gamma(t),$$

then (17) is a system of differential equations in which the unknown functions are the functions $x = F(t)$, $y = \Phi(t)$, and $z = \Psi(t)$, defining the motion of the point A. From equations (17) we can determine the functions F, Φ, Ψ, if we know the initial position of the point A, the instantaneous angular velocity $\boldsymbol{\omega}$, and the motion of the point O.

Therefore: *the motion of a rigid body is determined by giving the following*:

a) *the initial position of the body,*
b) *the motion of one of its points,*
c) *the instantaneous angular velocity of rotation* $\boldsymbol{\omega}$ *at each instant.*

§ 8. Rolling and sliding. Let a plane curve C, moving in its own plane Π, be in contact at each moment with a certain fixed curve C' lying in Π (Fig. 267).

If the instantaneous motion of the curve C is an instantaneous rotation about the point of contact O, then its instantaneous motion is called a *rolling* motion of the curve C on C'.

It follows from this that during a rolling motion the point of contact has a zero velocity. The point of contact is the instantaneous centre of rotation.

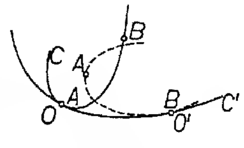

Fig. 267.

If the instantaneous motion of the curve C is an advancing motion, then its instantaneous motion is called a *sliding* motion of the curve C on C'.

The velocity of the advancing motion during sliding is equal to the velocity of the point of contact.

In the general case, the instantaneous motion of the curve C can be considered as the composition of two motions: a rotation about the point of contact O, and an advancing motion with a velocity of the point of contact.

Therefore: *an instantaneous motion is the composition of a rolling motion and of a sliding motion.*

It can be proved that during the rolling of curve C on curve C', the points of contact describe arcs of equal length on both curves (arcs OO' and AB in the Fig. 267). For example, when a circle rolls along a line l, the distance between the points of contact after one complete revolution is equal to the circumference of the circle.

The rolling and sliding of one surface on another is defined analogously.

Namely, let a surface Σ move in such a way that it is constantly tangent to a certain fixed surface Σ'.

If the point of contact has a zero velocity, then we say that the instantaneous motion of the surface Σ is a *rolling* motion on the surface Σ'.

During rolling the instantaneous motion is a rotation about an axis passing through the point of contact. In particular, if the surfaces Σ and Σ' are cylinders or cones, tangent along their generatrices, then during rolling the instantaneous axis of rotation is the generatrix along which the surfaces are in contact.

If the instantaneous motion of the surface Σ is an advancing motion, we say that the instantaneous motion of the surface Σ is a *sliding* motion on the surface Σ'.

Curve of instantaneous centres. Let a figure K move in the plane Π. At each instant let us consider the instantaneous centre of rotation in the plane Π. These centres will describe in Π a certain curve C', called the *fixed curve of instantaneous centres.*

At each instant let us now take under consideration on the figure K a point which coincides with the instantaneous centre of rotation at the given moment. These points will describe on the figure K a certain curve C which moves together with the figure. This curve is called the *moving curve of instantaneous centres.*

In general, the curves of instantaneous centres: the moving curve C and the fixed curve C' are tangent to each other at each instant, and their point of contact is the instantaneous centre of rotation. The moving curve therefore moves together with the figure K in such a way that its instantaneous motion is at each instant a rotation about its point of contact with the fixed curve C'.

Hence: *in a plane motion the moving curve of instantaneous centres rolls on the fixed curve of instantaneous centres.*

Cone of instantaneous axes. Let a body K rotate about the point O. Let us consider in space an instantaneous axis of rotation at each instant. These axes will generate a certain conical surface Σ' with vertex at O; it is called the *fixed cone of instantaneous axes.*

Let us next consider in the body K at each instant that line which coincides with the instantaneous axis of rotation at a given instant. The surface Σ which these lines form is called the *moving cone of instantaneous axes.*

The surface Σ moves together with the body and is in general tangent to Σ'.

Therefore: *during a rotation of a body about a point the moving surface of instantaneous axes rolls on the fixed surface of instantaneous axes.*

Surface of central axes. Let a body K move arbitrarily in space. Let us consider in space the axis of twist, i. e. the central axis at each instant. These axes form a certain surface Σ', called the *fixed surface of central axes.*

Let us also consider in the body at each moment a line which coincides with the central axis at a given moment. The surface Σ generated by these axes in the body is called the *moving surface of central axes.*

In general, the moving surface of central axes is tangent to the fixed surface at each moment along the central axis. The instantaneous motion of the moving surface is a twist about the axis of tangency.

Example I. A circle K of radius r, whose centre O moves with a uniform velocity $\mathbf{u}$, rolls along the line l (Fig. 268). Since the point of contact S is the centre of instantaneous rotation, denoting by ω the instantaneous angular velocity, we have $u = r\omega$ (where $u = |\mathbf{u}|$), whence

$$\omega = u / r. \tag{1}$$

The point A lying on the diameter SO has a velocity equal in magnitude to $2r\omega = 2u$; the point A therefore has the velocity $2\mathbf{u}$.

The wheels of a railway carriage have flanges on the inner sides of the rails in order to prevent the carriage from derailing. Therefore the lowest point of a railway

carriage wheel (in the figure the point B) is below the point of tangency S. Its instantaneous velocity consequently has a sense opposite to the velocity of the train (i. e. that of the centre O of the wheel) and is $SB \cdot \omega = u \cdot SB / r$.

For example, if a train moves with a velocity of 50 km/h, then at each instant there exist points on the wheels of the train having instantaneous velocities of 100 km/h (the point A in the figure), and even such that move in a direction opposite to that of the train (e. g. the point B).

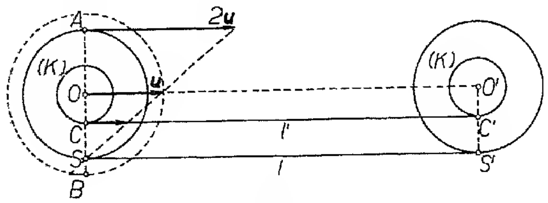

Fig. 268.

On the circle K let us consider a circle K' with centre O and radius $OC = r'$. The circle remains constantly tangent to the line $l' \parallel l$. Since the point C is not an instantaneous centre of rotation, this circle does not roll on l'. The motion of the wheel K' is a composition of the rolling motion and the sliding motion on the line l'. The rolling motion takes place with an angular velocity ω, and the sliding motion has the velocity of the point C, i. e. $SC \cdot \omega$.

Since the entire wheel K of radius r rolls on l, the segment SS' described by the points of contact during one revolution of the wheel is equal to the circumference of the wheel, i. e. $2\pi r$. The corresponding segment CC' for the wheel K is also $2\pi r$. It is not equal to the circumference $2\pi r'$ of the wheel K', since the motion of the circle K along the line l' is not a rolling motion, but a composition of the rolling motion and the sliding motion along this line.

Example 2. A circle (C) of radius r rolls on a fixed circle (C') of radius $2r$. The circle (C) is within the circle (C') (Fig. 269). Determine the paths of the points of circle (C).

Let A be an arbitrary point on the circumference of the circle (C). Since the point of contact S of both circles is an instantaneous centre of rotation, the velocity $\mathbf{v}$ of the point A is perpendicular to SA. The direction of the velocity of the point A therefore passes through the point O which lies on the circle (C) and on the diameter passing through S. Since $SO = 2r$, O is the centre of the circle (C'). The velocity of the point A is therefore constantly directed towards the fixed point O. Consequently the

point A moves along the line OA. Hence the points on the circumference of the circle (C) move along the diameters of the circle (C').

Now let P be an arbitrary point within the circle (C). Let us pass an arbitrary chord AB through P. The points A and B move along the lines l and l' passing through O. Since the segment AB does not change its length, by a well-known theorem from analytic geometry the point P describes an ellipse.

Example 3. A cone of revolution rolls on a plane Π (Fig. 270). The instantaneous motion of the cone is therefore an instantaneous rotation about a generatrix along which it is tangent to the plane Π.

The vertex W of the cone always lies on the instantaneous axis of rotation. Hence it constantly has a zero velocity, i. e. it remains at rest.

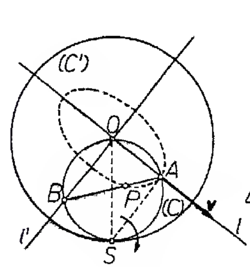

Fig. 269.

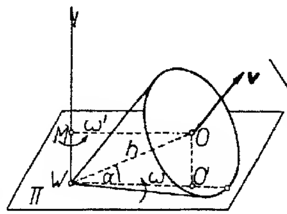

Fig. 270.

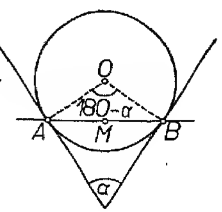

Fig. 271.

Let us denote by α the angle between the generatrices and the altitude h of the cone, by O the centre of the base of the cone, and by O' the projection of O on the plane Π. Since $O'O = h \sin\alpha = \text{const}$, the point O moves in a plane parallel to Π.

The distance of the point O from the line l, perpendicular to Π at the point W, is $MO = WO' = h\cos\alpha = \text{const}$. It follows from this that the point O moves in a circle with centre at M in a plane perpendicular to l, and hence it rotates about the line l.

Let ω be the angular velocity of the rolling cone, and ω' the angular velocity of the centre O during the rotation about the axis l. Finally, let $\mathbf{v}$ be the velocity of the point O. We therefore have $|\mathbf{v}| = O'O \cdot \omega$ and $|\mathbf{v}| = MO \cdot \omega'$, whence $\omega' = \omega \cdot O'O / MO = \omega \cdot O'O / WO'$, and hence

$$\omega' = \omega \tan\alpha. \tag{2}$$

Example 4. A sphere rolls in a trough formed by two planes (Fig. 271).

Since the points of contact A and B have a zero velocity, the instantaneous axis of rotation is the line AB.

Let us denote by r the radius of the sphere, by α the angle between the planes of the trough, by ω the angular velocity of the rolling motion, and finally by $\mathbf{v}$ the velocity of the centre O of the sphere.

The distance of the centre of the sphere from the axis of rotation is $OM = r \sin\frac{1}{2}\alpha$. Putting $v = |\mathbf{v}|$ we therefore obtain

$$v = r\omega \sin\tfrac{1}{2}\alpha. \tag{3}$$

Example 5. A segment AB of length d moves in such a way that its ends remain constantly on the lines l and m, perpendicular to each other and intersecting at the point M (Fig. 272).

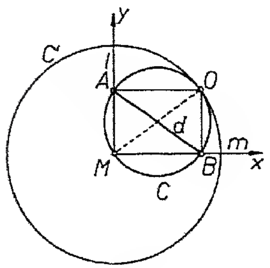

Fig. 272.

The centre of instantaneous rotation O is obtained by drawing perpendiculars l and m at the points A and B. Since $MO = AB = d$, the centres of instantaneous rotation form a circle C' with centre at M and radius d. The circle C' is therefore the fixed curve of instantaneous centres. Since the angle AOB is equal to $\frac{1}{2}\pi$ at every position of the segment AB, the moving curve of instantaneous centres will be the circle C of diameter AB (cf. example 2).

§ 9. Composition of motions of a body. Two simultaneous rotations. Let the instantaneous motion of a body K relative to the body K_1 (i. e. relative to the coordinate system attached rigidly to the body K_1) be a rotation about the axis l_1 with an angular velocity $\boldsymbol{\omega}_1$, and the instantaneous motion of the body K_1 relative to a body K' a rotation about the axis l_2 with an angular velocity $\boldsymbol{\omega}_2$. We then say that the body K makes, relative to the body K', two simultaneous instantaneous rotations about the axes l_1 and l_2 with angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$.

The instantaneous motion of the body K relative to the body K' is called the *resultant motion* of these two simultaneous rotations; we also say that it is *equivalent to the system of both rotations*.

Let us take a system of coordinates (ξ, η, ζ) in the body K_1, and a system of coordinates (x, y, z) in the body K' (Fig. 273). Let A be an arbitrary point of the body K. The velocity $\mathbf{v}$ of point A relative to the system (x, y, z) is the sum of its relative velocity $\mathbf{v}_r$ with respect to the system (ξ, η, ζ) and the velocity of transport $\mathbf{v}_t$:

$$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_t. \tag{1}$$

Since the instantaneous motion of the body K relative to the system (ξ, η, ζ) is a rotation about the axis l_1 with an angular velocity $\boldsymbol{\omega}_1$ (p. 323)

$$\mathbf{v}_r = \mathrm{Mom}_A\boldsymbol{\omega}_1. \tag{2}$$

The velocity of transport $\mathbf{v}_t$ of the point A is obtained by assuming that the point A is attached rigidly to the system (ξ, η, ζ). The system (ξ, η, ζ) rotates about the axis l_2 with an angular velocity $\boldsymbol{\omega}_2$; hence

$$\mathbf{v}_t = \mathrm{Mom}_A\boldsymbol{\omega}_2. \tag{3}$$

From (1), (2) and (3) we obtain

$$\mathbf{v} = \mathrm{Mom}_A\boldsymbol{\omega}_1 + \mathrm{Mom}_A\boldsymbol{\omega}_2. \tag{4}$$

As it is seen from this formula, the instantaneous velocity of the body K relative to K' is determined by the angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. It is not

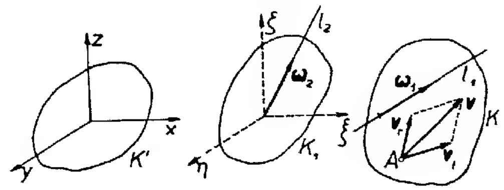

Fig. 273.

necessary to state, in addition, which of them is the velocity of rotation of the body K relative to K_1, and which the velocity of rotation of the body K_1 relative to K'.

Let us suppose that the axes l_1 and l_2 intersect at the point O. Let us consider the vector $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ with its origin at O. We have (p. 17)

$$\mathrm{Mom}_A\boldsymbol{\omega} = \mathrm{Mom}_A\boldsymbol{\omega}_1 + \mathrm{Mom}_A\boldsymbol{\omega}_2, \tag{5}$$

whence by (4)

$$\mathbf{v} = \mathrm{Mom}_A\boldsymbol{\omega}. \tag{6}$$

The instantaneous motion of the body K relative to K' is consequently an instantaneous rotation about an axis passing through the point O, with an angular velocity equal to the sum of the angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ of the component rotations.

Therefore: *a system of two simultaneous instantaneous rotations about the axes which intersect at the point O is equivalent to a rotation about an axis passing through O, with an angular velocity equal to the angular velocities of the component rotations.*

Let us suppose now that the axes l_1 and l_2 are parallel, where $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \neq 0$. Then the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ have a resultant vector $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ for which the equalities (5) and (6) hold.

Therefore: *a system of two simultaneous rotations about parallel axes, with angular velocities whose sum is different from zero, is equivalent to a rotation about an axis parallel to the preceding axes; the position of this axis and the angular velocity of rotation about it is determined by the resultant of the angular velocities of the component rotations.*

Finally, let us suppose that the axes l_1 and l_2 are parallel, but $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = 0$, i. e. that the senses of the rotations are opposite, and the absolute values of the angular velocities are equal. In this case the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ form a couple. Since the moment of the couple is a constant vector, by (4) all the points of the body K have one and the same velocity equal to the moment of the couple $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$. The instantaneous motion of the body K relative to the body K' is consequently an advancing motion.

Therefore: *a system of two simultaneous instantaneous rotations about parallel axes with angular velocities equal in magnitude, but opposite in sense, is equivalent to an instantaneous advancing motion.*

Let us now pass to the general case. Let O be an arbitrary point of the body K. In virtue of the theorem on reduction (p. 24), the system of vectors $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ is equipollent to a system composed of the vector $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ with its origin at O and a couple $\boldsymbol{\omega}', -\boldsymbol{\omega}'$ of moment equal to the moment of the system $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ with respect to O. For each point A of the body K we therefore have $\mathrm{Mom}_A\boldsymbol{\omega}_1 + \mathrm{Mom}_A\boldsymbol{\omega}_2 = \mathrm{Mom}_A\boldsymbol{\omega} + \mathbf{u}$, where $\mathbf{u}$ is the moment of the couple $\boldsymbol{\omega}', -\boldsymbol{\omega}'$. By (4) we then have

$$\mathbf{v} = \mathrm{Mom}_A\boldsymbol{\omega} + \mathbf{u}. \tag{7}$$

Consequently the instantaneous motion of the body K relative to K' is the composition of an advancing motion with a velocity $\mathbf{u}$ of the point O and a rotation with an angular velocity $\boldsymbol{\omega}$ about an axis passing through O.

Therefore: *a system of two simultaneous instantaneous rotations is equivalent to the composition of a rotation about an axis passing through an arbitrary point O of the body and an advancing motion with a velocity of the point O; the vector of angular velocity of the resultant motion is equal to the sum of the vectors of angular velocities of the component rotations.*

Composition of several simultaneous rotations. The results obtained can be generalized to the case of several simultaneous rotations. The resultant motion is defined in a manner similar to that for two rotations.

We shall therefore say, for instance, that the body K makes simultaneous rotations about the axes l_1, l_2, and l_3, with the velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$, and $\boldsymbol{\omega}_3$, relative to a certain body K', if the body K rotates about the axis l_1 with an angular velocity $\boldsymbol{\omega}_1$ relative to a certain body K_1, while K_1

rotates about the axis l_2 with an angular velocity $\boldsymbol{\omega}_2$ relative to a certain body K_2, and K_2 rotates about the axis l_3 with an angular velocity $\boldsymbol{\omega}_3$ relative to the body K'.

Similarly, the instantaneous motion of the body K relative to K' is defined as the resultant motion of the instantaneous rotations about the axes l_1, l_2, l_3, with the angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3$.

Let a body K make several simultaneous rotations with the angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$ As in the case of two rotations, one proves that the velocitity $\mathbf{v}$ of an arbitrary point A of the body K relative to the fixed body K' is

$$\mathbf{v} = \mathrm{Mom}_A\boldsymbol{\omega}_1 + \mathrm{Mom}_A\boldsymbol{\omega}_2 + \ldots \tag{8}$$

The velocity $\mathbf{v}$ of the point A is the total moment of the system of angular velocity vectors $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$:

$$\mathbf{v} = \mathrm{Mom}_A(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots). \tag{9}$$

Therefore, if the systems of angular velocities

$$\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots \quad \text{and} \quad \boldsymbol{\omega}_1', \boldsymbol{\omega}_2', \ldots$$

for two systems of simultaneous rotations are equipollent (p. 22), then the resultant motions of these rotations are the same.

This enables us to interpret the theorems of chapter I on systems of vectors as theorems on systems of simultaneous rotations.

The theorem on the reduction of a system of vectors (p. 26) can therefore be formulated as follows:

A system of several simultaneous rotations with angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$ is the composition of an advancing motion with a velocity of an arbitrary point O of the body and a rotation with an angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots$ about an axis passing through O.

According to this interpretation the theorems 1—4 on p. 26 will assume the form:

1. *A system of simultaneous rotations with angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$ about axes passing through a point O is equivalent to one rotation with an angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots$ about an axis passing through O.*

2. *A system of simultaneous rotations with angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$ about parallel axes (about axes lying in one plane Π) is equivalent to a rotation about a parallel axis (about an axis lying in the plane Π), when $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots \neq 0$, and to an advancing motion, when $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots = 0$.*

According to the definition of a parameter of a system of vectors (p. 20), the parameter of a system of angular velocities $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots$ is expressed as the scalar product $K = (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \ldots) \cdot \mathrm{Mom}_A(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots)$,

where A is an arbitrary point of the body. Denoting by $\boldsymbol{\omega}$ the sum of the angular velocities, we have by (9)

$$K = \boldsymbol{\omega} \cdot \mathbf{v}, \tag{10}$$

where $\mathbf{v}$ is the resultant velocity of the point A.

From theorem 4, p. 20, it follows that the scalar product $\boldsymbol{\omega} \cdot \mathbf{v}$ has a constant value. In particular, for $K = 0$ the system of simultaneous rotations is equivalent to a rotation or to an advancing motion (cf. table on p. 25).

As we know, every motion of a body is the composition of an advancing motion with a velocity $\mathbf{u}$ of an arbitrary point O of the body and of a rotation with an angular velocity $\boldsymbol{\omega}$ (p. 333). Therefore the instantaneous motion can be represented as the composition of a rotation $\boldsymbol{\omega}$ and of a couple $\boldsymbol{\omega}', -\boldsymbol{\omega}'$ of moment equal to $\mathbf{u}$.

Suppose that the motion of the body has been represented in another way as the composition of the rotation $\boldsymbol{\omega}_1$ and of the couple of rotations $\boldsymbol{\omega}_1', -\boldsymbol{\omega}_1'$. Since the systems $\boldsymbol{\omega}, \boldsymbol{\omega}', -\boldsymbol{\omega}'$, and $\boldsymbol{\omega}_1, \boldsymbol{\omega}_1', -\boldsymbol{\omega}_1'$, are equipollent, because they represent the same resolution of the velocities in the body, their sums are equal, i. e. $\boldsymbol{\omega} = \boldsymbol{\omega}_1$. Thus, we also obtain in this way the theorem (proved on p. 333), according to which the instantaneous axes of rotation are parallel and the instantaneous angular velocities are equal for of all representations of the instantaneous motion.

Let us notice in this connection that the parameter of the system $\boldsymbol{\omega}, \boldsymbol{\omega}', -\boldsymbol{\omega}'$ is $K = \boldsymbol{\omega} \cdot \mathrm{Mom}_O(\boldsymbol{\omega}', -\boldsymbol{\omega}') = \boldsymbol{\omega} \cdot \mathbf{u}$. Therefore, if $\boldsymbol{\omega} \cdot \mathbf{u} = 0$, then the instantaneous motion is an instantaneous rotation.

In particular, if $\boldsymbol{\omega} \perp \mathbf{u}$, and hence *if the instantaneous axis of rotation is perpendicular to the velocity of the advancing motion, then the instantaneous motion is equivalent to an instantaneous rotation.*

On p. 27 we proved that every system of vectors is equivalent to a wrench. From the definition of a wrench it follows that the instantaneous motion of a rigid body is a twist. We have therefore obtained a new proof of the theorem given on p. 334.

Relative motion of a body. Let the instantaneous motions of the two bodies K_1 and K_2 relative to a fixed system of coordinates (x, y, z) be given. We shall determine the instantaneous motion of the body K_2 relative to K_1, i. e. relative to a moving system of coordinates (ξ, η, ζ) attached rigidly to the body K_1.

Let us denote by $\boldsymbol{\omega}_1$ the instantaneous angular velocity vector of the body K_1, and let us represent the advancing motion as the composition of a couple of rotations with angular velocities $\boldsymbol{\omega}_1', -\boldsymbol{\omega}_1'$. Similarly, we shall represent the instantaneous motion of the body K_2 as the composition of the rotation $\boldsymbol{\omega}_2$ and of the couple of rotations $\boldsymbol{\omega}_2', -\boldsymbol{\omega}_2'$.

The absolute velocity $\mathbf{v}_a$ and the velocity of transport $\mathbf{v}_t$ of an arbitrary point A of the body K_2 are:

$$\mathbf{v}_a = \mathrm{Mom}_A(\boldsymbol{\omega}_2, \boldsymbol{\omega}_2', -\boldsymbol{\omega}_2'), \qquad \mathbf{v}_t = \mathrm{Mom}_A(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1', -\boldsymbol{\omega}_1').$$

Since $\mathbf{v}_r = \mathbf{v}_a - \mathbf{v}_t$ ((I), p. 57),

$$\mathbf{v}_r = \mathrm{Mom}_A(\boldsymbol{\omega}_2, \boldsymbol{\omega}_2', -\boldsymbol{\omega}_2', -\boldsymbol{\omega}_1, -\boldsymbol{\omega}_1', \boldsymbol{\omega}_1'). \tag{11}$$

The instantaneous motion of the body K_2 relative to the body K_1 is the composition of six simultaneous rotations. By the theorem on reduction (p. 24) the system of vectors referred to is equipollent to the vector $\boldsymbol{\omega}$ and the couple $\boldsymbol{\omega}', -\boldsymbol{\omega}'$. The vector $\boldsymbol{\omega}$ is the instantaneous angular velocity of the relative motion, and the moment of the couple $\boldsymbol{\omega}', -\boldsymbol{\omega}'$ is equal to the velocity of the instantaneous advancing motion.

Therefore: *the instantaneous relative motion of the body K_2 with respect to the body K_1 is obtained by adding the system of angular velocities with opposite senses, which determine the instantaneous motion of the body K_1, to the system of angular velocities, which determine the instantaneous motion of the body K_2.*

This theorem is usually stated more briefly by saying that the motion of the body K_2 relative to K_1 is obtained by compounding the instantaneous motion of the body K_2 with the instantaneous motion of the body K_1 with an opposite sense.

Example I. A cube makes two simultaneous twists about two skew edges. Let $\mathbf{u}_1, \boldsymbol{\omega}_1$ and $\mathbf{u}_2, \boldsymbol{\omega}_2$ denote the instantaneous angular velocities of the advancing and rotating motions of these twists. We shall determine the resultant twist.

Let us choose a system of coordinates in such a way that one of the edges (about which the twists take place) lies on the x-axis, and the other lies in the yz-plane and is parallel to the y-axis (Fig. 274).

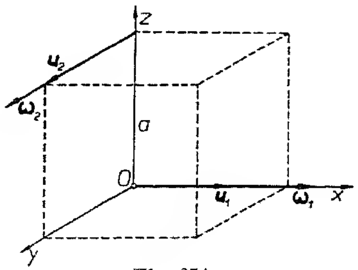

Fig. 274.

The advancing motions can be replaced by the couples of rotations $\boldsymbol{\omega}_1', -\boldsymbol{\omega}_1'$ and $\boldsymbol{\omega}_2', -\boldsymbol{\omega}_2'$ of moments $\mathbf{u}_1$ and $\mathbf{u}_2$. Therefore the instantaneous resultant motion is equivalent to the composition of six simultaneous rotations:

$$\boldsymbol{\omega}_1', -\boldsymbol{\omega}_1', \boldsymbol{\omega}_1, \quad \boldsymbol{\omega}_2', -\boldsymbol{\omega}_2', \boldsymbol{\omega}_2. \tag{12}$$

The sum of the system (12) is equal to $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$; consequently:

$$\omega_x = \omega_1, \quad \omega_y = \omega_2, \quad \omega_z = 0, \tag{13}$$

where ω_1 and ω_2 denote the corresponding projections of the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ on the x and y axes

In order to determine the moment of the rotations (12) with respect to the origin O of the system, let us note that the moments of the couples $\boldsymbol{\omega}_1', -\boldsymbol{\omega}_1'$ and $\boldsymbol{\omega}_2', -\boldsymbol{\omega}_2'$ are equal to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, respectively The moment $\boldsymbol{\omega}_1$ is zero because $\boldsymbol{\omega}_1$ lies on the x-axis. The moment $\boldsymbol{\omega}_2$ is perpendicular to the yz-plane and its projection on the x-axis is $a\omega_2$, where a is the length of an edge of the cube. Therefore, denoting the moment of the rotations (12) with respect to O by $\boldsymbol{u}$, we obtain:

$$u_x = u_1 + a\omega_2, \quad u_y = u_2, \quad u_z = 0, \tag{14}$$

where u_1 denotes the projection of $\boldsymbol{u}_1$ on the x-axis and u_2 the projection of $\boldsymbol{u}_2$ on the y-axis. Equations (14) obviously represent the projections of the velocity of the origin of the system.

The instantaneous motion is therefore the composition of an advancing motion with a velocity $\boldsymbol{u}$ and a rotation with a velocity $\boldsymbol{\omega}$ about an axis passing through O.

According to formula (I), p. 333, the velocity of an arbitrary point $A(x, y, z)$ is $\boldsymbol{v} = \boldsymbol{u} + \overline{OA} \times \boldsymbol{\omega}$, whence by (13) and (14)

$$v_x = u_1 + a\omega_2 - z\omega_2, \; v_y = u_2 + z\omega_1, \; v_z = x\omega_2 - y\omega_1. \tag{15}$$

In order to determine the central axis of twist we must find a point A such that its velocity has the direction of the vector $\boldsymbol{\omega}$, i. e. so that $\boldsymbol{v} = \lambda\boldsymbol{\omega}$ for a suitable numerical value of the factor λ. Hence the equations $v_x = \lambda\omega_1$, $v_y = \lambda\omega_2$, and $v_z = 0$, i. e. $v_x / \omega_1 = v_y / \omega_2$ and $v_z = 0$ must be satisfied. In view of (15) we therefore obtain the following equations of the central axis:

$$(u_1 + a\omega_2 - z\omega_2) / \omega_1 = (u_2 + z\omega_1) / \omega_2, \quad x\omega_2 - y\omega_1 = 0. \tag{16}$$

The velocity of the advancing screw motion is equal to the velocity of an arbitrary point of the central axis, e. g. of the point $A(0, 0, z)$; the value of z is calculated from the first of the equations (16), and then the velocity of the point A from formulae (15). We get:

$$v_x = k\omega_1, \quad v_y = k\omega_2, \quad v_z = 0,$$

where

$$k = [(u_1 + a\omega_2)\,\omega_1 + u_2\omega_2] / (\omega_1^2 + \omega_2^2) = \boldsymbol{u} \cdot \boldsymbol{\omega} / |\boldsymbol{\omega}|^2.$$

***Example* 2.** Steady precession. If a body moves in such a way that the instantaneous motion at each instant is the composition of two simultaneous rotations about two intersecting axes, of which the first l is fixed in space and the other m has a fixed position in the body, while the angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ of these rotations are constant in magnitude, then the motion of the body is called *steady precession*.

Since by hypothesis the axes l and m intersect, the instantaneous motion of the body is a rotation with an angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ about an axis p passing through the point of intersection of l and m (Fig. 275). Let us note that the instantaneous motion of the axis m is an instantaneous rotation about the axis l with an angular velocity $\boldsymbol{\omega}_1$ (for the rotation of the axis m about itself is left unconsidered). Consequently the axis m rotates about the axis l with a constant angular velocity ω_1. The point of intersection O of both axes is therefore fixed and the angle between the axes l and m is constant. It follows from this that the vector $\boldsymbol{\omega}$, and hence also the instantaneous axis of rotation p, make constant angles with the axes l and m.

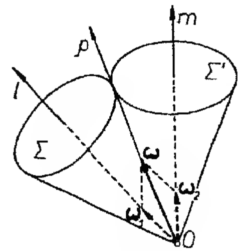

Fig. 275.

The axis p describes a cone of revolution Σ in space. The trace of the axis p in the body is also a cone of revolution Σ'.

Therefore: *the cones of instantaneous axes of rotation, fixed and moving, are cones of revolution with axes l and m.*

The earth's axis does not maintain a fixed direction in space, but describes a cone of revolution about an axis perpendicular to the ecliptic and passing through the centre of the earth. The time for a complete circuit of the earth's axis lasts about 26 000 years, and the angle between the earth's axis and the axis perpendicular to the ecliptic is $23\frac{1}{2}°$.

Let us take the centre of the earth as the origin of a system of coordinates moving with an advancing motion, and let us give the z-axis a direction perpendicular to the ecliptic. The x and y axes will then lie in the ecliptic. With respect to this system of coordinates the earth executes a steady precessional motion. The axis fixed in space is the z-axis and the axis fixed in the body is the earth's axis.

***Example* 3.** Two circles K_1 and K_2, lying in the plane Π, rotate about their centres with angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. Determine the instantaneous relative motion of the circle K_2 with respect to K_1 (Fig. 276).

Let $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ denote the angular velocity vectors, obviously per-

pendicular to Π (Fig. 277). The instantaneous motion of the circle K_2 relative to K_1 is the composition of two simultaneous rotations $\boldsymbol{\omega}_2$ and $-\boldsymbol{\omega}_1$.

If $\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 = 0$, then the instantaneous relative motion of the circle K_2 is an advancing motion with a velocity $\boldsymbol{u}$ equal to the moment of the couple $(\omega_2, -\omega_1)$. Denoting by d the distance between the centres, we have $|\boldsymbol{u}| = d\omega_1 = d\omega_2$. The velocity $\boldsymbol{u}$ is perpendicular to the line O_1O_2 joining the centres of the circles K_1 and K_2.

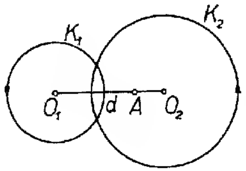

Fig. 276.

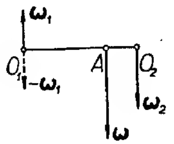

Fig. 277.

If $\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \neq 0$, then the vectors $\boldsymbol{\omega}_2$ and $-\boldsymbol{\omega}_1$ have a resultant $\boldsymbol{\omega} = \boldsymbol{\omega}_2 - \boldsymbol{\omega}_1$ whose origin is at A lying on the line O_1O_2 at the point with respect to which the moment of the system $\boldsymbol{\omega}_2, -\boldsymbol{\omega}_1$ is zero. The instantaneous relative motion of the circle K_2 is therefore an instantaneous rotation about the point A with an angular velocity $\boldsymbol{\omega}$.

If $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ have opposite senses (as in Fig. 277), then denoting by ω_1, ω_2, and ω, the absolute values of the angular velocities, we obtain:

$$\omega = \omega_1 + \omega_2, \quad O_1A = O_1O_2 \cdot \omega_2 / (\omega_1 + \omega_2).$$

§ 10. Analytic representation of the motion of a rigid body. Instantaneous angular velocity. Let us suppose that we are considering the motion of a rigid body relative to the system of coordinates (x, y, z). Let us choose a system of coordinates (ξ, η, ζ) with origin at M and attached rigidly to the body. The position of the body relative to the system (x, y, z) will be determined if the position of the system (ξ, η, ζ) is given, i. e. the coordinates x_0, y_0, z_0, of the point M and the angles $\alpha_1, \alpha_2, \alpha_3$, $\beta_1, \beta_2, \beta_3$, and $\gamma_1, \gamma_2, \gamma_3$, which the axes ξ, η, ζ make with the axes x, y, z.

Let A be an arbitrary point of the body. Let us denote its coordinates with respect to the moving system by ξ, η, ζ and with respect to the fixed system by x, y, z.

Knowing the coordinates ξ, η, ζ and the position of the moving frame, we can determine the coordinates x, y, z by means of the formulae (II), p. 54. If $\cos\alpha_i$, $\cos\beta_i$, and $\cos\gamma_i$, are denoted by a_i, b_i, and c_i (where $i = 1, 2, 3$), then these formulae will assume the form:

$$x = x_0 + a_1\xi + a_2\eta + a_3\zeta, \quad y = y_0 + b_1\xi + b_2\eta + b_3\zeta,$$
$$z = z_0 + c_1\xi + c_2\eta + c_3\zeta. \tag{1}$$

Let $\boldsymbol{v}$ be the velocity of the point A relative to the system (x, y, z). Since by hypothesis the system (ξ, η, ζ) is attached rigidly to the body, the coordinates ξ, η, ζ of the point A are constant (independent of the time). Differentiating (1) (and remembering that $a_1, a_2, \ldots, c_3$ are functions of the time t), we obtain:

$$\begin{aligned} v_x &= x^{\cdot} = x_0^{\cdot} + a_1^{\cdot}\xi + a_2^{\cdot}\eta + a_3^{\cdot}\zeta, \\ v_y &= y^{\cdot} = y_0^{\cdot} + b_1^{\cdot}\xi + b_2^{\cdot}\eta + b_3^{\cdot}\zeta, \\ v_z &= z^{\cdot} = z_0^{\cdot} + c_1^{\cdot}\xi + c_2^{\cdot}\eta + c_3^{\cdot}\zeta. \end{aligned} \tag{2}$$

From analytic geometry it is known that:

$$a_1^2 + a_2^2 + a_3^2 = 1, \quad b_1^2 + b_2^2 + b_3^2 = 1, \quad c_1^2 + c_2^2 + c_3^2 = 1, \tag{3}$$

$$a_1a_2 + b_1b_2 + c_1c_2 = 0, \quad a_1a_3 + b_1b_3 + c_1c_3 = 0,$$
$$a_2a_3 + b_2b_3 + c_2c_3 = 0. \tag{4}$$

Differentiating equations (3) and (4), we obtain:

$$a_1a_1^{\cdot} + a_2a_2^{\cdot} + a_3a_3^{\cdot} = 0, \quad b_1b_1^{\cdot} + b_2b_2^{\cdot} + b_3b_3^{\cdot} = 0,$$
$$c_1c_1^{\cdot} + c_2c_2^{\cdot} + c_3c_3^{\cdot} = 0, \tag{5}$$

$$\begin{aligned} a_1a_2^{\cdot} + b_1b_2^{\cdot} + c_1c_2^{\cdot} &= -a_1^{\cdot}a_2 - b_1^{\cdot}b_2 - c_1^{\cdot}c_2, \\ a_1a_3^{\cdot} + b_1b_3^{\cdot} + c_1c_3^{\cdot} &= -a_1^{\cdot}a_3 - b_1^{\cdot}b_3 - c_1^{\cdot}c_3, \\ a_2a_3^{\cdot} + b_2b_3^{\cdot} + c_2c_3^{\cdot} &= -a_2^{\cdot}a_3 - b_2^{\cdot}b_3 - c_2^{\cdot}c_3. \end{aligned} \tag{6}$$

Let $\boldsymbol{\omega}$ denote a vector whose projections on the axes of the system (ξ, η, ζ) are expressed by the formulae:

$$\omega_\xi = a_2a_3^{\cdot} + b_2b_3^{\cdot} + c_2c_3^{\cdot}, \quad \omega_\eta = a_3a_1^{\cdot} + b_3b_1^{\cdot} + c_3c_1^{\cdot},$$
$$\omega_\zeta = a_1a_2^{\cdot} + b_1b_2^{\cdot} + c_1c_2^{\cdot}. \tag{7}$$

Let us form the projections v_ξ, v_η, v_ζ of the velocity $\boldsymbol{v}$ on the axes ξ, η, ζ; we get $v_\xi = a_1v_x + b_1v_y + c_1v_z$, whence by substituting the values of v_x, v_y, v_z, from formulae (2):

$$v_\xi = (a_1x_0^{\cdot} + b_1y_0^{\cdot} + c_1z_0^{\cdot}) + (a_1a_1^{\cdot} + b_1b_1^{\cdot} + c_1c_1^{\cdot})\,\xi +$$
$$+ (a_1a_2^{\cdot} + b_1b_2^{\cdot} + c_1c_2^{\cdot})\,\eta + (a_1a_3^{\cdot} + b_1b_3^{\cdot} + c_1c_3^{\cdot})\,\zeta.$$

The coefficient of ξ is equal to zero by (5). The coefficients of η and ζ are by (6) and (7) equal to ω_ζ and $-\omega_\eta$, respectively. Consequently

$$v_\xi = (a_1x_0^{\cdot} + b_1y_0^{\cdot} + c_1z_0^{\cdot}) + \omega_\zeta\eta - \omega_\eta\zeta. \tag{8}$$

For the projections of the velocity $\boldsymbol{u}$ of the point M on the axes

x, y, z we get $u_x = x_0^{\cdot}$, $u_y = y_0^{\cdot}$, $u_z = z_0^{\cdot}$, and for the projections of this velocity on the axes ξ, η, ζ

$$u_\xi = a_1 u_x + b_1 u_y + c_1 u_z = a_1 x_0^{\cdot} + b_1 y_0^{\cdot} + c_1 z_0^{\cdot}, \qquad \text{etc.}$$

In virtue of (8), therefore, we obtain for v_ξ (and similarly for v_η, v_ζ) the formulae:

$$v_\xi = u_\xi + \omega_\zeta \eta - \omega_\eta \zeta, \quad v_\eta = u_\eta + \omega_\xi \zeta - \omega_\zeta \xi, \qquad (9)$$
$$v_\zeta = u_\zeta + \omega_\eta \xi - \omega_\xi \eta.$$

From formulae (9) it follows that the velocity $\mathbf{v}$ is the sum of two velocities: $\mathbf{v} = \mathbf{u} + \mathbf{w}$, of which the first is the velocity of the point M, and the other has the projections on the axes ξ, η, ζ:

$$w_\xi = \omega_\zeta \eta - \omega_\eta \zeta, \; w_\eta = \omega_\xi \zeta - \omega_\zeta \xi, \; w_\zeta = \omega_\eta \xi - \omega_\xi \eta. \qquad (10)$$

Comparing these formulae with formulae (V), p. 46, we see that $\mathbf{w}$ is the velocity the point A would have if the body were rotating with an angular velocity $\boldsymbol{\omega}$ about an axis passing through M. It follows from this that the vector $\boldsymbol{\omega}$ defined by formulae (7) is the instantaneous angular velocity vector.

Remark. Formulae (7) become simpler if we assume that the co-ordinate systems (x, y, z) and (ξ, η, ζ) coincide at a given moment t. Under this assumption we have $\alpha_1 = \beta_2 = \gamma_3 = 0$, and the remaining angles are equal to $\frac{1}{2}\pi$. Hence $a_1 = b_2 = c_3 = 1$, and the remaining cosines are zero. Then by (7) and (6):

$$\omega_\xi = -c_2^{\cdot}, \quad \omega_\eta = -a_3^{\cdot}, \quad \omega_\zeta = -b_1^{\cdot}. \qquad (11)$$

Since $c_2 = \cos \gamma_2$, $c_2^{\cdot} = -(\sin \gamma_2)\, \gamma_2^{\cdot} = -\gamma_2^{\cdot}$; consequently $\omega_\xi = \gamma_2^{\cdot}$. Proceeding similarly, we obtain

$$\omega_\xi = \omega_x = \gamma_2^{\cdot}, \quad \omega_\eta = \omega_y = \alpha_3^{\cdot}, \quad \omega_y = \omega_z = \beta_1^{\cdot}.$$

Therefore: *if the axes of a moving system of coordinates coincide at the instant t with the axes of a fixed coordinate system, then the projections of the instantaneous angular velocity vector on the axes of the moving system are the derivatives of the angles* $\sphericalangle\, \eta z$, $\sphericalangle\, \zeta x$, *and* $\sphericalangle\, \xi y$.

Central axis. In order to obtain the central axis it is necessary to determine the points whose velocities have the direction of the vector $\boldsymbol{\omega}$. Therefore, if the point $A(\xi, \eta, \zeta)$ lies on the central axis, then its velocity is equal to $\mathbf{v} = \lambda \boldsymbol{\omega}$, where λ is a certain constant. By substituting in equations (9), we consequently obtain the equations of the central axis:

$$\lambda \omega_\xi = u_\xi + \omega_\zeta \eta - \omega_\eta \zeta, \quad \lambda \omega_\eta = u_\eta + \omega_\xi \zeta - \omega_\zeta \xi, \qquad (12)$$
$$\lambda \omega_\zeta = u_\zeta + \omega_\eta \xi - \omega_\xi \eta,$$

whence

$$\frac{u_\xi + \omega_\zeta \eta - \omega_\eta \zeta}{\omega_\xi} = \frac{u_\eta + \omega_\xi \zeta - \omega_\zeta \xi}{\omega_\eta} = \frac{u_\zeta + \omega_\eta \xi - \omega_\xi \eta}{\omega_\zeta}. \qquad (13)$$

In order to obtain the velocity $\mathbf{v}$ of the instantaneous advancing motion during a twist, let us multiply both sides of equations (12) by $\omega_\xi, \omega_\eta, \omega_\zeta$, respectively. Adding them, we obtain

$$\lambda(\omega_\xi^2 + \omega_\eta^2 + \omega_\zeta^2) = u_\xi \omega_\xi + u_\eta \omega_\eta + u_\zeta \omega_\zeta = \mathbf{u} \cdot \boldsymbol{\omega}. \qquad (14)$$

Since $\mathbf{v} = \lambda \boldsymbol{\omega}$, putting $\omega = |\boldsymbol{\omega}| = \sqrt{\omega_\xi^2 + \omega_\eta^2 + \omega_\zeta^2}$, we get by (14)

$$\mathbf{v} = \frac{\mathbf{u} \cdot \boldsymbol{\omega}}{\omega^2}\, \boldsymbol{\omega},$$

whence

$$|\mathbf{v}| = |\mathbf{u} \cdot \boldsymbol{\omega}| \,/\, \omega. \qquad (15)$$

If $\mathbf{u} \perp \boldsymbol{\omega}$, then $\mathbf{u} \cdot \boldsymbol{\omega} = 0$, whence by (15) $\mathbf{v} = 0$.

Therefore: *if the velocity of an instantaneous advancing motion is perpendicular to the instantaneous axis of rotation, then the instantaneous motion is an instantaneous rotation about the central axis.*

Plane motion. Let us suppose that we are considering the motion of a figure in the plane Π. Let us select a fixed coordinate system (x, y) as well as a moving system (ξ, η) with origin at M and rigidly attached to the figure. Denote by x_0, y_0 the coordinates of the point M and by φ the angle between the axes ξ and x. Finally, let A be an arbitrary point of the figure having the coordinates x, y with respect to the fixed frame, and ξ, η with respect to the moving frame. The relations among these co-ordinates are given by formulae (II), p. 54:

$$x = x_0 + \xi \cos \varphi - \eta \sin \varphi, \quad y = y_0 + \xi \sin \varphi + \eta \cos \varphi. \qquad (16)$$

Let $\mathbf{v}$ be the velocity of the point A. Since the point A is attached rigidly to the moving system, ξ and η are constants. Differentiating (16), we obtain

$$v_x = x^{\cdot} = x_0^{\cdot} - (\xi \sin \varphi + \eta \cos \varphi)\, \varphi^{\cdot}, \qquad (17)$$
$$v_y = y^{\cdot} = y_0^{\cdot} + (\xi \cos \varphi - \eta \sin \varphi)\, \varphi^{\cdot}.$$

Comparing (17) with formulae (16) and putting

$$\omega = \varphi^{\cdot}, \qquad (18)$$

we get:

$$v_x = x_0^{\cdot} - (y - y_0)\, \omega, \quad v_y = y_0^{\cdot} + (x - x_0)\, \omega. \qquad (19)$$

Therefore: *the velocity* $\mathbf{v}$ *is the sum of two velocities:* $\mathbf{v} = \mathbf{u} + \mathbf{w}$, *of*

which the first is the velocity of the point M, and the other has the projections:

$$w_x = -(y - y_0)\,\omega, \quad w_y = (x - x_0)\,\omega \tag{20}$$

on the axes of the fixed system.

We see from this that $\mathbf{w}$ is the velocity the point A would have if the figure were rotating about the point $M(x_0, y_0)$ with an angular velocity ω (where the positive sense of the rotation agrees with the positive sense of the angle). Consequently $\omega = \varphi^{\cdot}$ is the instantaneous angular velocity.

In order to obtain the instantaneous centre of rotation it is necessary to determine the point whose velocity $\mathbf{v} = 0$. Denoting the coordinates of the instantaneous centre of rotation by x' and y', we obtain from (4):

$$0 = x_0^{\cdot} - (y' - y_0)\,\omega, \quad 0 = y_0^{\cdot} + (x' - x_0)\,\omega,$$

whence

$$x' = x_0 - y_0^{\cdot} / \omega, \quad y' = y_0 + x_0^{\cdot} / \omega. \tag{21}$$

Euler's angles. In some considerations it is convenient to define the position of the body by means of the so-called Euler's angles.

Let a body rotate about the point O. Let us choose two systems of coordinates with a common origin O: a fixed (x, y, z) and a moving (ξ, η, ζ) attached rigidly to the body (Fig. 278). The position of the moving system is determined as follows.

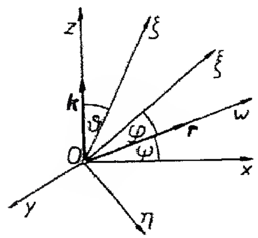

Fig. 278.

Let w denote the line of intersection of the planes xy and $\xi\eta$. This line is called the *line of nodes*.

The line w is perpendicular to the axes z and ζ. Let us give the line w such a sense that the system of axes (ζ, z, w) is left-handed, i. e. agrees with the assumed senses of the systems (x, y, z) and (ξ, η, ζ).

Let us denote by ϑ the angle through which it is necessary to rotate the z-axis about the w-axis in the positive direction (i. e. from right to left), in order that the positive direction of the z-axis coincides with the positive direction of the ζ-axis. Similarly, we denote by φ the angle through which it is necessary to rotate the w-axis about the ζ-axis in the positive direction, in order that the positive direction of the w-axis coincides with the positive direction of the ξ-axis. Finally, we denote by ψ the angle through which it is necessary to rotate the x-axis about the z-axis in the positive direction, in order that the positive directions of the x and w-axes coincide.

The angles ϑ, φ, ψ, are called *Euler's angles*.

These angles define the positions of the axes ξ, η, ζ, with the exception of the case when $\vartheta = 0$ or $\vartheta = \pi$, for then the position of the w-axis, and hence also the angles φ, ψ, are undefined.

However, if $\vartheta \neq 0$ and $\vartheta \neq \pi$, then the angle ψ defines the position of the w-axis in the xy-plane. Knowing the position of the w-axis already, we obtain the position of the ζ-axis by rotating the z-axis through the angle ϑ (in the positive direction) about the w-axis. Finally, we obtain the ξ-axis by rotating the w-axis about the ζ-axis in the positive direction through the angle φ.

Euler's angles vary between the following limits:

$$0 < \vartheta < \pi, \quad 0 \leqq \varphi \leqq 2\pi, \quad 0 \leqq \psi \leqq 2\pi.$$

The instantaneous motion of the system (ξ, η, ζ) is a rotation about a certain axis. Let $\boldsymbol{\omega}$ be its instantaneous angular velocity vector. Let us resolve $\boldsymbol{\omega}$ into three component vectors $\mathbf{o}_z$, $\mathbf{o}_w$, and $\mathbf{o}_\zeta$, in the direction of the axes z, w, and ζ. Consequently

$$\boldsymbol{\omega} = \mathbf{o}_z + \mathbf{o}_w + \mathbf{o}_\zeta. \tag{21}$$

Let us denote by o_z, o_w, and o_ζ, the coordinates of the corresponding vectors with respect to the axes z, w, and ζ. Let us note that if the system (ξ, η, ζ) rotates about the z-axis, then the angles ϑ and φ do not vary and the angle ψ is the angle of rotation. Therefore, during a rotation about the z-axis the magnitude of the instantaneous angular velocity of the system (ξ, η, ζ) is $\psi^{\cdot}$. Since $\mathbf{o}_z$ is the angular velocity vector for a rotation about the z-axis,

$$o_z = \psi^{\cdot} \quad \text{and similarly} \quad o_w = \vartheta^{\cdot}, \; o_\zeta = \varphi^{\cdot}. \tag{22}$$

Consequently the derivatives $\psi^{\cdot}, \vartheta^{\cdot}, \varphi^{\cdot}$ define the instantaneous angular velocity vector $\boldsymbol{\omega}$ if we know the position of the system (ξ, η, ζ), i. e. the angles ψ, ϑ, and φ.

We shall now derive formulae for the projections of the vector on the axes of the system (ξ, η, ζ).

Let us choose the unit vectors $\boldsymbol{k}$ and $\boldsymbol{r}$ on the axes z and w. Consequently $\mathbf{o}_z = o_z\boldsymbol{k}$ and $\mathbf{o}_w = o_w\boldsymbol{r}$; hence according to (22):

$$\mathbf{o}_z = \psi^{\cdot}\boldsymbol{k}, \quad \mathbf{o}_w = \vartheta^{\cdot}\boldsymbol{r}. \tag{23}$$

The projection of the vector $\boldsymbol{k}$ on the ζ-axis is $\cos\vartheta$. The vector $\boldsymbol{k}$ makes an angle $\frac{1}{2}\pi - \vartheta$ with the $\xi\eta$-plane; therefore the projection of $\boldsymbol{k}$

on the $\xi\eta$-plane has the length $\sin\vartheta$ and is perpendicular to w (since the z-axis is perpendicular to w). The projection of **k** makes angles $\frac{1}{2}\pi - \varphi$ and $\pi - \vartheta$ with the axes ξ and η; hence:

$$k_\xi = \sin\vartheta \sin\varphi, \quad k_\eta = -\sin\vartheta\cos\varphi, \quad k_\zeta = \cos\vartheta. \tag{24}$$

The projections of the vector **r** on the axes ξ and η are $\cos\varphi$ and $\sin\varphi$, while the projection on the ζ-axis is zero. Consequently:

$$r_\xi = \cos\varphi, \quad r_\eta = \sin\varphi, \quad r_\zeta = 0. \tag{25}$$

By (23) and (24) the projections of the vector $\mathbf{o}_z$ on the axes ξ, η, and ζ, are: $\psi^\cdot \sin\vartheta\sin\varphi$, $-\psi^\cdot \sin\vartheta\cos\varphi$, and $\psi^\cdot\cos\vartheta$, while the projections of the vector $\mathbf{o}_w$ on the axes ξ, η, and ζ, are by (23) and (25) equal to $\vartheta^\cdot\cos\varphi$, $\vartheta^\cdot\sin\varphi$, and 0, respectively; finally, the projections of the vector **o** on the axes ξ, η, and ζ, are obviously 0, 0 and $\varphi^\cdot$. From this we obtain in virtue of (21):

$$\omega_\xi = \vartheta^\cdot\cos\varphi + \psi^\cdot\sin\vartheta\sin\varphi, \quad \omega_\eta = \vartheta^\cdot\sin\varphi - \psi^\cdot\sin\vartheta\cos\varphi, \quad \omega_\zeta = \psi^\cdot\cos\vartheta + \varphi^\cdot. \tag{I}$$

Determining $\vartheta^\cdot$, $\psi^\cdot$, and $\varphi^\cdot$, from (I), we obtain:

$$\vartheta^\cdot = \omega_\xi\cos\varphi + \omega_\eta\sin\varphi, \quad \psi^\cdot = (\omega_\xi\sin\varphi - \omega_\eta\cos\varphi)\,/\,\sin\vartheta, \quad \varphi^\cdot = \omega_\zeta - (\omega_\xi\sin\varphi - \omega_\eta\cos\varphi)\cot\vartheta. \tag{II}$$

Knowing the projections ω_ξ, ω_η, and ω_ζ, of the angular velocity $\boldsymbol{\omega}$ on the axes ξ, η, and ζ, of the moving system at each instant, we can therefore determine ϑ, φ, and ψ, as a function of time by solving the system of differential equations (II).

Proceeding similarly, we obtain the following formulae for the projections of the angular velocity $\boldsymbol{\omega}$ on the axes x, y, z, of the fixed system:

$$\omega_x = \varphi^\cdot\sin\vartheta\sin\psi + \vartheta^\cdot\cos\psi, \quad \omega_y = \varphi^\cdot\sin\vartheta\cos\psi - \vartheta^\cdot\sin\psi, \quad \omega_z = \psi^\cdot + \varphi^\cdot\cos\vartheta, \tag{I'}$$

$$\vartheta^\cdot = \omega_x\cos\psi - \omega_y\sin\psi, \quad \varphi^\cdot = (\omega_x\sin\psi + \omega_y\cos\psi)\,/\,\sin\vartheta, \quad \psi^\cdot = \omega_z - (\omega_x\sin\psi + \omega_y\cos\psi)\cot\vartheta. \tag{II'}$$

Euler's angles in a steady precession. Let us assume that during the motion of a body we constantly have:

$$\vartheta^\cdot = 0, \quad \psi^\cdot = \text{const}, \quad \varphi^\cdot = \text{const}. \tag{26}$$

The instantaneous motion of the body is at each instant, therefore, the composition of two simultaneous rotations about the axes z and ζ with angular velocities $\psi^\cdot$ and $\varphi^\cdot$ of constant magnitudes. The z-axis has a fixed position in space, and the ζ-axis is rigidly attached to the body.

Under these conditions the motion of the body is a steady precession (cf. p. 349).

§ 11. Resolution of accelerations. Plane motion. If a plane figure rotates about a fixed point M with a variable angular velocity ω, then each point of the figure moves along the periphery of a circle. The acceleration $\mathbf{p}$ of an arbitrary point A of the figure is therefore the sum of the normal acceleration $\mathbf{p}_n$ directed towards M, and the tangential acceleration $\mathbf{p}_t$ perpendicular to MA. The normal and tangential accelerations are defined by the formulae (I) and (II), p. 45:

$$p_n = r\omega^2, \quad p_t = r\varepsilon, \tag{1}$$

where $r = MA$, and $\varepsilon = \omega^\cdot$ is the angular acceleration.

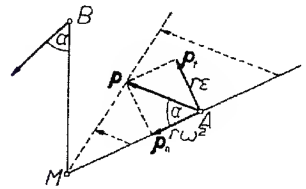

Fig. 279.

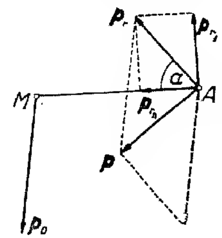

Fig. 280.

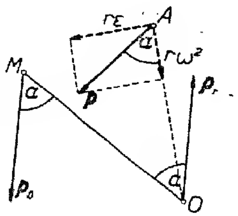

Fig. 281.

Let α denote the angle which the acceleration $\mathbf{p}$ of the point A makes with the line MA (Fig. 279). Then

$$\tan\alpha = p_t\,/\,p_n = \varepsilon\,/\,\omega^2. \tag{2}$$

We see from this that α is the same for all points.

Therefore: *in a rotation of a plane figure about a fixed point the accelerations of the points of the figure are proportional to their distances from the centre of rotation and make equal angles with the line joining these points with the centre of rotation.*

Let us assume now that the figure moves in the plane entirely arbitrarily (Fig. 280). Let us take an arbitrary point M of the figure as the origin of a system of coordinates (ξ, η), which moves with an advancing motion. The acceleration $\mathbf{p}$ of an arbitrary point A of the figure will be the sum of the accelerations: relative $\mathbf{p}_r$ and transport $\mathbf{p}_t$. The acceleration of transport is equal to the acceleration p_0 of the point M. The relative motion of the figure is a rotation about M with a variable angular velocity ω. We can therefore resolve the relative acceleration $\mathbf{p}_r$ into the sum of the (relative) accelerations: normal $\mathbf{p}_{r_n}$ and tangential $\mathbf{p}_{r_t}$. Consequently

$$\mathbf{p} = \mathbf{p}_0 + \mathbf{p}_r = \mathbf{p}_0 + \mathbf{p}_{r_n} + \mathbf{p}_{r_t}. \tag{3}$$

Therefore: *in a plane motion the accelerations of the points of a figure are the sums of the acceleration of an arbitrary point M of the figure and of the accelerations which these points would have during a rotation of the figure about the point M* (as a fixed point) *with an angular velocity* (of the instantaneous rotation of the figure) ω *and with an angular acceleration* $\varepsilon = \omega^{\cdot}$.

If $\omega \neq 0$ or $\varepsilon \neq 0$, the relative acceleration $\mathbf{p}_r$ makes with the line MA an angle α defined by formula (2) and this angle is constant for all points of the figure.

In this case let us pass through the point M a line l making an angle α with the direction of the acceleration $\mathbf{p}_0$ of the point M (Fig. 281). The relative accelerations of the points lying on the line l will have the direction of the acceleration $\mathbf{p}_0$. Since the relative accelerations are proportional to the distances from M, we shall find on l a point O whose relative acceleration will be equal to $-\mathbf{p}_0$. The acceleration of the point O will therefore be zero.

The point O is called the *centre of instantaneous accelerations*.

On the other hand, if $\omega = 0$ and $\varepsilon = 0$, then the accelerations of all the points of the figure are equal (namely, equal to the acceleration of the point M).

Therefore: *if the accelerations of the points of a figure in plane motion are not equal, then there exists a point whose acceleration is equal to zero.*

The accelerations of the points are hence such as if the figure were rotating about the instantaneous centre of accelerations (as a fixed point) with an angular velocity ω and with an angular acceleration $\varepsilon = \omega^{\cdot}$.

The accelerations of the points of the figure are proportional to the distances from the centre of instantaneous accelerations and make equal angles with the lines joining these points with the centre of instantaneous accelerations.

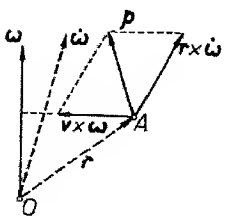

Fig. 282.

Motion in space. If a body rotates about a fixed point O with an instantaneous angular velocity $\boldsymbol{\omega}$ (Fig. 282), then the velocity of an arbitrary point A of the body is

$$\mathbf{v} = \mathbf{r} \times \boldsymbol{\omega}, \tag{4}$$

where $\mathbf{r} = \overline{OA}$. Calculating the derivative and denoting the acceleration of the point A by $\mathbf{p}$, we obtain

$$\mathbf{p} = \mathbf{r}^{\cdot} \times \boldsymbol{\omega} + \mathbf{r} \times \boldsymbol{\omega}^{\cdot}.$$

Since O is by hypothesis a fixed point, $\mathbf{r}^{\cdot} = \mathbf{v}$; consequently

$$\mathbf{p} = \mathbf{v} \times \boldsymbol{\omega} + \mathbf{r} \times \boldsymbol{\omega}^{\cdot}. \tag{5}$$

If a body rotates about a fixed axis with a constant angular velocity ω, then $\boldsymbol{\omega}^{\cdot} = 0$, whence by (5) $\mathbf{p} = \mathbf{v} \times \boldsymbol{\omega}$. On the other hand, each point then has a centripetal acceleration $\varrho|\boldsymbol{\omega}|^2$, where ϱ denotes the distance of the point from the axis of rotation. The product $\mathbf{v} \times \boldsymbol{\omega}$ is therefore the acceleration with which the points of the body would move if the body were rotating about a fixed axis with a constant angular velocity $\boldsymbol{\omega}$. The product $\mathbf{r} \times \boldsymbol{\omega}^{\cdot} = \mathrm{Mom}_A \boldsymbol{\omega}^{\cdot}$ represents the velocity which the point A would have if the body were rotating about a fixed axis with an angular velocity $\boldsymbol{\omega}^{\cdot}$. In general, the derivative $\boldsymbol{\omega}^{\cdot}$ has a direction different from $\boldsymbol{\omega}$. If $\boldsymbol{\omega}$ has a fixed direction (i. e. if the axis of rotation is fixed), then $\boldsymbol{\omega}^{\cdot}$ has the direction of $\boldsymbol{\omega}$; consequently $\mathbf{r} \times \boldsymbol{\omega}^{\cdot}$ has the direction of the velocity $\mathbf{v}$. In this case $\mathbf{r} \times \boldsymbol{\omega}^{\cdot}$ is the tangential acceleration and $\mathbf{v} \times \boldsymbol{\omega}$ the normal acceleration.

Let us now assume that the body moves arbitrarily in space. Then the resolution of the accelerations is obtained by taking an arbitrary point O of the body as the origin of the system of coordinates (ξ, η, ζ) moving with an advancing motion. The accelerations of the points will be the sums of the acceleration of transport (i. e. of the acceleration of the point O) and of the relative acceleration. The relative motion will be a rotation about the point O. Therefore the relative acceleration of an arbitrary point A is expressed according to (5) by the formula

$$\mathbf{p}_r = \mathbf{v}_r \times \boldsymbol{\omega} + \mathbf{r} \times \boldsymbol{\omega}^{\cdot}, \tag{6}$$

where $\mathbf{r} = \overline{OA}$, and $\mathbf{v}_r$ denotes the relative velocity of the point A.

The interpretation of the products $\mathbf{v}_r \times \boldsymbol{\omega}$ and $\mathbf{r} \times \boldsymbol{\omega}^{\cdot}$ is similar to that used in the case of the rotation of a body about a fixed point. Denoting the velocity of the point A by $\mathbf{v}$, the acceleration and the velocity of the point O by $\mathbf{p}_0$ and $\mathbf{v}_0$, respectively, we obtain $\mathbf{v}_r = \mathbf{v} - \mathbf{v}_0$, and hence by (6) the acceleration of the point A is

$$\mathbf{p} = \mathbf{p}_0 + (\mathbf{v} - \mathbf{v}_0) \times \boldsymbol{\omega} + \mathbf{r} \times \boldsymbol{\omega}^{\cdot}. \tag{7}$$

CHAPTER VIII

DYNAMICS OF A RIGID BODY

§ 1. Work and kinetic energy. In dynamics, as in statics, we shall also frequently assume that a rigid body is a rigid system of material points. As a result we shall be able to apply to a rigid body the theorems from dynamics concerning a system of material points.

Dynamical magnitudes. Dynamical magnitudes such as momentum, kinetic energy, angular momentum, etc., which we have met in connection with systems of material points, are also defined for rigid bodies by passing to the limit as was done in defining centres of gravity, statical moments and moments of inertia, of systems of material points. (*vide* chapt. IV, p. 169). For instance, in order to define the momentum of a body, we divide the body into parallelepipeds of volumes $\Delta\tau_1, \Delta\tau_2, \ldots$, and in each one of them we next consider, one at a time, the points $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$, ... Let $\varrho_1, \varrho_2, \ldots$ denote the densities of the body at the chosen points, and $\mathbf{v}_1, \mathbf{v}_2, \ldots$ the velocities of these points. The masses of the parallelepipeds are approximately $m_1 = \varrho_1 \Delta\tau_1$, $m_2 = \varrho_2 \Delta\tau_2, \ldots$ If the body is replaced by a system of material points of masses $m_1, m_2, \ldots$ placed at $A_1, A_2, \ldots$ and having velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots$, then the momentum of this system will be equal to

$$\mathbf{H} = \Sigma m_i \mathbf{v}_i = \Sigma \varrho_i \mathbf{v}_i \Delta\tau_i. \tag{1}$$

The limit of this sum as the dimensions of the parallelepipeds tend to zero is called the *momentum of the body*.

Denoting by $\varrho(x, y, z)$ the density, by $\mathbf{v}(x, y, z)$ the velocity of a point whose coordinates are x, y, z, and by $\mathbf{H}$ the momentum of the body, we get:

$$H_x = \iiint_D \varrho v_x \, d\tau, \quad H_y = \iiint_D \varrho v_y \, d\tau, \quad H_z = \iiint_D \varrho v_z \, d\tau, \tag{2}$$

where D denotes the region of space occupied by the body. We write the preceding formulae in vector form

$$\mathbf{H} = \iiint_D \varrho \mathbf{v} \, d\tau, \tag{3}$$

Proceeding similarly, we obtain the formula

$$E = \tfrac{1}{2}\iiint_D \varrho v^2 \, d\tau \tag{4}$$

for the kinetic energy, where $v = |\mathbf{v}|$.

The angular momentum ((I′), p. 199) with respect to the origin of the coordinate system has the projections:

$$K_x = \iiint_D \varrho(v_y z - v_z y) \, d\tau, \quad K_y = \iiint_D \varrho(v_z x - v_x z) \, d\tau, \tag{5}$$
$$K_z = \iiint_D \varrho(v_x y - v_y x) \, d\tau.$$

Denoting by $\mathbf{r}$ the radius vector $\overline{OA}$, where A has the coordinates x, y, z, we can write formulae (5) in the vector form:

$$\mathbf{K} = \iiint_D \varrho(\mathbf{v} \times \mathbf{r}) \, d\tau. \tag{6}$$

Work. Let a force $\mathbf{P}$, whose origin is at the point A, act on a rigid body, and let $\mathbf{v}$ denote the velocity of this point.

The work of the force $\mathbf{P}$ in the time from t' to t'' is expressed by the formula (p. 95)

$$L = \int_{t'}^{t''} (\mathbf{P}\mathbf{v}) \, dt. \tag{7}$$

Let us consider an arbitrary point O in the body. The instantaneous motion of the body can be considered as the composition of an advancing motion with a velocity $\mathbf{u}$ of the point O, and a rotation with an angular velocity $\boldsymbol{\omega}$ about an axis l passing through O. The velocity of the point A is therefore ((I), p. 333)

$$\mathbf{v} = \mathbf{u} + \overline{OA} \times \boldsymbol{\omega}, \tag{8}$$

whence

$$\mathbf{P}\mathbf{v} = \mathbf{P}\mathbf{u} + \mathbf{P}(\overline{OA} \times \boldsymbol{\omega}). \tag{9}$$

From formula (II), p. 13, we have (putting $\mathbf{a} = \mathbf{P}$, $\mathbf{b} = \overline{OA}$, and $\mathbf{c} = \boldsymbol{\omega}$)

$$\mathbf{P}(\overline{OA} \times \boldsymbol{\omega}) = \boldsymbol{\omega}(\mathbf{P} \times \overline{OA}). \tag{10}$$

Since $\mathrm{Mom}_O\mathbf{P} = \mathbf{P} \times \overline{OA}$ (p. 16), by (9) and (10)

$$\mathbf{P}\mathbf{v} = \mathbf{P}\mathbf{u} + \boldsymbol{\omega}\, \mathrm{Mom}_O\mathbf{P},$$

whence by (7)

$$L = \int_{t'}^{t''} \mathbf{P}\mathbf{u} \, dt + \int_{t'}^{t''} \boldsymbol{\omega}\, \mathrm{Mom}_O\mathbf{P} \, dt. \tag{11}$$

If the forces $\mathbf{P}_1, \mathbf{P}_2 \ldots$ act on a body, then their work is according to (11)

$$L = \int_{t'}^{t''} (\Sigma \boldsymbol{P}_i)\, \boldsymbol{u}\, \mathrm{d}t + \int_{t'}^{t''} \boldsymbol{\omega} (\Sigma \mathrm{Mom}_O \boldsymbol{P}_i)\, \mathrm{d}t.$$

Denoting by $\boldsymbol{P}$ the sum of the forces and by $\boldsymbol{M}_O$ the total moment with respect to O we get

$$L = \int_{t'}^{t''} (\boldsymbol{P}\boldsymbol{u})\, \mathrm{d}t + \int_{t'}^{t''} (\boldsymbol{\omega} \boldsymbol{M}_O)\, \mathrm{d}t. \tag{I}$$

From this formula it follows that *equipollent systems of forces do equal work.*

In particular, *the work done by a system of forces equipollent to zero* ($\boldsymbol{P} = 0$, $\boldsymbol{M} = 0$) *is zero.*

Denoting by α the angle which $\boldsymbol{M}_O$ makes with the axis l of instantaneous rotation, and by ω the component of the vector $\boldsymbol{\omega}$ with respect to the axis l, we have $\boldsymbol{\omega}\boldsymbol{M}_O = \omega|\boldsymbol{M}_O| \cos\alpha$. But $|\boldsymbol{M}_O| \cos\alpha$ is the projection of the moment $\boldsymbol{M}_O$ on the axis l. Consequently $|\boldsymbol{M}_O| \cos\alpha$ is equal to M_l, i. e. to the total moment of the forces with respect to the instantaneous axis of rotation; hence from (I) we get

$$L = \int_{t'}^{t''} (\boldsymbol{P}\boldsymbol{u})\, \mathrm{d}t + \int_{t'}^{t''} \omega M_l\, \mathrm{d}t. \tag{I$'$}$$

When a body moves with an advancing motion, then $\boldsymbol{\omega} = 0$; therefore by (I)

$$L = \int_{t'}^{t''} (\boldsymbol{P}\boldsymbol{u})\, \mathrm{d}t. \tag{12}$$

When a body rotates about a point O, then $\boldsymbol{u} = 0$; hence by (I$'$)

$$L = \int_{t'}^{t''} \omega M_l\, \mathrm{d}t. \tag{13}$$

Formula (13) also holds when a body rotates about a fixed axis l; we obtain it from formula (I$'$) by choosing the point O on the axis l. Denoting the angle of rotation in this case by φ, we get $\omega = \mathrm{d}\varphi / \mathrm{d}t$, whence $\omega\, \mathrm{d}t = \mathrm{d}\varphi$; hence by (13)

$$L = \int_{\varphi'}^{\varphi''} M_l\, \mathrm{d}\varphi, \tag{14}$$

where φ' and φ'' denote the angles at the initial and final positions of the body.

Kinetic energy. As we know (p. 331), the instantaneous motion of a body with respect to an arbitrary point O of the body is a rotation with an instantaneous angular velocity $\boldsymbol{\omega}$ about an axis passing through O. Let I denote the moment of inertia with respect to the instantaneous axis of rotation. The kinetic energy of the relative motion is $\frac{1}{2}I\omega^2$. Therefore by König's theorem (p. 215) the kinetic energy of the body is expressed by the formula

$$E = \tfrac{1}{2}mu^2 + \tfrac{1}{2}I\omega^2 + m\boldsymbol{u}(\boldsymbol{v}_0 - \boldsymbol{u}), \tag{II}$$

where u denotes the absolute value of the velocity $\boldsymbol{u}$ of the point O, $\boldsymbol{v}_0$ the velocity of the centre of mass, and m the mass of the body.

Therefore: *the kinetic energy of a rigid body is equal to the sum of:*

1. *the kinetic energy of an advancing motion with a velocity of an arbitrary point O of the body,*

2. *the kinetic energy of an instantaneous rotation with respect to an instantaneous axis passing through the point O and*

3. *the scalar product $m\boldsymbol{u}(\boldsymbol{v}_0 - \boldsymbol{u})$,*

where m denotes the mass of the body, $\boldsymbol{u}$ the velocity of the point O, and $\boldsymbol{v}_0$ the velocity of the centre of mass.

If the centre of mass is chosen as the point O, then $\boldsymbol{u} = \boldsymbol{v}_0$; putting $v_0 = |\boldsymbol{v}_0|$, we consequently get

$$E = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}I\omega^2. \tag{II$'$}$$

Therefore: *the kinetic energy of a rigid body is equal to the sum of the kinetic energy of an advancing motion with a velocity of the centre of mass and of the kinetic energy of an instantaneous rotation with respect to an instantaneous axis passing through the centre of mass.*

If the instantaneous motion is an instantaneous twist about the central axis passing through O, then the vectors $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are parallel. The velocity $\boldsymbol{v}_0$ of the centre of mass S is then $\boldsymbol{v}_0 = \boldsymbol{u} + \overline{OS} \times \boldsymbol{\omega}$. Since $\boldsymbol{v}_0 - \boldsymbol{u} = \overline{OS} \times \boldsymbol{\omega}$, the vector $\boldsymbol{v}_0 - \boldsymbol{u}$ is perpendicular to $\boldsymbol{\omega}$ and therefore also to $\boldsymbol{u}$. It follows from this that the scalar product $\boldsymbol{u}(\boldsymbol{v}_0 - \boldsymbol{u})$ is zero. By (II) we consequently have

$$E = \tfrac{1}{2}mu^2 + \tfrac{1}{2}I\omega^2. \tag{15}$$

Therefore: *if the instantaneous motion of a rigid body is represented as a twist, then the kinetic energy is the sum of the kinetic energies of the advancing and rotational motions.*

An instantaneous plane motion is an instantaneous rotation about the instantaneous centre of rotation with an instantaneous angular velocity.

The kinetic energy in a plane motion is $E = \frac{1}{2}I\omega^2$, where I is the moment of inertia with respect to the instantaneous centre of rotation, and ω the instantaneous angular velocity.

§ 2. Equations of motion. Motion of the centre of mass. Let m denote the mass of the body, $\boldsymbol{p}_0$ the acceleration of the centre of mass, and $\boldsymbol{P}$ the sum of the external forces acting on the body. Then (p. 196)

$$m\boldsymbol{p}_0 = \boldsymbol{P}. \tag{I}$$

Hence, knowing the sum of the external forces, we can determine the motion of the centre of mass of the body.

Principle of angular momentum. Let $\boldsymbol{K}$ denote the angular momentum, $\boldsymbol{M}$ the total moment of the external forces with respect to a fixed point or with respect to the centre of mass. Then (p. 202)

$$\boldsymbol{K}^{\cdot} = \boldsymbol{M}. \tag{II}$$

Therefore, knowing the total moment of the forces with respect to a fixed point or with respect to the centre of mass, we can determine the angular momentum.

If we calculate the acceleration $\boldsymbol{p}_0$ of the centre of mass and the angular momentum $\boldsymbol{K}$ from equations (I) and (II), the motion of the body will be determined. For having $\boldsymbol{p}_0$ given, we can determine the motion of the centre of mass. And knowing the angular momentum $\boldsymbol{K}$, we can (as we shall show later, p. 394) determine the instantaneous angular velocity $\boldsymbol{\omega}$. Since the motion of one point and the instantaneous angular momentum define the motion of a body (p. 337), equations (I) and (II) are sufficient to determine this motion.

Principle of kinetic energy. Let us consider a rigid body as a rigid system of material points. Then by the theorem given on p. 208 the internal forces do no work. Consequently, only the external forces do work. From the theorem on kinetic energy (p. 216) it follows that *the increase in kinetic energy of a rigid body is equal to the work of the external forces.*

If the external forces possess a potential, then (p. 216) *the sum of the kinetic and potential energies is constant.*

D'Alembert's principle. As we know (p. 188), the forces of inertia balance the forces acting on the points of a system. Since the internal forces have a sum and total moment equal to zero, *the forces of inertia balance the external forces.*

This principle reduces the investigation of the motion of a rigid body to problems of statics.

Advancing motion of a body. If the instantaneous motion of a rigid body is an advancing motion, then the angular momentum with respect to the centre of mass is zero (p. 200). Conversely:

If the angular momentum with respect to the centre of mass is zero at some instant, then the instantaneous motion of a rigid body is an advancing motion.

Proof. Let us assume that the angular momentum $\boldsymbol{K}$ with respect to the centre of mass is zero. The instantaneous motion of the rigid body can be considered as the composition of an instantaneous advancing motion with a velocity of the centre of mass and a rotation with an angular velocity ω about an axis l passing through the centre of mass. Since $\boldsymbol{K}$ is the sum of the angular momenta of the advancing and rotational motions, $\boldsymbol{K}$ is equal to the angular momentum of the rotational motion because the angular momentum of the advancing motion with respect to the centre of mass is zero (p. 200).

Let us denote by K the angular momentum with respect to the instantaneous axis of rotation l. From formula (7), p. 201, we have $K = I\omega$, where I is the moment of inertia with respect to the axis l. Since K is the projection of the angular momentum $\boldsymbol{K}$ on the instantaneous axis of rotation, $K = 0$. Consequently $I\omega = 0$, whence $\omega = 0$, q. e. d.

If a body moves with an advancing motion during a certain interval of time, then the angular momentum $\boldsymbol{K}$ with respect to the centre of mass is constantly zero during this time. Because of this the derivative $\boldsymbol{K}^{\cdot}$ of the angular momentum is also zero. From formula (II), p. 364, it follows that $\boldsymbol{M} = 0$, which means that the moment of the forces with respect to the centre of mass is zero. Hence by theorem 1, p. 26, the forces have a resultant acting at the centre of mass.

Conversely, if the forces have a resultant acting constantly at the centre of mass, then $\boldsymbol{M} = 0$; hence $\boldsymbol{K}^{\cdot} = 0$ constantly, i. e. $\boldsymbol{K} =$ const. If we assume that the instantaneous motion at the initial moment was an advancing motion, i. e. that $\boldsymbol{K} = 0$ at that moment, then $\boldsymbol{K} = 0$ constantly, which means that the body will move with an advancing motion.

Therefore: *in order that a body move with an advancing motion, it is necessary and sufficient that the following conditions be satisfied:*

1° *the instantaneous motion is an instantaneous advancing motion at the initial moment,*

2° *the forces have a resultant acting at the centre of mass at each moment.*

Conditions of equilibrium. The necessary and sufficient conditions which must be satisfied by a system of forces in equilibrium follow easily from conditions 1° and 2° (cf. p. 244).

If a body is at rest, then the acceleration $\boldsymbol{p}_0$ of the centre of mass and the angular momentum $\boldsymbol{K}$ are equal to zero; consequently by (I) and (II), p. 364, $\boldsymbol{P} = 0$ and $\boldsymbol{M} = 0$.

Conversely, if we assume that the body was at rest at $t = t_0$ and that $\boldsymbol{P} = 0$ and $\boldsymbol{M} = 0$ constantly, then from conditions 1° and 2° it follows that the body will move with an advancing motion. The centre of mass will be at rest because $\boldsymbol{p}_0 = 0$ and the initial velocity $\boldsymbol{v}_0$ is zero; the whole body will consequently be at rest. The system of forces is therefore in equilibrium.

Hence we have proved that *the necessary and sufficient condition for the equilibrium of forces is the vanishing of the sum and of the total moment of the forces.*

Reactions of bodies in contact. Two rigid bodies I and II, in contact at the point A, act on each other with certain forces subject to the law of action and reaction. The forces with which body II acts on body I can be replaced by one force $\boldsymbol{R}$ with its origin at A and a force couple of moment $\boldsymbol{M}$.

At present we do not possess a general theory for the moment $\boldsymbol{M}$. Only in some particular cases have certain laws been established concerning $\boldsymbol{M}$. For the force $\boldsymbol{R}$, however, experiments have yielded rather general laws (although approximate), which give sufficiently accurate results in practice.

Let us assume that the bodies have a common tangent plane Π at A. The component vector $\boldsymbol{N}$ (of the reaction $\boldsymbol{R}$) perpendicular to Π is called the *normal reaction*, and the tangential component $\boldsymbol{T}$ the *friction*.

Experiments show that certain relations between $\boldsymbol{N}$ and $\boldsymbol{T}$ obtain. We shall consider two cases:

1° The poins of contact of the two bodies have equal velocities: these points therefore have a zero velocity relative to each other. The instantaneous motion of one body relative to the other is a rotation about an axis passing through the point of contact, it is consequently a relative rolling motion. In this case (putting $T = |\boldsymbol{T}|$ and $N = |\boldsymbol{N}|$), we have

$$T \leqq fN, \tag{1}$$

where f denotes the coefficient of static friction (cf. p. 268), depending only on the nature of the surfaces of two bodies at the point of contact.

2° The velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of the points of contact of the two bodies are different. The relative velocity $\boldsymbol{v}_r$ of the point of contact of body I with respect to body II is $\boldsymbol{v}_r = \boldsymbol{v}_1 - \boldsymbol{v}_2$ and lies in the tangent plane. (Fig. 283). In this case the friction has the direction of the relative velocity $\boldsymbol{v}_r$, but an opposite sense. Moreover, we have

$$T = \mu N, \tag{2}$$

where μ denotes the so-called *coefficient of dynamic friction*, depending only on the nature of the surfaces of the body and not on the velocities of the points. We can therefore assume that $\mu = \text{const}$ during motion as long as the bodies are in contact and as long as the points of contact have different velocities.

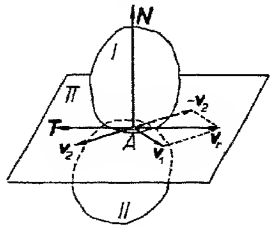

Fig. 283.

In general μ is somewhat smaller than f.

The laws given in 1° and 2° are approximate.

If the friction is zero the surfaces of the bodies are said to be *smooth*.

The surfaces of bodies are said to be *perfectly rough* if the bodies can move only in such a way that their points of contact have equal velocities. The relative motion of one body with respect to the other is then a rolling motion.

Work of the friction. Let $\boldsymbol{R}$ denote the reaction of body II on body I; then $-\boldsymbol{R}$ denotes the reaction of body I on body II. Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ denote the velocities of the material points of contact of the bodies. The work which the reactions at the point of contact do in the time from t' to t'' is

$$L = \int_{t'}^{t''} \boldsymbol{R}\boldsymbol{v}_1 \, dt - \int_{t'}^{t''} \boldsymbol{R}\boldsymbol{v}_2 \, dt = \int_{t'}^{t''} \boldsymbol{R}(\boldsymbol{v}_1 - \boldsymbol{v}_2) \, dt.$$

Since $\boldsymbol{v}_1 - \boldsymbol{v}_2$ lies in a tangent plane (or is zero), the normal reactions do no work because $\boldsymbol{N}(\boldsymbol{v}_1 - \boldsymbol{v}_2) = 0$. The work of the reactions is therefore reduced to the work of the forces of friction. Consequently

$$L = \int_{t'}^{t''} \boldsymbol{T}(\boldsymbol{v}_1 - \boldsymbol{v}_2) \, dt.$$

When $\boldsymbol{v}_r = \boldsymbol{v}_1 - \boldsymbol{v}_2 \neq 0$, the friction $\boldsymbol{T}$ has in virtue of 2° the direction of $\boldsymbol{v}_r$, but an opposite sense; hence the scalar product $\boldsymbol{T}(\boldsymbol{v}_1 - \boldsymbol{v}_2)$ is negative. When $\boldsymbol{v}_1 - \boldsymbol{v}_2 = 0$, then $\boldsymbol{T}(\boldsymbol{v}_1 - \boldsymbol{v}_2) = 0$. Therefore in both cases $\boldsymbol{T}(\boldsymbol{v}_1 - \boldsymbol{v}_2) \leqq 0$, whence $L \leqq 0$.

When the motion of the bodies relative to each other is a rolling motion, then the work of the friction is zero, and when this motion is a sliding motion, then the work of the friction is negative and causes a decrease in the kinetic energy of the bodies.

Example 1. A circular disk falls under the influence of its own weight in a vertical plane along a circle K. Determine the motion of the disk.

Let O denote the centre of the circle K and O' the centre of the disk, R and r their radii, m the mass of the disk, I its moment of inertia with respect to the centre of mass O', and finally φ the angle between the vertical and the segment OO' (Fig. 284).

We shall first consider the case when the disk and the circle K are smooth, and then when they are perfectly rough.

1° Let us assume that the disk as well as the circle are smooth and that at the initial moment $t = 0$ the disk was at rest. The forces acting on the disk during motion are: the weight $\mathbf{Q}$ with its initial point at the centre of mass O' and the reaction $\mathbf{N}$ whose direction passes through O'. The moment of these forces with respect to the centre of mass is therefore constantly zero. Since the disk was initially at rest, it will move with an advancing motion (p. 365), i. e. it will slide along the circle K (p. 337). The centre O' of the disk will therefore move along a circle with centre at O and of radius $a = R - r$. Consequently all the points of the disk will move along circles of radius a (cf., e. g. the path of the point A shown Fig. 284).

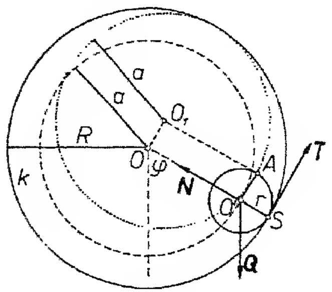

Fig. 284.

Denoting by $\mathbf{p}_0$ the acceleration of the centre of mass of the disk, we have by the theorem on the motion of the centre of mass (p. 364)

$$m\mathbf{p}_0 = \mathbf{Q} + \mathbf{N}. \tag{3}$$

Forming the projections on the tangent and normal to the path at the point O', we get:

$$ma\varphi^{\cdot\cdot} = -mg\sin\varphi, \quad ma\varphi^{\cdot 2} = N, \tag{4}$$

where $N = |\mathbf{N}|$. The first of the equations (4) can be written in the form

$$\varphi^{\cdot\cdot} = -\frac{g}{a}\sin\varphi. \tag{5}$$

Comparing equation (5) with the equation of the simple pendulum $\varphi^{\cdot\cdot} = -\frac{g}{a}\sin\varphi$ (p. 130), we see that the centre of mass of the disk will execute an oscillatory motion like that of a simple pendulum of length $l = a$.

2° Let us now assume that the disk and circle are perfectly rough. The disk will therefore (p. 367) roll along the circle K.

Let us denote by ω the instantaneous angular velocity of the disk. The kinetic energy of the disk is ((II′), p. 363)

$$E = \tfrac{1}{2}ma^2\varphi^{\cdot 2} + \tfrac{1}{2}I\omega^2$$

(because $a\varphi^{\cdot}$ is the velocity of the centre of mass). If the body is at rest at $t = 0$ and $\varphi = \varphi_0$, then from the principle of the equivalence of work and kinetic energy (p. 364) we get

$$\tfrac{1}{2}ma^2\varphi^{\cdot 2} + \tfrac{1}{2}I\omega^2 = mga(\cos\varphi - \cos\varphi_0), \tag{6}$$

because the friction (p. 367) and the reaction $\mathbf{N}$ do no work.

The velocity v of the point of contact S is zero. Consequently $v = a\varphi^{\cdot} - r\omega = 0$, whence $\omega = a\varphi^{\cdot} / r$. Substituting in (6), we therefore get

$$\tfrac{1}{2}a^2(m + I / r^2)\,\varphi^{\cdot 2} = mga(\cos\varphi - \cos\varphi_0). \tag{7}$$

From this we obtain $\varphi^{\cdot}$ in terms of φ and then ω. Differentiating equation (7), we get $a^2[m + I / r^2]\,\varphi^{\cdot}\varphi^{\cdot\cdot} = -mga\varphi^{\cdot}\sin\varphi$, whence after simplifying

$$\varphi^{\cdot\cdot} = -\frac{mg}{a(m + L / r^2)}\sin\varphi. \tag{8}$$

Comparing equation (8) with the equation of the simple pendulum $\varphi^{\cdot\cdot} = -\frac{g}{l}\sin\varphi$ (p. 130), we see that the centre of the disk will move like a pendulum of length $l = a(1 + I / mr^2)$.

Since $I = \frac{1}{2}mr^2$ for a homogeneous circle, $l = \frac{3}{2}a$. The period of oscillation will therefore be longer than that of a pendulum of length a.

Example 2. A heavy rigid body hangs on a horizontal axis l (Fig. 285) about which it can only rotate. The position of the body is determined by giving the position of the axis l and the angle φ which the line SG, passing through the centre of mass S and cutting the axis l at right angles in the point G, makes with the vertical. The axis l cuts the vertical axis k in the point O and rotates about it with a constant angular velocity ω. What is the relation between ω and φ if φ is constant?

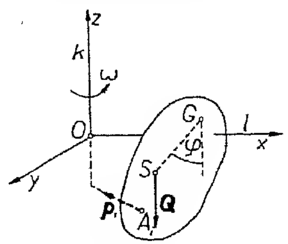

Fig. 285.

We shall solve the problem by d'Alembert's principle (p. 364). The forces of inetia balance the acting forces. Consequently the

total moment of the forces of inertia and that of the acting forces with respect to the axis l is zero.

At a certain time t let us choose a coordinate system, taking the axis l as the x-axis and the axis k as the z-axis. Since φ = const, each point of the body moves with an angular velocity ω along a horizontal circle (whose centre lies on the z-axis). The acceleration of each point of the body is therefore directed towards the centre of the circle and is $r\omega^2$, where r denotes the radius of the circle.

Let us assume that the body is a set of material points. If $\mathbf{p}_i$ denotes the acceleration of the point A_i of mass m_i and coordinates x_i, y_i, z_i, then

$$p_{i_x} = -x_i\omega^2, \quad p_{i_y} = -y_i\omega^2, \quad p_{i_z} = 0. \tag{9}$$

Consequently the forces of inertia of the point A have the projections:

$$-m_i p_{i_x} = m_i x_i \omega^2, \quad -m_i p_{i_y} = m_i y_i \omega^2, \quad -m_i p_{i_z} = 0.$$

The moment of the forces of inertia with respect to the axis l (i. e. the x-axis) is:

$$B_x = \Sigma(-m_i p_{i_y})\, z_i = \omega^2\, \Sigma m_i y_i z_i = \omega^2 D_x, \tag{10}$$

where D_x denotes the product of inertia with respect to the planes xy and xz.

The centre of gravity S has the coordinates $x_0 = OG$, $y_0 = l_0 \sin\varphi$, $z_0 = -l_0 \cos\varphi$ (where $l_0 = SG$). The moment of the weight with respect to the x-axis is

$$M_x = mgl_0 \sin\varphi, \tag{11}$$

where m denotes the mass of the body.

The moment of the reactions with respect to the x-axis is zero, because the reactions have their points of application on the z-axis. Therefore $B_x + M_x = 0$; hence by (10) and (11)

$$\omega^2 D_x + mgl_0 \sin\varphi = 0. \tag{12}$$

This equation is the sought for relation and can be satisfied only when $D_x \leqq 0$ (e. g., when the body is in the quadrant in which $y > 0$ and $z < 0$).

Example 3. A horizontal rod OA is attached rigidly at the point O on a vertical axis which is fixed at the points K and L (Fig. 286). A material point (a small sphere) B, which is strung on the rod, is capable of moving freely along the rod. At the initial moment $t = 0$ the rod OA

revolves about KL with an angular velocity ω_0, while the point B has a zero velocity relative to the rod and is situated at a distance x_0 from O. Determine the motion of the rod OA and of the material point B.

Let I denote the moment of inertia of the rod with respect to the axis KL, m the mass of the point B, ω the angular velocity of the rod, and x the length of the segment OB.

Let us assume that there is no friction. The external forces acting on the system consisting of the axis KL, the rod OA, and the point m, are: the reactions at L and K as well as the force of gravity. The moment of these forces with respect to KL is zero. The angular momentum of the system with respect to the axis KL is therefore constant. The angular momentum of the rod with respect to KL is $I\omega$ (p. 201).

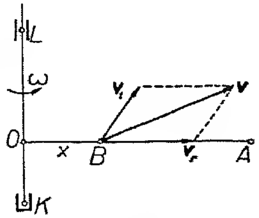

Fig. 286.

The velocity $\mathbf{v}$ of the point B is the sum of its relative velocity $\mathbf{v}_r$ with respect to the rod and the velocity of transport $\mathbf{v}_t$. The relative velocity has the component (with respect to OA) $v_r = x^{\cdot}$ and the direction OA; its moment with respect to KL is therefore zero. The velocity of transport is perpendicular to OA, has a horizontal direction, and $|\mathbf{v}_t| = x\omega$. Consequently the moment of momentum of the point B with respect to KL is equal to $mx^2\omega$. The angular momentum of the entire system is therefore $I\omega + mx^2\omega$, whence

$$(I + mx^2)\, \omega = k = \text{const.} \tag{13}$$

From the given initial conditions we have $k = (I + mx_0^2)\, \omega_0$. Let us note that the work of the acting forces is zero. Therefore the kinetic energy of the system is constant. The kinetic energy of the rod is $\frac{1}{2}I\omega^2$ (p. 363) and that of the point B

$$\tfrac{1}{2}m\mathbf{v}^2 = \tfrac{1}{2}m(\mathbf{v}_r^2 + \mathbf{v}_t^2) = \tfrac{1}{2}m(x^{\cdot 2} + x^2\omega^2).$$

Consequently

$$\tfrac{1}{2}I\omega^2 + \tfrac{1}{2}m(x^{\cdot 2} + x^2\omega^2) = h = \text{const.} \tag{14}$$

Calculating ω from (13) and substituting in (14), we obtain

$$k^2 / (I + mx^2) + mx^{\cdot 2} = 2h. \tag{15}$$

Equation (15) defines the relative motion of the point along the rod. Knowing x, we determine ω from (13).

If the point reaches A and then leaves the rod, then the rod will revolve with a constant angular velocity ω' which is obtained from (13)

by putting $x = OA = l$. The velocity of the point at the moment it leaves the rod is $v^2 = l^2\omega'^2 + \dot{x}^2$, where $\dot{x}^2$ is obtained from (15) by putting $x = l$.

Example 4. A material point M rolls down the hypotenuse of a material right-angled triangle ABC (Fig. 287). The triangle lies in a vertical plane and rests on a smooth horizontal line l. Determine the motion of the system consisting of the point M and the triangle ABC.

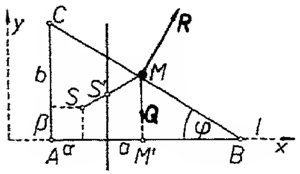

Fig. 287.

Let us take the axis l as the x-axis of the coordinate system. Let a and b denote the lengths of the legs of the triangle, S the centre of its mass, α and β the projections of the segment AS on the legs, φ the angle whose vertex is at B, m' the mass of the triangle, m'' the mass of the point, x_1 and $y_1 = \beta$ the coordinates of the point S, and x_2, y_2, the coordinates of the material point M.

Let us assume that the system consisting of the triangle and the point M was at rest at $t = 0$.

The external forces acting on the system are the reaction of the line l as well as the weight of the triangle and that of the point M. These forces have a constant vertical direction. Consequently their sum also has a constant vertical direction. From the principle of centre of mass (p. 364) it follows that the centre of mass S' of the whole system will move along a vertical (since the initial velocity was equal to zero). The coordinates of the centre of mass S' of the system are:

$$x_0 = (m'x_1 + m''x_2) / m, \quad y_0 = (m'\beta + m''y_2) / m, \tag{16}$$

where $m = m' + m''$. Hence

$$m'x_1 + m''x_2 = c = \text{const}. \tag{17}$$

The centre of mass S of the triangle moves along the line $y = \beta$; therefore its velocity is $\dot{x}_1$. The kinetic energy of the triangle ABC is equal to $\frac{1}{2}m'\dot{x}_1^2$, and the kinetic energy of the point M is $\frac{1}{2}m''(\dot{x}_2^2 + \dot{y}_2^2)$. Only the weight of the point M does work. Since the weight has the potential $-m''gy_2$, from the principle of conservation of total energy we obtain

$$\tfrac{1}{2}m'\dot{x}_1^2 + \tfrac{1}{2}m''(\dot{x}_2^2 + \dot{y}_2^2) + m''gy_2 = h = \text{const}. \tag{18}$$

The point B has the abscissa $x_1 - \alpha + a$. Denoting by M' the projection of M on the x-axis, we have $\tan\varphi = MM' / M'B$, from which $\tan\varphi = y_2 / (x_1 - \alpha + a - x_2)$. Consequently

$$y_2 - (x_1 - x_2 - \alpha + a)\tan\varphi = 0. \tag{19}$$

From equations (17)—(19) we can obtain x_1, x_2, and y_2, as functions of time.

Equations (17) and (19) are equations of the first degree. We can therefore determine from them x_2 and y_2 as linear functions of x_1. We obtain

$$x_2 = Ax_1 + B, \quad y_2 = A'x_1 + B', \tag{20}$$

where A, B and A', B' are certain constants. Differentiating and substituting in (18) we get

$$\tfrac{1}{2}[m' + m''(A^2 + A'^2)]\,\dot{x}_1^2 + m''g(A'x_1 + B') = h.$$

Calculating the derivative, we obtain

$$[m' + m''(A^2 + A'^2)]\,\dot{x}_1\ddot{x}_1 + m''gA'\dot{x}_1 = 0, \tag{21}$$

whence, after dividing by $\dot{x}_1$,

$$\ddot{x}_1 = \text{const}. \tag{22}$$

Therefore the triangle will move with a uniformly accelerated advancing motion.

From equations (20) we obtain, knowing x_1,

$$A'x_2 - Ay_2 = A'B - AB'. \tag{23}$$

Hence the point M will move along a straight line.

In virtue of (20) we have $\ddot{x}_2 = A\ddot{x}_1$ and $\ddot{y}_2 = A'\ddot{x}_1$; therefore according to (22) $\ddot{x}_2 = \text{const}$ and $\ddot{y}_2 = \text{const}$. The projections of the acceleration of the point M are constants; hence the acceleration of the point M is constant. The relative acceleration of the point M with respect to the triangle is also constant, because we obtain it by subtracting the acceleration of the triangle from the acceleration of the point M. The point M will therefore roll down the hypotenuse with a uniformly accelerated motion (relative to the hypotenuse).

Let us also examine whether M does not leave the triangle before reaching the point B.

Let us denote by **R** the reaction of the triangle on the point M. Since the weight **Q** and the force **R** act on the point M, forming projections on the axes x and y, we obtain:

$$m''\ddot{x}_2 = R_x \quad \text{and} \quad m''\ddot{y}_2 = -m''g + R_y.$$

But $\ddot{x}_2 = \text{const}$ and $\ddot{y}_2 = \text{const}$; hence $R_x = \text{const}$ and $R_y = \text{const}$. Consequently **R** is constant. The force **R** is therefore directed constantly towards the point M which cannot consequently fall away from the triangle ABC before reaching the point B.

§ 3. Rotation about a fixed axis. If a rigid body has a fixed axis l, then it can only rotate about this axis. Let us assume that the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ act on the body. Let us give the axis l an arbitrary sense and denote by I the moment of inertia of the body with respect to l, by M the moment of the forces with respect to l, and by ω the angular velocity.

The angular momentum with respect to the axis l is $K = I\omega$ ((7), p. 201). According to the theorem about angular momentum with respect to an axis (p. 202) we have $K^{\cdot} = M$; hence

$$I\omega^{\cdot} = M. \tag{1}$$

The angular acceleration is $\varepsilon = \omega^{\cdot}$; therefore

$$I\varepsilon = M. \tag{I}$$

Let Π and Π' be two planes passing through the axis l; let Π be fixed and Π' attached rigidly to the body and rotating together with it. Finally, let φ denote the angle between the planes Π and Π'. Then $\varphi^{\cdot} = \omega$ and $\omega^{\cdot\cdot} = \varepsilon$, whence by (I)

$$I\varphi^{\cdot\cdot} = M. \tag{2}$$

Differential equation (2) has the same form as the equation $mx^{\cdot\cdot} = P$, which defines the motion of a material point along the x-axis.

If the forces $\mathbf{P}_i$ or the moment M are given as functions of φ, $\varphi^{\cdot}$, and t (i. e. of the position of the body, the angular velocity, and the time), then equation (2) is a differential equation of the second order, and determines the motion if φ and $\varphi^{\cdot}$ (i. e. the position and angular velocity of the body) are known at the initial moment $t = t_0$.

The kinetic energy of a body rotating about an axis l is $E = \frac{1}{2}I\omega^2$ (p. 363). Let us denote by ω_0 and ω the angular velocities at t_0 and t, and by L_{t_0t} the work of the forces from t_0 to t. Since the forces of reaction holding the axis at rest have their points of application on the axis, they do no work. From the theorem on the equivalence of work and kinetic energy (p. 364) we therefore get

$$\tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2 = L_{t_0t}. \tag{3}$$

The work of the forces is expressed by formula (14), p. 362,

$$L_{t_0t} = \int_{\varphi_0}^{\varphi} M \,\mathrm{d}\varphi, \tag{4}$$

where φ_0 and φ denote the angles of rotation at t_0 and t. If M is a function of φ only, we can obtain L_{t_0t} from formula (4) as a function of the angle φ. Substituting in (3), we obtain a differential equation of the first order.

Example 1. Atwood's machine.[1]) At the ends of an inextensible (weightless) string, passing over a perfectly rough material pulley, are hung two heavy points of masses m_1 and m_2. Let r denote the radius of the pulley.

Let us assume that the points move vertically. The paths traversed by the points are equal, and therefore the accelerations of the points are equal in magnitude, but opposite in sense.

Let us denote by p the projection of the acceleration of the point m_1 on the z-axis, directed vertically downwards (*vide* Fig. 132). The pulley is perfectly rough, and consequently the string does not slide along it. Therefore, if the pulley rotates through an angle φ (where we assume $\varphi > 0$, when the point m_1 falls), then the point m_1 will cover a distance $s = r\varphi$. From this $s^{\cdot\cdot} = r\varphi^{\cdot\cdot}$ and hence

$$p = r\varepsilon. \tag{5}$$

Let us denote by $\mathbf{R}_1$ and $\mathbf{R}_2$ the reactions of the string on the points m_1 and m_2, and by R_1 and R_2 their absolute values. The reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ are not equal, because the string does not pass over a smooth body.

The reactions of the string and the weights act on the points m_1 and m_2. Consequently:

$$m_1p = m_1g - R_1, \quad -m_2p = m_2g - R_2. \tag{6}$$

The part of the string from the point m_1 to the pulley acts on the pulley with a force $-\mathbf{R}_1$; similarly the part of the string from the point m_2 to the pulley acts on the pulley with a force $-\mathbf{R}_2$. The moments of these forces with respect to the axis of the pulley are R_1r and $-R_2r$, where the moment of the force $-\mathbf{R}_1$ is positive, as the force $-\mathbf{R}_1$ tends to rotate the pulley in the direction assumed previously as positive for the angle φ.

Therefore, denoting by I the moment of inertia of the pulley with respect to its axis, we obtain by (I), p. 374,

$$I\varepsilon = (R_1 - R_2)\,r. \tag{7}$$

From equations (5)—(7) we can determine ε, p, R_1, and R_2. From equations (6) we get $R_1 - R_2 = (m_1 - m_2)\,g - (m_1 + m_2)\,p$. Substituting in formula (7), we obtain $I\varepsilon = (m_1 - m_2)\,rg - (m_1 + m_2)\,rp$, whence by (5)

$$\varepsilon = \frac{(m_1 - m_2)\,rg}{I + (m_1 + m_2)\,r^2}. \tag{8}$$

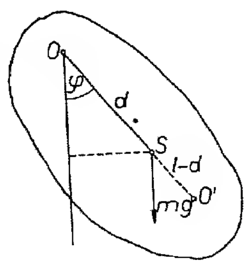

Fig. 288.

Hence we see that $\varepsilon = \text{const}$; consequently in

[1]) cf. p. 193, example 4.

view of (5) $p = \text{const}$. The points m_1 and m_2 will therefore move with a uniformly accelerated motion.

The acceleration p is obtained from (8) and (5). The reactions R_1 and R_2 can be calculated from equations (6).

Compound pendulum. A *compound pendulum* is a rigid body rotating about a horizontal axis under the influence of the force of gravity.

Through the centre of mass S of the pendulum (Fig. 288) let us pass a vertical plane, perpendicular to the axis and cutting it in the point O. Let φ denote the angle between OS and the vertical; the positive sense of rotation is chosen from left to right. Let us finally denote by M the moment of the force of gravity, by I the moment of inertia of the pendulum with respect to the axis of rotation, and let $d = OS$.

The angular acceleration is equal to $\varepsilon = \varphi^{\cdot\cdot}$, and the moment of the force of gravity

$$M = -mgd \sin\varphi$$

(where m denotes the mass of the body), whence by (2), p. 374, $I\varphi^{\cdot\cdot} = -mgd \sin\varphi$, whence

$$\varphi^{\cdot\cdot} = -\frac{mgd}{I}\sin\varphi. \tag{9}$$

Comparing equation (9) with the equation of the simple pendulum ((I), p. 130): $\varphi^{\cdot\cdot} = -\frac{g}{l}\sin\varphi$, we see that if the length l of the simple pendulum satisfies the condition

$$-mgd / I = -g / l, \tag{10}$$

then the motion of the compound pendulum is the same as that of the simple pendulum. From (10) we obtain

$$l = I / md. \tag{11}$$

Therefore: *the motion of a compound pendulum is the same as the motion of a simple pendulum of length $l = I / md$*, where I denotes the moment of inertia with respect to the axis of rotation, m the mass of the pendulum, and d the distance of the centre of mass from the axis of rotation.

Denoting by K the radius of gyration with respect to the axis of rotation, we have $I = mK^2$, whence by (11)

$$l = K^2 / d. \tag{12}$$

The length l is called the *reduced length* of the compound pendulum with respect to the axis of rotation.

Let I_0 denote the moment of inertia, and K_0 the radius of gyration with respect to the axis passing through the centre of gravity and parallel to the axis of rotation. Then (p. 158) $I = I_0 + md^2$, whence $mK^2 = mK_0^2 + md^2$ or $K^2 = K_0^2 + d^2$, and hence by (12)

$$l = \frac{K_0^2}{d} + d. \tag{13}$$

On the line OS let us consider the point O' at a distance l from O. Since $l > d$ by (13), O' will fall beyond the point S. Let us calculate the reduced length l' with respect to the axis of rotation passing through O' and parallel to the axis l passing through O.

Since $O'S = l - d$, we have by (13)

$$l' = \frac{K_0^2}{l-d} + l - d. \tag{14}$$

In view of (13) $l - d = K_0^2 / d$; hence after substitution we get from (14)

$$l' = d + \frac{K_0^2}{d}.$$

Comparing with (13), we see that

$$l' = l.$$

The reduced lengths with respect to the axes passing through O and through O' are consequently equal.

Therefore, if a body is hung on an axis passing through O' and parallel to an axis passing through O, the period of oscillation in both cases is the same (under the same initial angular displacement).

The point O' is called the *centre of oscillation* with respect to the point O.

Determination of the reaction on an axis of rotation. Let us assume that the axis of rotation l is fixed by means of reactions (frictionless) acting on the axis l. Taking an arbitrary point O on the axis l as the centre of reduction, we can replace the reactions by one force $\mathbf{R}$ with its origin at O and a force couple of moment $\mathbf{H}$. In general $\mathbf{R}$ and $\mathbf{H}$ change during the motion. To compute $\mathbf{R}$ and $\mathbf{H}$ we shall use d'Alembert's principle.

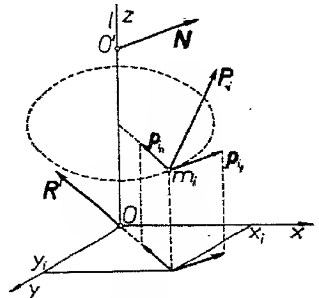

Fig. 289.

Let us select O as the origin of the coordinate system, taking the axis l as the z-axis (Fig. 289). Let us divide the body into small pieces and replace each of them by a material point of equal mass. In this manner we obtain a system of material points $m_1, m_2, \ldots$ having the coordinates $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ We shall denote the accelerations of the points of the system by $\mathbf{p}_1, \mathbf{p}_2, \ldots$, and the forces acting on these points by $\mathbf{P}_1, \mathbf{P}_2, \ldots$

The forces of inertia $-m_i\mathbf{p}_i$ balance the reactions and the forces $\mathbf{P}_i$; hence the sum and total moment (with respect to O) are equal to zero. Consequently:

$$\Sigma \mathbf{P}_i + \mathbf{R} + \Sigma(-m_i\mathbf{p}_i) = 0, \tag{15}$$

$$\Sigma \mathrm{Mom}_O \mathbf{P}_i + \mathbf{H} + \Sigma \mathrm{Mom}_O(-m_i\mathbf{p}_i) = 0. \tag{16}$$

Let the body have an angular velocity ω and an angular acceleration ε at the instant t. Let us consider an arbitrary point $m_i(x_i, y_i, z_i)$. Let us resolve the acceleration $\mathbf{p}_i$ of this point into a tangential acceleration $\mathbf{p}_{i_t}$ and a normal acceleration $\mathbf{p}_{i_n}$ (p. 40). We obviously have $\mathbf{p}_i = \mathbf{p}_{i_t} + \mathbf{p}_{i_n}$. The component accelerations are expressed by the formulae $p_{i_t} = r_i\varepsilon$, and $p_{i_n} = r_i\omega^2$ (p. 45), where r_i denotes the distance of the point from the axis of rotation. Consequently:

$$p_{i_{t_x}} = y_i\varepsilon, \qquad p_{i_{t_y}} = -x_i\varepsilon, \qquad p_{i_{t_z}} = 0; \tag{17}$$

$$p_{i_{n_x}} = -x_i\omega^2, \quad p_{i_{n_y}} = -y_i\omega^2, \quad p_{i_{n_z}} = 0. \tag{18}$$

Denoting by m the mass of the body, by x_0, y_0, z_0, the coordinates, and by $\mathbf{p}_0$ the acceleration of the centre S of its mass, we obtain from formula (III), p. 195,

$$\Sigma m_i\mathbf{p}_i = m\mathbf{p}_0. \tag{19}$$

Equation (15) will therefore assume the form

$$\Sigma \mathbf{P}_i + \mathbf{R} - m\mathbf{p}_0 = 0. \tag{20}$$

For the tangential acceleration $\mathbf{p}_{0_t}$ and the normal acceleration $\mathbf{p}_{0_n}$ of the centre of mass S, we have $\mathbf{p}_0 = \mathbf{p}_{0_t} + \mathbf{p}_{0_n}$. Therefore by (17) and (18) we obtain (putting $i = 0$):

$$p_{0_x} = y_0\varepsilon - x_0\omega^2, \quad p_{0_y} = -x_0\varepsilon - y_0\omega^2, \quad p_{0_z} = 0. \tag{21}$$

Let us now calculate the moments of the forces of inertia. The force of inertia is

$$-m_i\mathbf{p}_i = -m_i\mathbf{p}_{i_t} - m_i\mathbf{p}_{i_n}. \tag{22}$$

Let us denote by $\mathbf{B}$ the moment of the forces of inertia with respect to O. by $\mathbf{B}_t$ the moment of the tangential forces of inertia (i. e. the forces $-m_i\mathbf{p}_{i_t}$), and by $\mathbf{B}_n$ the moment of the normal forces of inertia (i. e. the forces $-m_i\mathbf{p}_{i_n}$). By (22)

$$\mathbf{B} = \mathbf{B}_t + \mathbf{B}_n. \tag{23}$$

The projection of $\mathbf{B}_t$ on the x-axis is (cf. (2), p. 232)

$$B_{t_x} = \Sigma(-m_i p_{i_{t_y}} z_i + m_i p_{i_{t_z}} y_i), \tag{24}$$

whence by (17)

$$\begin{aligned} B_{t_x} &= \varepsilon\, \Sigma m_i x_i z_i \text{ and similarly } B_{t_y} = \varepsilon\, \Sigma m_i y_i z_i, \\ B_{t_z} &= -\varepsilon\, \Sigma m_i(x_i^2 + y_i^2). \end{aligned} \tag{25}$$

Proceeding in the same way, we obtain:

$$B_{n_x} = \omega^2\, \Sigma m_i y_i z_i, \quad B_{n_y} = \omega^2\, \Sigma m_i x_i z_i, \quad B_{n_z} = 0. \tag{26}$$

By dividing the body into smaller and smaller pieces the sums in formula (24) tend to the products of inertia D_y and D_x, as well as to the moment of inertia I_z with respect to the z-axis (p. 158). In the limit we therefore get from (25) and (26):

$$B_{t_x} = \varepsilon D_y, \quad B_{t_y} = \varepsilon D_x, \qquad B_{t_z} = -\varepsilon I_z, \tag{27}$$

$$B_{n_x} = \omega^2 D_x, \ B_{n_y} = -\omega^2 D_y, \ B_{n_z} = 0, \tag{28}$$

whence, by (23),

$$B_x = \varepsilon D_y + \omega^2 D_x, \quad B_y = \varepsilon D_x - \omega^2 D_y, \quad B_z = -\varepsilon I_z. \tag{29}$$

Forming projections on the coordinate axes, we get from equations (20) and (21):

$$\begin{aligned} &\Sigma P_{i_x} + R_x - my_0\varepsilon + mx_0\omega^2 = 0, \\ &\Sigma P_{i_y} + R_y + mx_0\varepsilon + my_0\omega^2 = 0, \\ &\Sigma P_{i_z} + R_z = 0. \end{aligned} \tag{II}$$

The forces of reaction have their points of application on the axis l; consequently $H_z = 0$. Denoting by $\mathbf{M}$ the moment of the forces $\mathbf{P}_i$ with respect to O, by (16) and (29) we therefore obtain for the projections on the coordinate axes:

$$\begin{aligned} &M_x + H_x + \varepsilon D_y + \omega^2 D_x = 0, \\ &M_y + H_y + \varepsilon D_x - \omega^2 D_y = 0, \\ &M_z - \varepsilon I_z = 0. \end{aligned} \tag{III}$$

The last of the equations (III) was derived previously from the principle of angular momentum (p. 375, formula (I)). From equations (II) and (III) we can calculate **R** and **H**.

Let us now assume that the axis is fixed at two points O and O'. Denote the reactions at these points by **R'** and **N**, put $d = OO'$, and give the z-axis the direction OO'. We obtain:

$$R_x = R'_x + N_x, \quad R_y = R'_y + N_y, \quad R_z = R'_z + N_z; \tag{30}$$

$$H_x = N_y d, \qquad H_y = -N_x d, \qquad H_z = 0. \tag{31}$$

If we determine **R** and **H** from (II) and (III), then we can calculate from (30) and (31) only the components N_x, N_y, R'_x, R'_y, and the sum $N_z + R'_z$.

Let us assume that O' is in a frictionless bearing (Fig. 204). Then **N** is perpendicular to the axis of rotation; consequently $N_z = 0$.

Therefore, if the point O' is in a frictionless bearing, the reactions can be determined.

Axis of rotation as a central axis of inertia. Under the assumption that the centre of gravity lies on the axis of rotation and that the axis of rotation is one of the central axes of inertia, we have (p. 164):

$$x_0 = 0, \quad y_0 = 0, \quad D_x = 0, \quad D_y = 0.$$

Hence equations (II) and (III) assume the form:

$$\Sigma P_{i_x} + R_x = 0, \ \Sigma P_{i_y} + R_y = 0, \ \Sigma P_{i_z} + R_z = 0; \tag{32}$$

$$M_x + H_x = 0, \quad M_y + H_y = 0, \quad M_z - \varepsilon I_z = 0. \tag{33}$$

We see from this that under this assumption the reactions do not depend on the angular velocity or on the angular acceleration. Therefore they are such as if the body were at rest.

If the forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$ are equal to zero, then from equations (32) and (33) we obtain $\mathbf{R} = 0$, $\mathbf{H} = 0$, and $\varepsilon = 0$; hence $\omega =$ const. Since the reactions are equipollent to zero, we can assume that the axis l is not fixed, i. e. that the axis of rotation is free.

Conversely, if we assume that no forces act on the body, and therefore that $\mathbf{P}_i = 0$, $\mathbf{R} = 0$, and $\mathbf{H} = 0$, then equations (II) and (III) assume the form:

$$\begin{aligned} &-y_0\varepsilon + x_0\omega^2 = 0, \quad x_0\varepsilon + y_0\omega^2 = 0, \\ &\varepsilon D_y + \omega^2 D_x = 0, \quad \varepsilon D_x - \omega^2 D_y = 0, \quad \varepsilon I_z = 0. \end{aligned} \tag{34}$$

From the last of the equations (34) $\varepsilon = 0$; consequently $\omega =$ const. If $\omega \neq 0$ (i. e. when the body is not at rest), we obtain from (34) $x_0 = 0$,

$y_0 = 0$, and $D_x = 0$, $D_y = 0$. The z-axis (i. e. the axis of rotation) is therefore the central axis of inertia.

Hence we have derived the following property of the central axes of inertia:

If a free rigid body on which no forces act rotates about a fixed axis, then this axis is one of the central axes of inertia.

Example 2. A heavy rectangular door $OABC$ (Fig. 290) (where $\overline{OA}$ and $\overline{CB}$ have a vertically upward sense) can rotate about the vertical axis OA which is fixed at the points O_1 and O_2 of the side OA, where $OO_1 = O_2A = d$. At the instant $t = 0$ the door is at rest and a force **P** of constant absolute value P, perpendicular to the door and applied constantly at the centre D of the side BC, begins to act. Determine the reactions at the points O_1 and O_2 at the moment t.

Let us denote the mass of the door by m and the moment of inertia with respect to the axis OA by I. Let $OA = a$ and $OC = b$. Let us assume that the force **P** revolves the door from right to left with respect to the axis of rotation, which is directed upwards.

The moment of the force **P** with respect to the axis of rotation is constant and equal to bP. Consequently $I\varepsilon = bP$, whence $\varepsilon = bP / I$. If the door is homogeneous, then by (8), p. 180, $I = \frac{1}{3}mb^2$. Hence

$$\varepsilon = 3P / mb. \tag{35}$$

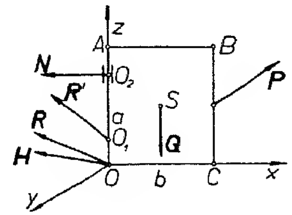

Fig. 290.

The angular acceleration $\varepsilon =$ const; consequently

$$\omega = \varepsilon t = 3Pt / mb. \tag{36}$$

The door will therefore rotate with a uniformly accelerated motion.

We shall now determine the reactions. At the instant t let us take O as the origin of the coordinate system, giving the axes z and x the directions OA and OC. Let us denote by **R'** the reaction at O_1, by **N** the reaction at O_2, finally by **R** the sum, and by **H** the moment, of the reactions with respect to O. Assuming that there is a bearing at O_2 (p. 278), we obtain:

$$R_x = R'_x + N_x, \ R_y = R'_y + N_y, \ R_z = R'_z; \tag{37}$$

$$H_x = R'_y d + N_y(a - d), \ H_y = -R'_x d - N_x(a - d), \ H_z = 0. \tag{38}$$

From equations (II) and (III), p. 379, we obtain **R** and **H**, and then we calculate **R'** and **N** from (37) and (38).

The acting forces are the weight **Q** and the force **P**. The coordinates of the centre of mass S of the door are: $x_0 = \frac{1}{2}b$, $y_0 = 0$, $z_0 = \frac{1}{2}a$, and those of the initial point of the force $\mathbf{P}: x = b, y = 0, z = \frac{1}{2}a$. Since the door lies in the zx-plane, $D_x = 0$. The product of inertia D_y is (p. 175)

$$D_y = \int_0^b\int_0^a \varrho xz \, dx \, dz = \int_0^b \tfrac{1}{2}\varrho a^2 x \, dx = \tfrac{1}{4}a^2b^2\varrho,$$

where ϱ denotes the density. Since $ab\varrho = m$, $D_y = \frac{1}{4}mab$. From equations (II) and (III), p. 379, we obtain:

$$R_x + \tfrac{1}{2}mb\omega^2 = 0, \quad -P + R_y + \tfrac{1}{2}mb\varepsilon = 0, \quad -mg + R_z = 0; \quad (39)$$

$$-\tfrac{1}{2}aP + H_x + \tfrac{1}{4}mab\varepsilon = 0, \quad -\tfrac{1}{2}mbg + H_y - \tfrac{1}{4}mab\omega^2 = 0. \quad (40)$$

From equations (39) and (40) we calculate **R** and **H** by means of (35) and (36), and then we determine the reactions **R′** and **N** from equations (37) and (38).

Example 3. A heavy rod AO hangs from a horizontal axis and can only rotate in a vertical plane about its end O. The rod is released freely from a horizontal position. What is the reaction at O when the rod makes an angle φ with the vertical?

Let us denote by **R** the reaction at O, by **Q** the weight of the rod, by m the mass of the body, and by $\boldsymbol{p}_0$ the acceleration of the centre of mass S of the rod (Fig. 291).

From the theorem on the motion of the centre of mass it follows that

$$m\boldsymbol{p}_0 = \mathbf{R} + \mathbf{Q}. \quad (41)$$

From this equation one can determine **R** if $\boldsymbol{p}_0$ is known. Let us put $l = OS$ and denote by ω and ε the angular velocity and the angular acceleration of the rod, respectively; then the accelerations: tangential p_{0_t} and normal p_{0_n} are:

$$p_{0_t} = l\varepsilon \text{ and } p_{0_n} = l\omega^2.$$

The angular acceleration is obtained from the equation $I\varepsilon = M$. Since the moment of the force of gravity with respect to O is equal to $mgl \sin\varphi$,

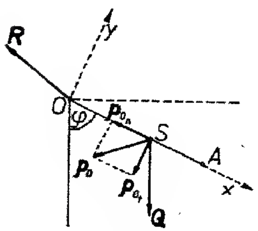

Fig. 291.

$$\varepsilon = mgl \sin\varphi \,/\, I, \quad (42)$$

where I denotes the moment of inertia of the rod with respect to O.

The angular velocity ω is calculated by appealing to the principle of equivalence of work and kinetic energy (p. 364). The increase

in kinetic energy of the rod is equal to the work of the force of gravity. The kinetic energy of the rod is $E = \frac{1}{2}I\omega^2$, and the work of the weight $L = mgl\cos\varphi$, because the level of the centre of mass was lowered by $h = l\cos\varphi$. Initially the kinetic energy was equal to zero. Therefore $\frac{1}{2}I\omega^2 = mgl\cos\varphi$, whence

$$\omega^2 = 2mgl\cos\varphi \,/\, I. \quad (43)$$

Let us take the direction OS as the positive direction of the x-axis of the coordinate system (x, y). Forming the projections on the axes x and y, we obtain from equations (41):

$$-ml\omega^2 = mg\cos\varphi + R_x, \quad -ml\varepsilon = -mg\sin\varphi + R_y.$$

From (42) and (43) we therefore get:

$$R_x = -mg\cos\varphi[1 + 2ml^2/I], \; R_y = mg\sin\varphi[1 - ml^2/I].$$

If the rod is homogeneous, then its length is equal to $a = 2l$, and the moment of inertia $I = \frac{4}{3}ml^2$ (p. 179). Consequently:

$$R_x = -\tfrac{5}{2}mg\cos\varphi, \; R_y = \tfrac{1}{4}mg\sin\varphi,$$

whence

$$|\mathbf{R}| = \tfrac{1}{4}mg\sqrt{1 + 99\cos^2\varphi}.$$

The maximum value of $|\mathbf{R}|$ therefore occurs for $\varphi = 0$ and is

$$|\mathbf{R}| = \tfrac{5}{2}mg.$$

Centre of percussion. Let us assume that the axis of rotation z is a principal axis of inertia at the point O and that the centre of mass lies in the yz-plane (where $y_0 > 0$) at a certain instant t_0 (Fig. 292). Consequently:

$$x_0 = 0, \; y_0 > 0, \quad D_x = 0, \; D_y = 0. \quad (44)$$

Equations (II) and (III), p. 379, then assume the form:

$$\Sigma P_{i_x} + R_x - my_0\varepsilon = 0, \quad \Sigma P_{i_y} + R_y + my_0\omega^2 = 0, \quad \Sigma P_{i_z} + R_z = 0, \quad (45)$$

$$M_x + H_x = 0, \quad M_y + H_y = 0, \quad M_z - \varepsilon I_z = 0. \quad (46)$$

Let us assume that the force **P**, with its point of application at A whose coordinates are x, y, z (Fig. 292), suddenly began to act at the instant t_0. As a result of the action of the force **P**, the reaction **R** and the moment **H** changed to **R** + **R′** and **H** + **H′**; the acceleration ε assumed the value $\varepsilon + \varepsilon'$; the angular velocity, equal to ω at the instant t_0, did not undergo a sudden change. At the instant when the force **P** begins to act, equations (II) and (III) assume (in view of (44)) the form:

$$
\begin{aligned}
&P_x + \Sigma P_{i_x} + R_x + R'_x - m y_0(\varepsilon + \varepsilon') = 0,\\
&P_y + \Sigma P_{i_y} + R_y + R'_y + m y_0 \omega^2 = 0,\\
&P_z + \Sigma P_{i_z} + R_z + R'_z = 0,
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
&M'_x + M_x + H_x + H'_x = 0,\\
&M'_y + M_y + H_y + H'_y = 0,\\
&M'_z + M_z - (\varepsilon + \varepsilon')\, I_z = 0,
\end{aligned}
\tag{48}
$$

where $\boldsymbol{M}'$ denotes the moment of the force $\mathbf{P}$ with respect to O. Comparing equations (45) and (46) with (47) and (48), we get:

$$
\begin{aligned}
&P_x + R'_x - m y_0 \varepsilon' = 0, \qquad P_y + R'_y = 0, \qquad P_z + R'_z = 0,\\
&M'_x + H'_x = 0, \qquad M'_y + H'_y = 0, \qquad M'_z - \varepsilon' I_z = 0.
\end{aligned}
\tag{49}
$$

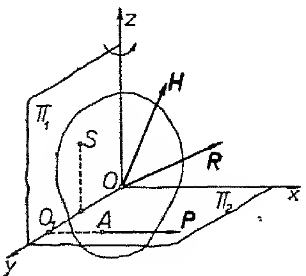

Fig. 292.

From equations (49) we can determine $\boldsymbol{R}'$ and $\boldsymbol{H}'$.

Let us assume that $\mathbf{P}$ has the direction of the x-axis and lies in the xy-plane. Then:

$$
\begin{aligned}
&z = 0, \quad P_y = 0, \quad P_z = 0,\\
&M'_x = 0, \quad M'_y = 0, \quad M'_z = P_x y.
\end{aligned}
\tag{50}
$$

From (49) we obtain:

$$\boldsymbol{H}' = 0, \tag{51}$$

$$\varepsilon' = P_x y \,/\, I_z. \tag{52}$$

From (49), (50) and (52) we get:

$$R'_x = P_x (m y y_0 \,/\, I_z - 1), \;\; R'_y = 0, \;\; R'_z = 0. \tag{53}$$

On the y-axis let us consider a point O_1 whose ordinate l is defined by the formula

$$l = I_z \,/\, m y_0. \tag{54}$$

If the direction of the force $\mathbf{P}$ passes through O_1, then $y = l$ and hence by (53) $R'_x = 0$, $R'_y = 0$, and $R'_z = 0$, whence

$$\boldsymbol{R} = 0. \tag{55}$$

The point O_1 is called the *centre of percussion*.

The centre of percussion lies on the line of intersection of the plane Π_1 passing through the axis of rotation and the centre of mass, with the plane Π_2 perpendicular to the axis of rotation at the point O. The centre of percussion lies in Π_1 on the same side of the axis of rotation as the

centre of mass. The distance of the centre of percussion from the axis of rotation is defined by formula (54).

If a body is acted upon suddenly by a force whose direction passes through the centre of percussion and is perpendicular to the plane passing through the axis of rotation and the centre of mass, then the reactions supporting the axis of rotation do not change suddenly (the axis does not quiver).

By (53) and (54) we have

$$R'_x = P_x (y \,/\, l - 1). \tag{56}$$

Hence if $y > l$, then $\boldsymbol{R}'$ and $\boldsymbol{P}$ have the same senses, and if $y < l$, then $\mathbf{R}$ and $\mathbf{P}$ have opposite senses.

Let us assume that in a compound pendulum a certain plane Π_2, perpendicular to the axis, passes through the centre of mass S and is a central plane. Consequently the axis of rotation is a principal axis of inertia with respect to the point O in which the axis pierces the plane Π_2. The line OS is the intersection of the plane Π_2 with the plane Π_1, passing through the axis of rotation and the centre of mass. The centre of percussion hence lies on the line OS at a distance l from O, defined by formula (54), where obviously $y_0 = OS$. Therefore putting $OS = d$ and $I_z = I$, we get $l = I \,/\, md$. Comparing with formula (11), p. 376, we see that the centre of percussion coincides with the centre of oscillation.

Let us assume that the pendulum is at rest and that the axis of the pendulum lies on two smooth horizontal rods. The plane Π_1 (passing through the axis and the centre of mass) is therefore vertical.

If the pendulum is struck at the centre of percussion in a horizontal direction perpendicular to Π_1, then the axis will not quiver.

If it is struck above the centre of percussion, then by (56) a reaction having a sense opposite to that of striking is necessary to maintain the axis at rest. Since this reaction cannot appear (because the rods on which the axis lies are smooth and cannot therefore induce a horizontal reaction), the axis will move in the direction of striking.

On the other hand, it the pendulum is struck below the centre of percussion, then the axis will move in the direction opposite to that of striking.

§ 4. Plane motion. Plane motion of a plane figure. Let a material figure move in the plane Π and let the acting forces $\mathbf{P}_1, \mathbf{P}_2, \ldots$, also lie in this plane (Fig. 293). Let us denote by $\boldsymbol{p}_0$ the acceleration of the centre

of mass S of the figure, by m the mass, and by $\boldsymbol{P}$ the sum of the forces. Then according to (I), p. 364,

$$m\boldsymbol{p}_0 = \boldsymbol{P}. \tag{I}$$

Let us further denote by ω the instantaneous angular velocity and by I_0 the moment of inertia with respect to the centre of mass. The instantaneous motion of the figure can be considered as the composition of an advancing motion with a velocity of the centre of mass and a rotating motion with a velocity ω about the centre of mass. Since the angular momentum of an advancing motion with respect to the centre of mass is zero (p. 200), the angular momentum K of the instantaneous motion with respect to the centre of mass is equal to the angular momentum of the rotating motion. By formula (7), p. 201, we therefore have

$$K = I\omega, \tag{1}$$

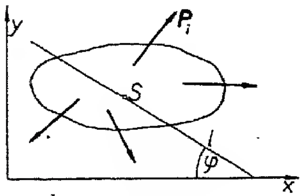

Fig. 293.

whence $K^{\cdot} = I\varepsilon$, where $\varepsilon = \omega^{\cdot}$. Denoting by M the moment of the forces with respect to the centre of mass, we obtain from (II), p. 364,

$$I\varepsilon = M. \tag{II}$$

Equations (I) and (II) define the motion of the material figure in the plane.

Let us consider a fixed line l in the figure, and denote by φ the angle between l and the x-axis, measured clockwise (Fig. 293). The coordinates x_0, y_0, of the centre of mass and the angle φ define the position of the figure.

Forming projections on the coordinate axes, we obtain from equation (I):

$$mx_0^{\cdot\cdot} = P_x, \quad my_0^{\cdot\cdot} = P_y. \tag{I'}$$

Since $\varphi^{\cdot} = \omega$ and $\varphi^{\cdot\cdot} = \varepsilon$, by (II)

$$I\varphi^{\cdot\cdot} = M. \tag{II'}$$

From equations (I′) and (II′) we can determine x_0, y_0, and φ.

Plane motion of a body. Let a body move with a plane motion, i. e. let its points move in planes parallel to a certain fixed plane Π, called the directional plane (p. 312). Let us resolve the forces $\{\boldsymbol{P}_i\}$ acting on the body into the components $\{\boldsymbol{P}_i'\}$ parallel to Π and into the components $\{\boldsymbol{P}_i''\}$ perpendicular to Π (Fig. 294). Since the centre of gravity S moves in a plane parallel to Π, its acceleration $\boldsymbol{p}_0$ lies in the plane Π. By the principle of the motion of the centre of mass we have

$$m\boldsymbol{p}_0 = \Sigma\boldsymbol{P}_i = \Sigma\boldsymbol{P}_i' + \Sigma\boldsymbol{P}_i'',$$

and hence after forming projections on the directional plane

$$m\boldsymbol{p}_0 = \Sigma\boldsymbol{P}_i'. \tag{2}$$

Denoting by l the axis perpendicular to Π and passing constantly through the center of gravity, by I the moment of inertia, and by K the angular momentum with respect to the axis l, we obtain (p. 364) $K^{\cdot} = \Sigma\mathrm{Mom}_l\boldsymbol{P}_i$. But the moment of the forces $\boldsymbol{P}_i''$ with respect to l is zero, because $\boldsymbol{P}_i'' \parallel l$; consequently

$$K^{\cdot} = \Sigma\mathrm{Mom}_l P_i'. \tag{3}$$

The instantaneous axis of rotation in a plane motion is perpendicular to the directional plane; the axis l is therefore an instantaneous axis of rotation. Since it passes constantly through the centre of mass, $K = I\omega$ ((7), p. 201), from which $K^{\cdot} = I\omega^{\cdot} = I\varepsilon$ and by (3)

$$I\varepsilon = \Sigma\mathrm{Mom}_l\boldsymbol{P}_i'. \tag{4}$$

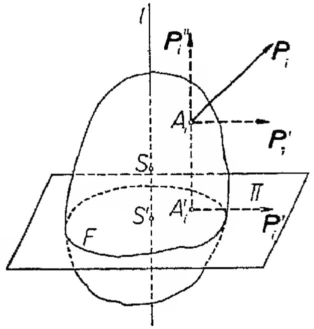

Fig. 294.

In the plane Π let us consider an arbitrary plane figure F attached rigidly to the body: this can be, for example, a section of the body made by the plane Π or the projection of the body on this plane. The motion of the figure F obviously determines the motion of the body. Let us form the projections of the forces $\{\boldsymbol{P}_i\}$ and the centre of gravity S on the plane Π. Equations (2) and (4) define the plane motion of the figure F under the assumption that:

1. the mass of the figure F is equal to the mass of the entire body,

2. the centre of gravity of the figure F is the projection of the centre of mass of the body,

3. the moment of inertia of the figure F with respect to S' is equal to the moment of inertia I of the body with respect to the axis l (Fig. 294).

It follows from this that the plane motion of a body will be determined if we give the projections of the forces on the directional plane, the mass of the body, the projection of its centre of gravity and the moment of inertia of the body with respect to a line passing through the centre of mass and perpendicular to the directional plane.

Example 1. A heavy rod AB slides down in a vertical plane with its ends resting on two smooth planes: horizontal and vertical (floor and wall). Hence the forces acting on the rod are: the weight at the centre of the rod and the reactions **N**, **R**, perpendicular to the planes. The components N and R of the reactions with respect to the axes x and y are non-negative (as in Fig. 295).

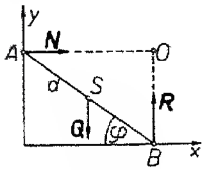

Fig. 295.

Let us denote by x_0, y_0, the coordinates of the centre of mass S of the body, by φ the angle which the rod makes with the coordinate axes, by I_0 the moment of inertia with respect to the centre of mass, and by $2d$ the length of the rod.

From equations (I′) and (II′), p. 386, we obtain:

$$m\ddot{x}_0 = N,\ \ m\ddot{y}_0 = -mg + R,\ \ I_0\ddot{\varphi} = d(N \sin\varphi - R\cos\varphi). \tag{5}$$

In addition to the above equations the following relations hold:

$$x_0 = d\cos\varphi, \quad y_0 = d\sin\varphi. \tag{5′}$$

From equations (5) and (5′) we can obtain a differential equation determining φ as a function of the time t. We shall obtain this equation by applying the principle of equivalence of work and kinetic energy.

The forces **R** and **N** do no work; only the force of gravity does work. Let us note that the velocities of the points A and B have the directions of the axes y and x. Consequently the instantaneous centre of rotation O is the point of intersection of the lines perpendicular to the axes x and y at the points B and A (p. 326). The moment of inertia with respect to O is (cf. (I), p. 159)

$$I = I_0 + md^2, \tag{6}$$

and hence I has a constant value. The kinetic energy E is therefore expressed by the formula $E = \frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{\varphi}^2$.

If $\varphi = \varphi_0$ initially, then the work of the force of gravity is $L = mgd(\sin\varphi_0 - \sin\varphi)$. Consequently, under the assumption that $\dot{\varphi} = 0$ initially, we obtain

$$\tfrac{1}{2}I\dot{\varphi}^2 = mgd(\sin\varphi_0 - \sin\varphi). \tag{7}$$

The solution of equation (7) requires a knowledge of the theory of elliptic functions. Nevertheless, we can determine the reactions **N** and **R** without solving the equation if we know φ. With this in view, differentiating equation (7), we obtain $I\dot{\varphi}\ddot{\varphi} = -mgd\dot{\varphi}\cos\varphi$, whence

$$I\ddot{\varphi} = -mgd\cos\varphi. \tag{8}$$

By (5′) we have after differentiating:

$$\ddot{x}_0 = -d\dot{\varphi}^2\cos\varphi - d\ddot{\varphi}\sin\varphi,\ \ \ddot{y}_0 = -d\dot{\varphi}^2\sin\varphi + d\ddot{\varphi}\cos\varphi. \tag{9}$$

From equations (5) we obtain:

$$N = m\ddot{x}_0,\ \ R = m\ddot{y}_0 + mg. \tag{10}$$

From equations (7) and (8) we can determine $\dot{\varphi}$ and $\ddot{\varphi}$. From equations (9) we then obtain $\ddot{x}_0$ and $\ddot{y}_0$, whence by (10) we get the reactions **R** and **N**.

Let us calculate the value of the reaction $N = |\mathbf{N}|$. In virtue of (7), (8), and (9),

$$\ddot{x}_0 = -d\cos\varphi\,\frac{2mgd(\sin\varphi_0 - \sin\varphi)}{I} + d\sin\varphi\,\frac{mgd\cos\varphi}{I};$$

hence by (10)

$$N = m\ddot{x}_0 = \frac{m^2gd^2\cos\varphi}{I}(3\sin\varphi - 2\sin\varphi_0). \tag{11}$$

Since N must be a non-negative number,

$$3\sin\varphi - 2\sin\varphi_0 \geq 0. \tag{12}$$

The point A will therefore slide down along the vertical wall as long as the angle φ satisfies the inequality (12). The moment the angle φ reaches the value φ_1 satisfying the equation

$$3\sin\varphi_1 - 2\sin\varphi_0 = 0, \tag{13}$$

the point A will stop sliding along the vertical wall, since the reaction N would then have to become negative, i. e. the wall would have to attract the point A. At that moment, therefore, the rod will fall away from the vertical wall.

Let h_0 denote the initial height of the point A, and h_1 the height of this point at the moment it falls away from the vertical wall. Since $h_0 = 2d\sin\varphi_0$ and $h_1 = 2d\sin\varphi_1$, it follows by (13) that

$$h_1 = \tfrac{2}{3}h_0. \tag{14}$$

Consequently the point A will fall away at $\frac{2}{3}$ of its initial height. After falling away from the wall the motion of the rod will be defined by equations (5) under the assumption that $N = 0$ and $y_0 = d\sin\varphi$.

Example 2. A cylinder of revolution moves down a perfectly rough inclined plane; it will therefore roll. The instantaneous motion of the cylinder will hence be a rotation about a generatrix along which the cylinder is in contact with the plane. Let us assume that this generatrix is horizontal.

The friction does no work because the points of application of the friction (i. e. the points of contact of the cylinder and the plane) have a zero velocity (p. 210). The only force doing work is the weight of the cylinder.

Let us denote by I the moment of inertia of the cylinder with respect to the generatrix, and by ω the angular velocity of rolling at the instant t. The kinetic energy is consequently

$$E = \tfrac{1}{2} I \omega^2. \tag{15}$$

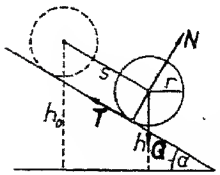

Fig. 296.

Further, let m denote the mass of the cylinder, h the height of the centre of mass at the instant t, and h_0 its height at the instant t_0 (Fig. 296). The work of the force of gravity from t_0 to t is therefore

$$L = mg(h_0 - h). \tag{16}$$

Assuming that the initial angular velocity was ω_0 at the instant t_0, we obtain from (15) and (16) by the principle of the equivalence of work and kinetic energy

$$\tfrac{1}{2} I \omega^2 - \tfrac{1}{2} I \omega_0^2 = mg(h_0 - h), \tag{17}$$

from which we can determine ω.

Finally, let s denote the path traversed by the centre of mass from the initial instant t_0 to the instant t. Let us assume that the centre of mass lies on the axis of the cylinder, and let α be the angle made by the inclined plane with the horizontal. Then

$$h_0 - h = s \sin \alpha. \tag{18}$$

Since the velocity of the centre of mass S at the instant t is $v = s^{\cdot} = r\omega$ (where r is the radius of the cylinder), and at the initial instant t_0 it was $v_0 = s_0^{\cdot} = r\omega_0$, by (17) and (18) we obtain

$$I s^{\cdot 2} / 2r^2 - I s_0^{\cdot 2} / 2r^2 = mgs \sin \alpha, \tag{19}$$

whence, differentiating with respect to t, we get $I s^{\cdot} s^{\cdot\cdot} / r^2 = mgs^{\cdot} \sin \alpha$. Consequently

$$p = s^{\cdot\cdot} = mgr^2 \sin\alpha / I. \tag{20}$$

The centre of mass will therefore fall with a uniformly accelerated motion.

The moment of inertia of a solid cylinder (of constant density) with respect to a generatrix is (p. 183, formula (23)) $I = \tfrac{3}{2} mr^2$. Hence by (20)

$p = \tfrac{2}{3} g \sin\alpha$. The centre of mass will therefore fall with an acceleration smaller than that for a free point, for which $p = g \sin\alpha$ (p. 122).

The moment of inertia of a hollow cylinder (e. g. of a pipe) with respect to the axis is mr^2, and with respect to a generatrix it is $I = 2mr^2$. Hence by (6) $p = \tfrac{1}{2} g \sin\alpha$. A solid cylinder will therefore fall faster than a hollow one.

If the initial velocity was $\omega_0 = 0$, then the centre of mass traverses a path s in the time

$$t = \sqrt{2s / p} = \sqrt{2sI / mgr^2 \sin\alpha}. \tag{21}$$

Formula (21) can be used to determine the moment of inertia I experimentally.

Let us denote by $\mathbf{T}$ the sum of the frictional forces, by $\mathbf{N}$ the sum of the normal reactions, by $\mathbf{Q}$ the weight, and by $\mathbf{p}$ (as above) the acceleration of the centre of mass of the cylinder. From the theorem on the motion of the centre of mass we have $m\mathbf{p} = \mathbf{T} + \mathbf{N} + \mathbf{Q}$. Since $\mathbf{p}$, $\mathbf{N}$, and $\mathbf{Q}$, are perpendicular to the axis of the cylinder, $\mathbf{T}$ is also perpendicular to the axis of the cylinder. Forming projections on the inclined plane and on the normal to the inclined plane (and putting $T = |\mathbf{T}|$ and $N = |\mathbf{N}|$), we obtain:

$$mp = -T + mg \sin\alpha, \quad 0 = N - mg \cos\alpha,$$

whence by (20):

$$T = mg \sin\alpha (1 - mr^2 / I), \quad N = mg \cos\alpha.$$

Example 3. A circle moves in a vertical plane Π, always remaining tangent to a horizontal line l (Fig. 297). At $t = 0$ the instantaneous motion of the circle was a rotation about the centre of the circle with an angular velocity ω_0. Determine the motion of the circle taking friction into consideration.

Let us take the line l as the x-axis and give to the y-axis a sense vertically upwards. Let us assume that the centre of the circle S is at the same time the centre of mass. Let us denote by r, m, and I, the radius, the mass, and the moment of inertia, of the circle with respect to S, by x_0, y_0, the coordinates of the centre of the circle, by ω and ε the angular velocity and the angular acceleration of the centre S, finally by T and N the components (with respect to the axes x and y) of the friction $\mathbf{T}$ and of the normal reaction $\mathbf{N}$, acting at the point of tangency A. From equations (I) and (II), p. 386, we get:

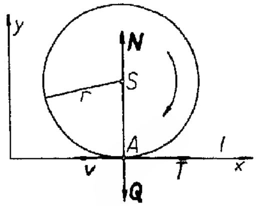

Fig. 297.

$$m\ddot{x}_0 = T, \quad m\ddot{y}_0 = N - mg, \quad I\varepsilon = -Tr. \tag{22}$$

Since $y_0 = r$ constantly, $\ddot{y}_0 = 0$, whence by (22)

$$N = mg. \tag{23}$$

Let v be the component (with respect to the x-axis) of the velocity $\mathbf{v}$ of the point A. Since the instantaneous motion of the circle at the time t is a composition of the advancing motion of the centre of mass with a velocity $\dot{x}_0$ and a rotation about the centre of mass with an angular velocity ω,

$$v = \dot{x}_0 - r\omega, \tag{24}$$

whence by differentiation

$$\dot{v} = \ddot{x}_0 - r\varepsilon. \tag{25}$$

By (22) we have $\ddot{x}_0 = T / m$ and $\varepsilon = -Tr / I$, from which by substitution in equation (25)

$$\dot{v} = (1 / m + r^2 / I)\, T. \tag{26}$$

If $\mathbf{v} \neq 0$, then $\mathbf{T}$ has a sense opposite to that of $\mathbf{v}$ (p. 367); consequently $vT \leqq 0$; whereas if $\mathbf{v} = 0$, then $vT = 0$. Hence we always have $vT \leqq 0$. Therefore, multiplying both sides of the equation (26) by v, we obtain $v\dot{v} = (1 / m + r^2 / I)\, Tv$; consequently

$$v\dot{v} \leqq 0. \tag{27}$$

But $2v\dot{v}$ is the derivative of v^2; hence by (27) the derivative of v^2 is not positive, and consequently v^2 is a non-increasing function. If, therefore, at a certain instant t_1 the value of v reaches 0, then from this instant on $v = 0$ constantly, i. e. from this instant on the circle will roll along the line l.

Let us first examine the motion of the circle from the time $t = 0$ to $t = t_1$. In this interval of time $v \neq 0$; hence $T = \mu N$, where μ denotes the coefficient of friction (p. 367). Taking the sense of the rotation as in the figure, we have $T > 0$; hence according to (23) $T = \mu mg$, whence by substitution in (22)

$$\ddot{x}_0 = \mu g, \quad \varepsilon = -\mu mrg / I. \tag{28}$$

Integrating equations (28) and making use of the conditions $\dot{x}_0 = 0$ and $\omega = \omega_0$ at $t = 0$, we obtain:

$$\dot{x}_0 = \mu gt, \quad \omega = \omega_0 - \mu mgrt / I \tag{29}$$

By (24) we have

$$v = \mu gt(1 + mr^2 / I) - r\omega_0; \tag{30}$$

hence $v = 0$ occurs at the instant

$$t_1 = \frac{r\omega_0}{\mu g(1 + mr^2 / I)}. \tag{31}$$

Since, as we have shown, we shall have $v = 0$ constantly from the moment t_1 on, $\dot{v} = 0$, and consequently for $t \geqq t_1$ in virtue of (26) $T = 0$ constantly, or by (22) $\ddot{x}_0 = 0$ and $\varepsilon = 0$.

Therefore for $t \geqq t_1$ the circle will roll with a constant angular velocity ω_1 and the centre of mass will move with a uniform motion with a velocity $v_0 = r\omega_1$.

From formulae (29) and (31) we get

$$\omega_1 = \omega_0 / (1 + mr^2 / I). \tag{32}$$

From the instant t_1 on the kinetic energy is $E_1 = \text{const}$. Since E_1 is equal to the sum of the kinetic energies of the advancing motion with the velocity of the centre of mass $v_0 = r\omega_1$ and of the rotational motion,

$$E_1 = \tfrac{1}{2}mr^2\omega_1^2 + \tfrac{1}{2}I\omega_1^2 = \tfrac{1}{2}I\omega_0^2 / (1 + mr^2 / I). \tag{33}$$

$E_0 = \tfrac{1}{2}I\omega_0^2$ at $t = 0$; consequently

$$E_1 = E_0 / (1 + mr^2 / I). \tag{34}$$

§ 5. Angular momentum. Let O be an arbitrary point of a moving body. The instantaneous motion of the body is the composition of an instantaneous advancing motion with a velocity $\mathbf{u}$ of the point O and a rotation with an angular velocity $\boldsymbol{\omega}$ about an axis passing through O.

Let us divide the body into small pieces and replace each of them by a material point of the same mass. We shall obtain a system of points $A_1, A_2, \ldots$, of masses $m_1, m_2, \ldots$ At a given moment t let us choose an arbitrary system of coordinates (ξ, η, ζ) whose origin is at O. Let us denote the coordinates of the points $A_1, A_2, \ldots$, by $\xi_1, \eta_1, \zeta_1, \xi_2, \eta_2, \zeta_2, \ldots$

The velocity of the point A_i can be represented in the form (Fig. 298)

$$\mathbf{v}_i = \mathbf{u} + \mathbf{w}_i, \tag{1}$$

where $\mathbf{w}_i$ is the velocity of the instantaneous rotation. By (V), p. 46, we have

$$w_{i\xi} = \eta_i\omega_\zeta - \zeta_i\omega_\eta, \quad w_{i\eta} = \zeta_i\omega_\xi - \xi_i\omega_\zeta,$$
$$w_{i\zeta} = \xi_i\omega_\eta - \eta_i\omega_\xi. \tag{2}$$

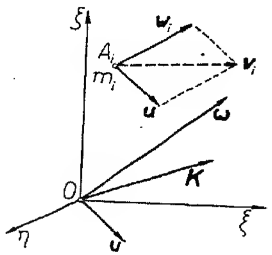

Fig. 298.

Let $\mathbf{K}_i$ be the moment with respect to O of the momentum $m_i\mathbf{v}_i$ of the point A_i, i. e. $\mathbf{K}_i = \mathrm{Mom}_O(m_i\mathbf{v}_i)$. By (II), p. 18, the projection of K_i on the ξ-axis is $K_{i\xi} = m_i(v_{i\eta}\zeta_i - v_{i\zeta}\eta_i)$, whence by (1) and (2)

$$K_{i\xi} = m_i[(u_\eta + \zeta_i\omega_\xi - \xi_i\omega_\zeta)\,\zeta_i - (u_\zeta + \xi_i\omega_\eta - \eta_i\omega_\xi)\,\eta_i],$$

i. e.

$$K_{i\xi} = \omega_\xi m_i(\zeta_i^2 + \eta_i^2) - \omega_\eta m_i\xi_i\eta_i - \omega_\zeta m_i\xi_i\zeta_i + u_\eta m_i\zeta_i - u_\zeta m_i\eta_i.$$

Since the angular momentum with respect to the point O is $\mathbf{K} = \Sigma \mathbf{K}_i$,

$$K_\xi = \omega_\xi\,\Sigma m_i(\zeta_i^2 + \eta_i^2) - \omega_\eta\,\Sigma m_i\xi_i\eta_i - \omega_\zeta\,\Sigma m_i\xi_i\zeta_i + u_\eta\,\Sigma m_i\zeta_i - u_\zeta\,\Sigma m_i\eta_i.$$

As the body is subdivided into smaller and smaller pieces the sums appearing in the last formula will tend respectively to:

$$I_\xi,\quad D_\zeta,\quad D_\eta,\quad m\zeta_0,\quad m\eta_0,$$

where m denotes the mass of the body, and ξ_0, η_0, ζ_0, the coordinates of the centre of mass. We therefore obtain (after carrying out a similar calculation for the projections K_η and K_ζ):

$$\begin{aligned} K_\xi &= \omega_\xi I_\xi - \omega_\eta D_\zeta - \omega_\zeta D_\eta + m(\zeta_0 u_\eta - \eta_0 u_\zeta), \\ K_\eta &= \omega_\eta I_\eta - \omega_\zeta D_\xi - \omega_\xi D_\zeta + m(\xi_0 u_\zeta - \zeta_0 u_\xi), \\ K_\zeta &= \omega_\zeta I_\zeta - \omega_\xi D_\eta - \omega_\eta D_\xi + m(\eta_0 u_\xi - \xi_0 u_\eta). \end{aligned} \tag{I}$$

Angular momentum with respect to the centre of mass of a body or with respect to its fixed point. If O is the centre of mass, then $\xi_0 = 0$, $\eta_0 = 0$, $\zeta_0 = 0$. On the other hand, if O is fixed, then $u_\xi = 0$, $u_\eta = 0$, $u_\zeta = 0$. In both cases we have by (I):

$$\begin{aligned} K_\xi &= \omega_\xi I_\xi - \omega_\eta D_\zeta - \omega_\zeta D_\eta, \\ K_\eta &= \omega_\eta I_\eta - \omega_\zeta D_\xi - \omega_\xi D_\zeta, \\ K_\zeta &= \omega_\zeta I_\zeta - \omega_\xi D_\eta - \omega_\eta D_\xi. \end{aligned} \tag{II}$$

In particular, if the axes of the coordinate system are principal axes of inertia at the point O, then $D_\xi = 0$, $D_\eta = 0$, $D_\zeta = 0$, and consequently:

$$K_\xi = \omega_\xi I_\xi,\quad K_\eta = \omega_\eta I_\eta,\quad K_\zeta = \omega_\zeta I_\zeta. \tag{III}$$

From formulae (III) it follows that *we can determine the angular momentum if we know the instantaneous angular velocity and conversely.*

The directions of the angular momentum and the angular velocity are in general different. The scalar product $\mathbf{K}\cdot\boldsymbol{\omega}$ is by (III)

$$\mathbf{K}\cdot\boldsymbol{\omega} = K_\xi\omega_\xi + K_\eta\omega_\eta + K_\zeta\omega_\zeta = \omega_\xi^2 I_\xi + \omega_\eta^2 I_\eta + \omega_\zeta^2 I_\zeta;$$

consequently if $\boldsymbol{\omega} \neq 0$, then $\mathbf{K}\boldsymbol{\omega} > 0$.

Therefore: *the angular momentum forms an acute angle with the angular velocity vector.*

We shall now prove the following

Theorem. *If the angular momentum* $\mathbf{K}$ *or the angular velocity vector* $\boldsymbol{\omega}$ *have the direction of one of the principal axes of inertia, then the angular momentum and the angular velocity have the same direction and conversely.*

Proof. Let us take as the ξ-axis that principal axis of inertia whose direction is the direction of the angular momentum $\mathbf{K}$. Then $K_\eta = 0$ and $K_\zeta = 0$, whence by (III) $\omega_\eta = 0$ and $\omega_\zeta = 0$. Therefore the vector $\boldsymbol{\omega}$ has the direction of the ξ-axis, i. e. of the angular momentum.

The proof is carried out in a similar manner if $\boldsymbol{\omega}$ has the direction of one of the principal axes of inertia.

Conversely, if $\mathbf{K}$ and $\boldsymbol{\omega}$ have the same direction, we take this direction as the direction of the ξ-axis. Then $\omega_\xi = 0$ and $\omega_\zeta = 0$, as well as $K_\eta = 0$ and $K_\zeta = 0$, whence by (II) $K_\xi = I_\xi\omega_\xi$, $0 = -\omega_\xi D_\zeta$, and $0 = -\omega_\xi D_\eta$, and hence $D_\zeta = 0$, $D_\eta = 0$. The ξ-axis is therefore a principal axis of inertia, q. e. d.

If the point O is a spherical point, i. e. if $I_\xi = I_\eta = I_\zeta$, then, denoting the moments of inertia by I, we have by (III) $K_\xi = I\omega_\xi$, $K_\eta = I\omega_\eta$, $K_\zeta = I\omega_\zeta$, whence

$$\mathbf{K} = I\boldsymbol{\omega}. \tag{3}$$

Therefore: *if the centre of mass (or a fixed point of a body) is a spherical point, then the angular momentum constantly has the direction and sense of the angular velocity.*

Derivative of the angular momentum. Let $\mathbf{K}$ be the angular momentum of a body with respect to an arbitrary point O of this body and let (x, y, z) be a fixed system of coordinates with its origin at O', and (ξ, η, ζ) an arbitrary moving system of coordinates with its origin at O (Fig. 299). Let us denote by $\mathbf{u}$ the velocity of the point O, and by $\boldsymbol{\omega}'$ the instantaneous angular velocity of the system (ξ, η, ζ). Let us draw a vector $\overline{OA} = \mathbf{K}$ from the point O. Putting $\overline{O'O} = \mathbf{r}$ and $\overline{O'A} = \boldsymbol{\rho}$, we obtain $\boldsymbol{\rho} = \mathbf{r} + \mathbf{K}$, whence $\mathbf{K} = \boldsymbol{\rho} - \mathbf{r}$. Calculating the derivative, we obtain $\mathbf{K}^{\cdot} = \boldsymbol{\rho}^{\cdot} - \mathbf{r}^{\cdot}$. But $\mathbf{r}^{\cdot} = \mathbf{u}$, and $\boldsymbol{\rho}^{\cdot}$ is equal to the absolute velocity $\mathbf{v}_a$ of the point A with respect to the fixed system $O'(x, y, z)$. Consequently

$$\mathbf{K}^{\cdot} = \mathbf{v}_a - \mathbf{u}. \tag{4}$$

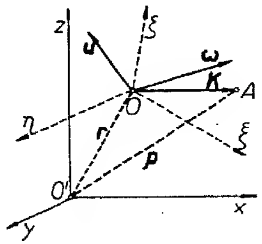

Fig. 299.

Let $\mathbf{v}_r$ be the relative velocity of the point A with respect to the system (ξ, η, ζ), and $\mathbf{v}_t$ the velocity of transport. Hence (p. 57)

$$\mathbf{v}_a = \mathbf{v}_r + \mathbf{v}_t, \tag{5}$$

whence by (4)

$$\boldsymbol{K}^{\cdot} = \boldsymbol{v}_r + (\boldsymbol{v}_t - \boldsymbol{u}). \tag{6}$$

In the system (ξ, η, ζ) the point A has the coordinates K_ξ, K_η, K_ζ. Consequently

$$v_{r_\xi} = K_\xi^{\cdot},\ v_{r_\eta} = K_\eta^{\cdot},\ v_{r_\zeta} = K_\zeta^{\cdot}. \tag{7}$$

The instantaneous motion of the system (ξ, η, ζ) is the composition of an advancing motion with a velocity $\boldsymbol{u}$ of the point O and a rotation with an instantaneous angular velocity $\boldsymbol{\omega}'$ about an axis passing through O. Hence (p. 62)

$$\boldsymbol{v}_t = \boldsymbol{u} + \overline{OA} \times \boldsymbol{\omega}' = \boldsymbol{u} + \boldsymbol{K} \times \boldsymbol{\omega}'. \tag{8}$$

Taking

$$\boldsymbol{w} = \boldsymbol{K} \times \boldsymbol{\omega}' \tag{9}$$

we therefore obtain by (8) and (6)

$$\boldsymbol{K}^{\cdot} = \boldsymbol{v}_r + \boldsymbol{w}. \tag{10}$$

From (9) we get:

$$w_\xi = K_\eta \omega_\zeta' - K_\zeta \omega_\eta',\ w_\eta = K_\zeta \omega_\xi' - K_\xi \omega_\zeta,\ w_\zeta = K_\xi \omega_\eta' - K_\eta \omega_\xi'. \tag{11}$$

Let us denote the projections of the vector $\boldsymbol{K}^{\cdot}$ on the axes ξ, η, and ζ, by $(\boldsymbol{K}^{\cdot})_\xi, (\boldsymbol{K}^{\cdot})_\eta$, and $(\boldsymbol{K}^{\cdot})_\zeta$. From (10) in virtue of (7) and (11) we obtain:

$$\begin{gathered}(\boldsymbol{K}^{\cdot})_\xi = K_\xi^{\cdot} + K_\eta \omega_\zeta' - K_\zeta \omega_\eta',\ (\boldsymbol{K}^{\cdot})_\eta = K_\eta^{\cdot} + K_\zeta \omega_\xi' - K_\xi \omega_\zeta', \\ (\boldsymbol{K}^{\cdot})_\zeta = K_\zeta^{\cdot} + K_\xi \omega_\eta' - K_\eta \omega_\xi'.\end{gathered} \tag{IV}$$

Formulae (IV) determine the projections of the derivative of the angular momentum $\boldsymbol{K}^{\cdot}$ with respect to O on the axes ξ, η, ζ of the moving system in terms 1° of the projections K_ξ, K_η, K_ζ, of the angular momentum $\boldsymbol{K}$ on these axes, 2° of the derivatives $K_\xi^{\cdot}, K_\eta^{\cdot}, K_\zeta^{\cdot}$, of the projections K_ξ, K_η, K_ζ, and 3° of the projections $\omega_\xi', \omega_\eta', \omega_\zeta'$, of the instantaneous velocity of the system (ξ, η, ζ) — but not of the body! — on the axes of this system.

One should note the difference between the symbols $(\boldsymbol{K}^{\cdot})_\xi$ and $K^{\cdot}_\xi$. The value of the first symbol is obtained by first calculating the derivative and then forming the projection on the ξ-axis; whereas the value of the second symbol is obtained, conversely, by projecting first the vector $\boldsymbol{K}$ on the ξ-axis and then calculating the derivative of the projection. As formula (IV) indicates, in general $(\boldsymbol{K}^{\cdot})_\xi \neq K^{\cdot}_\xi$.

Let us assume that O is a fixed point or the centre of mass and that the axes ξ, η, ζ constantly have the directions of the principal axes of inertia of the body at the point O. In this case the instantaneous angular velocity $\boldsymbol{\omega}$ of the body is equal to the instantaneous angular velocity of the coordinate system (ξ, η, ζ):

$$\boldsymbol{\omega} = \boldsymbol{\omega}'. \tag{12}$$

Since by (III), p. 394,

$$K_\xi = I_\xi \omega_\xi,\quad K_\eta = I_\eta \omega_\eta,\quad K_\zeta = I_\zeta \omega_\zeta, \tag{13}$$

as I_ξ, I_η, I_ζ, are fixed, we get:

$$K_\xi^{\cdot} = I_\xi \omega_\xi^{\cdot},\quad K_\eta^{\cdot} = I_\eta \omega_\eta^{\cdot},\quad K_\zeta^{\cdot} = I_\zeta \omega_\zeta^{\cdot}. \tag{14}$$

Substituting the values from (12)—(14) in (IV), we obtain:

$$\begin{gathered}(\boldsymbol{K}^{\cdot})_\xi = I_\xi \omega_\xi^{\cdot} + (I_\eta - I_\zeta)\, \omega_\eta \omega_\zeta,\ (\boldsymbol{K}^{\cdot})_\eta = I_\eta \omega_\eta^{\cdot} + (I_\zeta - I_\xi)\, \omega_\zeta \omega_\xi, \\ (\boldsymbol{K}^{\cdot})_\zeta = I_\zeta \omega_\zeta^{\cdot} + (I_\xi - I_\eta)\, \omega_\xi \omega_\eta.\end{gathered} \tag{V}$$

Formulae (V) refer to a system of coordinates whose origin is the centre of mass or a fixed point of a body and whose axes constantly have the directions of the principal axes of inertia.

§ 6. Euler's equations. We shall now consider the motion a body acted on by forces executes if it has one fixed point O, and is therefore only capable of rotating about this point. For it is to this case that we can reduce the investigation of the motion of a rigid body under the influence of forces in the most general case.

Let $\boldsymbol{K}$ be the angular momentum and $\boldsymbol{M}$ the total moment of the forces with respect to O. Then according to the principle of angular momentum (II), p. 364,

$$\boldsymbol{K}^{\cdot} = \boldsymbol{M}. \tag{1}$$

Let us note that $\boldsymbol{M}$ does not depend on a force applied at O, because its moment with respect to O is zero.

Let us choose two coordinate systems having the origin O: a fixed system (x, y, z) and a moving system (ξ, η, ζ) whose axes are the principal axes of inertia with respect to O. Let A, B, C, denote the moments of inertia of the body with respect to the principal axes of inertia (i. e. to the axes ξ, η, ζ), and let $\boldsymbol{\omega}$ denote the instantaneous angular velocity of the body. By (1) and (V) we get:

$$\begin{aligned} A\omega_\xi^{\cdot} + (B - C)\, \omega_\eta \omega_\zeta &= M_\xi, \\ B\omega_\eta^{\cdot} + (C - A)\, \omega_\zeta \omega_\xi &= M_\eta, \\ C\omega_\zeta^{\cdot} + (A - B)\, \omega_\xi \omega_\eta &= M_\zeta. \end{aligned} \tag{I}$$

Equations (I) are called *Euler's equations.*

Equations (I) serve to determine $\omega_\xi, \omega_\eta, \omega_\zeta$, as functions of the time t. Knowing $\omega_\xi, \omega_\eta, \omega_\zeta$, we can define the position of the moving system (ξ, η, ζ) and hence also the position of the body by means of Euler's angles ϑ, φ, ψ (p. 354), calculated from the differential equations (II), p. 356. In this manner, by means of Euler's equations (I) and equations

(II), p. 356, we can determine the motion of the body. The solution of these equations presents many difficulties and not always can it be carried out. However, we shall meet with some cases in which these solutions can be obtained. The most important of these is the case when no forces except the reaction at the point O act on the body.

If we know the motion of the body, then we can calculate the reaction $\boldsymbol{R}$ applied at O. For let us denote by $\boldsymbol{P}$ the sum of the acting forces, by m the mass of the body, and by $\boldsymbol{p}_0$ the acceleration of the centre of mass. By the theorem on the motion of the centre of mass ((I), p. 364), we hence have $m\boldsymbol{p}_0 = \boldsymbol{P} + \boldsymbol{R}$, whence

$$\boldsymbol{R} = m\boldsymbol{p}_0 - \boldsymbol{P}. \tag{2}$$

Remark 1. Euler's equations (I) also hold when the point O is not a fixed point, but the centre of mass of the body, for then the theorem concerning the angular momentum $\boldsymbol{K}^{\cdot} = \boldsymbol{M}$ (p. 364) holds, and the formulae (II), p. 356, are true for any point O.

Remark 2. Making use of formulae (IV), p. 396, we can give equations which are more general than Euler's equations.

Let O be a fixed point or the centre of mass, and (ξ, η, ζ) an arbitrary system of coordinates with origin at O and having an instantaneous angular velocity $\boldsymbol{\omega}'$. Since $\boldsymbol{K}^{\cdot} = \boldsymbol{M}$, by formulae (IV), p. 396, we obtain:

$$\begin{aligned} K_\xi^{\cdot} + K_\eta\,\omega_\zeta' - K_\zeta\,\omega_\eta' &= M_\xi, \\ K_\eta^{\cdot} + K_\zeta\,\omega_\xi' - K_\xi\,\omega_\zeta' &= M_\eta, \\ K_\zeta^{\cdot} + K_\xi\,\omega_\eta' - K_\eta\,\omega_\xi' &= M_\zeta. \end{aligned} \tag{II}$$

Motion of an unconstrained rigid body. Let us take the centre of mass S of a body as the origin of the coordinate system (x, y, z), moving with an advancing motion relative to an inertial frame. The motion of the body in space will be defined if we determine the motion of the centre of mass S and the motion of the body relative to S, i. e. relative to the system (x, y, z).

The motion of the centre of mass can be obtained from equations (I), p. 364.

On the other hand, in order to determine the motion of the body relative to the system (x, y, z), we can assume that this system is at rest (p. 135) and that in addition to the forces acting on the body, only the forces of transport act on it (because the forces of Coriolis are zero (p. 136)). The acceleration of transport is equal to the acceleration $\boldsymbol{p}_0$ of the centre of mass and is common to all the points of the body (p. 60). If we consider the body as a system of material points $m_1, m_2, \ldots$, then the forces of transport are $-m_1\boldsymbol{p}_0, -m_2\boldsymbol{p}_0, \ldots$ The forces of transport are therefore

proportional to the masses and have the same directions as well as senses. Consequently (p. 239) the forces of transport have a resultant $\boldsymbol{R}$ whose origin is at the centre of mass:

$$\boldsymbol{R} = -m_1\boldsymbol{p}_0 - m_2\boldsymbol{p}_0 - \ldots = -m\boldsymbol{p}_0,$$

where m denotes the mass of the body. Denoting the sum of the acting forces by $\boldsymbol{P}$, we obtain from the theorem on the motion of the centre of mass $m\boldsymbol{p}_0 = \boldsymbol{P}$, whence $\boldsymbol{R} = -\boldsymbol{P}$.

Since the centre of mass S is fixed relative to the system (x, y, z), and the force of transport has its origin at S, *the motion of the body relative to the centre of mass* (and consequently also relative to the system (x, y, z)) *is such as if the centre of mass were fixed and the body were acted upon by the same forces.*

The motion of a body relative to the centre of mass is therefore independent of the motion of the centre of mass itself and we can determine it by means of Euler's equations.

We see from this that the investigation of the motion of a body in the most general case does indeed reduce to the investigation of the motion of the centre of mass and the rotation of the body about a fixed point.

§ 7. Rotation of a body about a point under the action of no forces. Let us assume that no forces act on a rigid body having a fixed point O. In this case the moment of the forces is $\boldsymbol{M} = 0$; hence Euler's equations (I), p. 397, assume the form:

$$\begin{gathered} A\omega_\xi^{\cdot} + (B - C)\,\omega_\eta\omega_\zeta = 0, \quad B\omega_\eta^{\cdot} + (C - A)\,\omega_\zeta\omega_\xi = 0, \\ C\omega_\zeta^{\cdot} + (A - B)\,\omega_\xi\omega_\eta = 0. \end{gathered} \tag{I'}$$

Equations (I′) also hold under the assumption alone that $\boldsymbol{M} = 0$ constantly, i. e. that the forces acting on the body have a resultant whose direction constantly passes through the point O. It follows from this that equations (I′) also apply to the motion of a heavy rigid body having a fixed centre of gravity when no forces other than gravity act on the body.

The solution of equations (I′) in the general case requires a knowledge of the theory of elliptic functions. Here we shall give the solutions of those equations only in the cases when the ellipsoid of inertia with respect to O is a sphere or an ellipsoid of revolution, i. e. when all three or at least two of the numbers A, B, C, are equal.

At present we shall deduce certain general propositions from equations (I′).

Angular momentum and kinetic energy. Since $\mathbf{M} = 0$, from the theorem concerning angular momentum it follows that *the angular momentum* $\mathbf{K}$ *is a constant vector.*

Since $|\mathbf{K}|^2 = K_\xi^2 + K_\eta^2 + K_\zeta^2$, by (III), p. 394, we get, putting $I_\xi = A$, $I_\eta = B$, and $I_\zeta = C$,

$$|\mathbf{K}|^2 = A^2\omega_\xi^2 + B^2\omega_\eta^2 + C^2\omega_\zeta^2 = \text{const.} \tag{1}$$

Let us multiply both sides of Euler's equations by $\omega_\xi, \omega_\eta, \omega_\zeta$, and add. We obtain $A\dot{\omega}_\xi\omega_\xi + B\dot{\omega}_\eta\omega_\eta + C\dot{\omega}_\zeta\omega_\zeta = 0$, which can be written in the form $\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}(A\omega_\xi^2 + B\omega_\eta^2 + C\omega_\zeta^2) = 0$, whence

$$A\omega_\xi^2 + B\omega_\eta^2 + C\omega_\zeta^2 = \text{const.} \tag{2}$$

In order to give equation (2) a meaning, let us consider the angles α, β, γ, which $\boldsymbol{\omega}$ makes with the axes ξ, η, ζ. We therefore have $\omega_\xi = |\boldsymbol{\omega}|\cos\alpha$, $\omega_\eta = |\boldsymbol{\omega}|\cos\beta$, $\omega_\zeta = |\boldsymbol{\omega}|\cos\gamma$, whence

$$A\omega_\xi^2 + B\omega_\eta^2 + C\omega_\zeta^2 = (A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma)\,|\boldsymbol{\omega}|^2. \tag{3}$$

Let I be the moment of inertia of a body with respect to the instantaneous axis of rotation. In virtue of formula (I), p. 162, $I = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma$, and consequently by (3) the left side of (2) is equal to $I|\boldsymbol{\omega}|^2$. Now, since the kinetic energy of the body is $E = \frac{1}{2}I|\boldsymbol{\omega}|^2$ (p. 364),

$$2E = A\omega_\xi^2 + B\omega_\eta^2 + C\omega_\zeta^2 = \text{const.} \tag{4}$$

From this it is apparent, that equation (2) expresses the fact that the *kinetic energy of the body is constant.*

By (III), p. 394, we further have $\mathbf{K}\boldsymbol{\omega} = A\omega_\xi^2 + B\omega_\eta^2 + C\omega_\zeta^2$, whence, by (4), $\mathbf{K}\boldsymbol{\omega} = \text{const}$. Since $\mathbf{K}\boldsymbol{\omega} = |\mathbf{K}|\,\mathrm{Proj}_{\mathbf{K}}\boldsymbol{\omega}$ and according to (1) $|\mathbf{K}| = \text{const}$, $\mathrm{Proj}_{\mathbf{K}}\boldsymbol{\omega} = \text{const}$.

Therefore: *the projection of the instantaneous angular velocity on the direction of the angular momentum is constant.*

Let us suppose that the direction of the angular momentum at $t = 0$ was the same as the direction of the instantaneous angular velocity. Therefore $\mathbf{K}$ and $\boldsymbol{\omega}$ had (p. 394) the direction of one of the principal axes of inertia, e. g. the ζ-axis. Consequently at $t = 0$:

$$\omega_\xi = 0, \quad \omega_\eta = 0 \quad \text{and} \quad \omega_\zeta = \omega_\zeta^0, \tag{5}$$

where ω_ζ^0 denotes the projection of $\boldsymbol{\omega}$ on the ζ-axis at $t = 0$.

Euler's equations (I′) are differential equations of the first order. From the theory of differential equations it is known that there exists only one solution satisfying the conditions (5) at $t = 0$. This unique solution is

$$\omega_\xi = \text{const} = 0, \quad \omega_\eta = \text{const} = 0, \quad \omega_\zeta = \text{const} = \omega_\zeta^0,$$

because it satisfies the conditions (5) for $t = 0$, and, as it is easily verified, Euler's equations (I′). The vector $\boldsymbol{\omega}$ therefore has a constant magnitude and it always has the direction of the principal axis of inertia ζ; consequently (p. 394) $\boldsymbol{\omega}$ likewise has the direction of the angular momentum $\mathbf{K}$. And, since the angular momentum $\mathbf{K}$ maintains a constant direction in space, the direction of the vector $\boldsymbol{\omega}$ is also constant.

Therefore: *if the instantaneous angular velocity initially has the direction of a principal axis of inertia, then* (under the assumption that the moment of the forces with respect to a fixed point is zero) *the motion of the body is a rotation about a fixed axis with a constant angular velocity.*

Rotation about a spherical point. Let us assume that the point O is a spherical point, i. e. that $A = B = C$. Euler's equations (I′) then assume the form $\dot{\omega}_\xi = 0$, $\dot{\omega}_\eta = 0$, $\dot{\omega}_\zeta = 0$, i. e.

$$\omega_\xi = c_1, \quad \omega_\eta = c_2, \quad \omega_\zeta = c_3. \tag{II′}$$

It follows from this that the angular velocity is constant in magnitude. The point O is by hypothesis a spherical point; therefore by (3), p. 395 (putting $I = A$), we have $\mathbf{K} = A\boldsymbol{\omega}$, and consequently the instantaneous axis of rotation has the direction of the angular momentum; and because $\mathbf{K}$ is a constant vector, the instantaneous axis of rotation has a constant direction in space. In view of this the body rotates about a fixed axis with a constant angular velocity.

Therefore: *if the point O is a spherical point* (i. e. if $A = B = C$), *then the motion of a body under the action of forces whose moment with respect to O is zero, is a rotation about a fixed axis with a constant angular velocity.*

Rotation about a point whose ellipsoid of inertia is an ellipsoid of revolution. Let us assume that $A = B$, i. e. that the ellipsoid of inertia with respect to the point is one of revolution. This case occurs if the body possesses e. g. an axis of symmetry passing through O. Euler's equations (I′) then assume the form:

$$\dot{\omega}_\xi + \frac{A - C}{A}\omega_\zeta\omega_\eta = 0, \quad \dot{\omega}_\eta - \frac{A - C}{A}\omega_\zeta\omega_\xi = 0, \quad \dot{\omega}_\zeta = 0. \tag{II″}$$

The third of the equations (II″) gives

$$\omega_\zeta = c = \text{const.} \tag{6}$$

From equation (4), putting $A = B$, we obtain

$$A(\omega_\xi^2 + \omega_\eta^2) + C\omega_\zeta^2 = \text{const};$$

hence in virtue of (6) we have

$$\omega_\xi^2 + \omega_\eta^2 = c_1^2 = \text{const}. \tag{7}$$

Since $|\boldsymbol{\omega}|^2 = \omega_\xi^2 + \omega_\eta^2 + \omega_\zeta^2$, by (6) and (7)

$$|\boldsymbol{\omega}|^2 = c^2 + c_1^2 = \text{const}. \tag{8}$$

Hence: *the instantaneous angular velocity is constant in magnitude.* Let us set

$$\frac{A - C}{A}\,\omega_\zeta = h. \tag{9}$$

Since $\omega_\zeta = \text{const}$ in view of (6), $h = \text{const}$ and the first two equations (II′) assume the form:

$$\omega_\xi^{\cdot} + h\omega_\eta = 0, \quad \omega_\eta^{\cdot} - h\omega_\xi = 0. \tag{10}$$

Let $h \neq 0$. Differentiating the first of the equations (10), we get $\omega_\xi^{\cdot\cdot} + h\omega_\eta^{\cdot} = 0$ or $\omega_\eta^{\cdot} = -\omega_\xi^{\cdot\cdot}/h$, whence by substituting in the second of the equations (10) we obtain after multiplying by h

$$\omega_\xi^{\cdot\cdot} + h^2\omega_\xi = 0. \tag{11}$$

The general solution of equation (11) has the form

$$\omega_\xi = a\sin ht + b\cos ht, \tag{12}$$

where a and b are arbitrary constants. The first of the equations (10) gives $\omega_\eta = -\omega_\xi^{\cdot}/h$, whence by (12)

$$\omega_\eta = -a\cos ht + b\sin ht. \tag{13}$$

Equations (6), (12), and (13), represent the general solution of Euler's equations (II″) also when $h = 0$. The solution contains three arbitrary constants a, b, c, which are determined from the initial conditions.

Determination of Euler's angles. We shall now consider the determination of Euler's angles by means of equations (6), (12), and (13), and equations (II), p. 356.

Since the angular momentum $\mathbf{K}$ is a constant vector, we can take the direction of the angular momentum as the direction of the z-axis. The projection of the angular momentum on the ζ-axis is $K_\zeta = |\mathbf{K}|\cos\vartheta$. On the other hand (putting $I_\zeta = C$) we have by (III), p. 394, $K_\zeta = C\omega_\zeta$; therefore $|\mathbf{K}|\cos\vartheta = C\omega_\zeta$, whence

$$\cos\vartheta = C\omega_\zeta / |\mathbf{K}|. \tag{14}$$

Since $|\mathbf{K}| = \text{const}$ and $\omega_\zeta = \text{const}$,

$$\vartheta = \vartheta_0 = \text{const}. \tag{15}$$

If $\vartheta_0 = 0$ or $\vartheta_0 = \pi$, then the angular momentum $\mathbf{K}$ constantly has the direction of the axis of inertia ζ; consequently according to the theorem given on p. 395 the instantaneous angular velocity has the direction of the angular momentum. Similarly, if $\vartheta_0 = \frac{1}{2}\pi$, then by (14) $\omega_\zeta = 0$; hence ((III), p. 394) $K_\xi = A\omega_\xi$, $K_\eta = A\omega_\eta$, $K_\zeta = 0$, whence $\mathbf{K} = A\boldsymbol{\omega}$; therefore the angular momentum has the direction of the instantaneous angular velocity. From the theorem given on p. 395 we conclude, therefore, that *if $\vartheta_0 = 0$ or π or $\frac{1}{2}\pi$, then the motion of a body is a rotation about a fixed axis with a constant angular velocity.*

Let us now assume that $\vartheta_0 \neq 0$, $\vartheta_0 \neq \pi$, and $\vartheta_0 \neq \frac{1}{2}\pi$. Since $\vartheta = \vartheta_0 = \text{const}$, the ζ-axis describes a cone of revolution whose axis is the z-axis. Substituting the values ω_ξ and ω_η from equations (12) and (13) in equations (II), p. 356, we obtain (because $\vartheta^{\cdot} = 0$):

$$a\sin(ht - \varphi) + b\cos(ht - \varphi) = 0, \tag{16}$$

$$\psi^{\cdot} = [a\cos(ht - \varphi) - b\sin(ht - \varphi)] / \sin\vartheta_0, \tag{17}$$

$$\varphi^{\cdot} = \omega_\zeta - [a\cos(ht - \varphi) - b\sin(ht - \varphi)]\cos\vartheta_0. \tag{18}$$

Were $a = 0$ and $b = 0$, then by (12) and (13) we should have $\omega_\xi = 0$ and $\omega_\eta = 0$; hence $\boldsymbol{\omega}$ would have the direction of the axis of inertia ζ, and consequently of the angular momentum $\mathbf{K}$ (in virtue of the theorem on p. 395). The ζ-axis would therefore have the direction of the z-axis and ϑ_0 would be zero or π, contrary to hypothesis. One of the numbers a and b is therefore different from zero, whence by (16) $ht - \varphi = \text{const}$.

Let $\varphi = \varphi_0$ for $t = 0$. Consequently $ht - \varphi = -\varphi_0$, i. e.

$$\varphi = ht + \varphi_0. \tag{19}$$

Substituting this value of φ in (17) and (18), we get:

$$\psi^{\cdot} = (a\cos\varphi_0 + b\sin\varphi_0) / \sin\vartheta_0, \tag{20}$$

$$h = \omega_\zeta - (a\cos\varphi_0 + b\sin\varphi_0)\cot\vartheta_0. \tag{21}$$

Since $\vartheta_0 \neq 0$, $\vartheta_0 \neq \pi$, and $\vartheta_0 \neq \frac{1}{2}\pi$, from (21) we obtain $a\cos\varphi_0 + b\sin\varphi_0 = (\omega_\zeta - h)\tan\vartheta_0$, whence by (20)

$$\psi^{\cdot} = (\omega_\zeta - h) / \cos\vartheta_0. \tag{22}$$

Substituting in equations (19) and (22) the value of h from equation (9), we obtain together with equation (15):

$$\varphi^{\cdot} = \frac{A - C}{A}\,\omega_\zeta, \quad \psi^{\cdot} = \frac{C}{A\cos\vartheta_0}\,\omega_\zeta, \quad \vartheta^{\cdot} = 0. \tag{23}$$

Integrating equations (23) and assuming that $\varphi = \varphi_0$, $\psi = \psi_0$, and $\vartheta = \vartheta_0$, for $t = 0$, we get:

$$\varphi = \frac{A - C}{A}\,\omega_\zeta t + \varphi_0, \quad \psi = \frac{C}{A \cos\vartheta_0}\,\omega_\zeta t + \psi_0, \quad \vartheta = \vartheta_0. \tag{24}$$

Since according to (6) $\omega_\zeta = \text{const}$, from (23) it follows that $\varphi^\cdot = \text{const}$ and $\psi^\cdot = \text{const}$.

Consequently: *the motion of a body is the composition of two rotations, one of which is about the fixed axis ζ in the body, and the other about the fixed axis z in space. The angular velocity of both rotations is constant.*

Such a motion was called a *steady precession* (p. 356). The relation between the two angular velocities is according to (23)

$$\varphi^\cdot \,/\, \psi^\cdot = (A - C) \cos\vartheta_0 \,/\, C. \tag{25}$$

We have therefore proved the

Theorem. *If the ellipsoid of inertia at the point O is an ellipsoid of revolution, then the motion of the body is either a rotation about a fixed axis with a constant angular velocity or it is a steady precession.*

Rotation of a body about a point in the general case. We shall now make certain remarks concerning a body which rotates about a point O, under the assumption that the moment of the forces with respect to the point O is zero.

Let us retain the notations used up to the present. The axes ξ, η, ζ, have the directions of the principal axes of inertia at the point O, and hence the equation of the ellipsoid of inertia with respect to O has in the system (ξ, η, ζ) the form (formula (8), p. 164) $A\xi^2 + B\eta^2 + C\zeta^2 = c^2$, where c is an arbitrary constant. Since the kinetic energy E is constant, we can assume $c^2 = 2E$. Hence the ellipsoid of inertia will have the equation

$$A\xi^2 + B\eta^2 + C\zeta^2 = 2E. \tag{26}$$

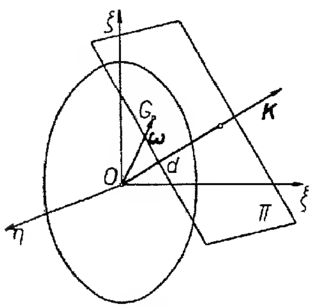

Fig. 300.

Let us denote by G the terminus of the angular velocity vector $\boldsymbol{\omega}$ (Fig. 300). The point G consequently has the coordinates ω_ξ, ω_η, and ω_ζ. By formula (4), p. 400, the coordinates of the point G satisfy equation (26). It follows from this that *the terminus of the vector $\boldsymbol{\omega}$ lies on the ellipsoid of inertia* (26).

The equation of the plane Π, tangent to the ellipsoid (26) at the point G, has the form

$$A\omega_\xi \xi + B\omega_\eta \eta + C\omega_\zeta \zeta = 2E. \tag{27}$$

Since by (III), p. 394, the angular momentum $\mathbf{K}$ has the projections $K_\xi = A\omega_\xi$, $K_\eta = B\omega_\eta$, and $K_\zeta = C\omega_\zeta$, on the axes ξ, η, ζ, the angular momentum $\mathbf{K}$ is perpendicular to the plane Π. The distance of the plane Π from the point O is $d = 2E/\sqrt{A^2\omega_\xi^2 + B^2\omega_\eta^2 + C^2\omega_\zeta^2}$; consequently by (1), p. 400,

$$d = 2E \,/\, |\mathbf{K}| = \text{const}. \tag{28}$$

The distance of the plane Π from the point O is therefore constant. Moreover, since the plane Π is constantly perpendicular to the fixed vector $\mathbf{K}$, *Π is a fixed plane in space.*

The ellipsoid of inertia is constantly tangent to the plane Π. The instantaneous motion of the body is an instantaneous rotation with an angular velocity $\boldsymbol{\omega}$, while G is the terminus of the vector $\boldsymbol{\omega}$, and consequently the velocity of the point G is zero. It follows from this that *the ellipsoid of inertia rolls on the plane Π.*

We shall now consider the question, what positions can the vector $\boldsymbol{\omega}$ assume in the body, i. e. what curve does the point G describe on the ellipsoid of inertia.

Let us denote the coordinates of the point G by ξ, η, ζ. Consequently $\xi = \omega_\xi$, $\eta = \omega_\eta$, and $\zeta = \omega_\zeta$. Hence by (1), p. 400, we have

$$A^2\xi^2 + B^2\eta^2 + C^2\zeta^2 = K^2, \tag{29}$$

where $K = |\mathbf{K}|$. The coordinates of the point G also satisfy equation (26) of the ellipsoid of inertia. Multiplying both sides of equation (26) by K^2, and of equation (29) by $2E$, and subtracting, we obtain

$$(AK^2 - 2EA^2)\,\xi^2 + (BK^2 - 2EB^2)\,\eta^2 + (CK^2 - 2EC^2)\,\zeta^2 = 0. \tag{30}$$

Equation (30) is the equation of a cone with its vertex at O. The point G therefore describes a curve which is the intersection of the ellipsoid of inertia (26) and the cone (30). These curves are closed and (in general) of the fourth degree. In particular, the cone (30) is a cone of revolution when, e. g. $A = B$ (or $A = C$ or $B = C$). If A, B, and C, are different, the cone can degenerate into two planes.

The angular velocities traced in the body form the cone (30). Consequently the cone defined by equation (30) is the moving cone of instantaneous angular velocities (p. 339).

If we trace the positions of the point G on the fixed plane Π, then we

shall obtain a certain curve. The cone for which this curve is the directrix, and O the vertex, is the fixed cone of instantaneous angular velocities (p. 339).

§ 8. Rotation of a heavy body about a point. We shall now consider the motion of a heavy body in which an arbitrary point O other than the centre of gravity is fixed.

Such a motion is executed, for example, by a top rotating about an axis one of whose ends rests on a sufficiently rough floor, so that the sliding of the end of the axis on the floor is impossible.

For simplicity's sake, let us assume that the body has an axis of symmetry passing through O (Fig. 301). The centre of mass S obviously lies on this axis. Let us take the axis of symmetry as the ζ-axis of the moving coordinate system and give it a sense towards the centre of mass S. Let us put $OS = l$. Let the z-axis of the fixed coordinate system have a sense vertically upwards. The weight of the body $\mathbf{Q}$ therefore has the direction of the z-axis. Denoting by $\mathbf{k}$ the unit vector lying on the z-axis, and by m the mass of the body, we have $\mathbf{Q} = -mg\mathbf{k}$. Forming the projections on the axes ξ, η, ζ, we obtain by (24), p. 356:

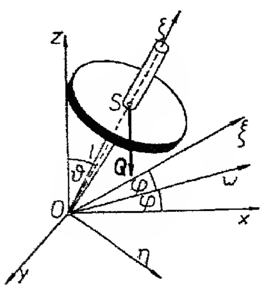

Fig. 301.

$$Q_\xi = -mg \sin\vartheta \sin\varphi, \quad Q_\eta = mg \sin\vartheta \cos\varphi, \quad Q_\zeta = -mg \cos\vartheta.$$

Since $\mathbf{Q}$ has its origin at the centre of mass S whose coordinates in the system (ξ, η, ζ) are $0, 0, l$, denoting by $\boldsymbol{M}$ the moment of the weight $\mathbf{Q}$ with respect to O, we get:

$$M_\xi = mgl \sin\vartheta \cos\varphi, \quad M_\eta = mgl \sin\vartheta \sin\varphi, \quad M_\zeta = 0.$$

As $A = B$, Euler's equations (I), p. 397, assume the form:

$$\begin{gathered} A\dot{\omega}_\xi + (A - C)\, \omega_\eta \omega_\zeta = mgl \sin\vartheta \cos\varphi, \\ A\dot{\omega}_\eta - (A - C)\, \omega_\zeta \omega_\xi = mgl \sin\vartheta \sin\varphi, \\ C\dot{\omega}_\zeta = 0. \end{gathered} \tag{1}$$

If we express $\omega_\xi, \omega_\eta, \omega_\zeta$, in equations (1) in terms of Euler's angles according to formulae (I), p. 356, we obtain a system of differential equations of the second order, where the unknowns will be ϑ, φ, and ψ, as functions of the time.

System (1) can be reduced to a system of differential equations of

the first order in a simple way. We get one equation from the third equation of system (1). Integrating this equation, we obtain $C\omega_\zeta = \text{const}$; hence

$$\omega_\zeta = r = \text{const}. \tag{2}$$

Two other equations of the first order are obtained from the principle of conservation of energy (for the weight possesses a potential) and from the principle of angular momentum, according to which the angular momentum with respect to the fixed axis z is in this case constant, because the moment of the weight with respect to this axis is zero (for the weight has the direction of the z-axis).

The centre of mass has the coordinate $z = l \cos\vartheta$; therefore the potential of the weight is $V = -mgz = -mgl \cos\vartheta$.

The kinetic energy of the body ((4), p. 400) is

$$E = \tfrac{1}{2}[A(\omega_\xi^2 + \omega_\eta^2) + C\omega_\zeta^2],$$

and from the principle of conservation of energy we have $E - V = \text{const}$; consequently

$$A(\omega_\xi^2 + \omega_\eta^2) + C\omega_\zeta^2 + 2mgl \cos\vartheta = h = \text{const}. \tag{3}$$

Denoting by $\mathbf{K}$ the angular momentum with respect to O, we have ((III), p. 394) $K_\xi = A\omega_\xi$, $K_\eta = A\omega_\eta$, and $K_\zeta = C\omega_\zeta$. The z-axis makes with the axes ξ, η, and ζ, angles whose cosines are k_ξ, k_η, and k_ζ (because $\mathbf{k}$ is the unit vector having the sense and the direction of the z-axis). Therefore the projection of the angular momentum $\mathbf{K}$ on the z-axis is $K_z = K_\xi k_\xi + K_\eta k_\eta + K_\zeta k_\zeta$. Substituting into this formula the values k_ξ, k_η, k_ζ, from formulae (24), p. 356, and remembering that $K_z = \text{const}$, because the moment of the weight with respect to the vertical axis z is zero, we obtain

$$K_z = A(\omega_\xi \sin\vartheta \sin\varphi - \omega_\eta \sin\vartheta \cos\varphi) + C\omega_\zeta \cos\vartheta = \text{const}. \tag{4}$$

Let us now express the projections ω_ξ, ω_η, ω_ζ, in formulae (2)—(4) in terms of Euler's angles according to formulae (I), p. 356. We obtain:

$$\begin{gathered} \dot{\psi} \cos\vartheta + \dot{\varphi} = r, \quad \dot{\vartheta}^2 + \dot{\psi}^2 \sin^2\vartheta + a \cos\vartheta = b, \\ \dot{\psi} \sin^2\vartheta + \alpha \cos\vartheta = \beta, \end{gathered} \tag{5}$$

where

$$\begin{gathered} r = \omega_\zeta, \quad a = 2mgl / A, \quad b = (h - Cr^2) / A, \quad \alpha = Cr / A, \\ \beta = K_z / A. \end{gathered} \tag{6}$$

The third of the equations (5) gives $\dot{\psi} = (\beta - \alpha \cos\vartheta) / \sin^2\vartheta$. Substituting this value of $\dot{\psi}$ into the second of the equations (5) and multiplying by $\sin^2\vartheta$, we get

$$\dot{\vartheta}^2 \sin^2\vartheta + (\beta - \alpha \cos\vartheta)^2 = (b - a \cos\vartheta) \sin^2\vartheta. \tag{7}$$

Let us substitute

$$u = \cos\vartheta \quad \text{or} \quad u^{\cdot} = -\vartheta^{\cdot}\sin\vartheta \tag{8}$$

into (7), and then into the first and third of the equations (5).

We obtain:

$$u^{\cdot 2} = (b - au)(1 - u^2) - (\beta - \alpha u)^2, \tag{9}$$

$$\psi^{\cdot} = (\beta - \alpha u) \,/\, (1 - u^2), \tag{10}$$

$$\varphi^{\cdot} = r - (\beta - \alpha u)\, u \,/\, (1 - u^2). \tag{11}$$

From (9) we can determine u, i. e. $\cos\vartheta$, and then from (10) and (11) the angles ψ and φ.

Let us denote the right side of equation (9) by $f(u)$. Assuming that $\beta - \alpha \neq 0$ and $\beta + \alpha \neq 0$ we have:

$$f(+1) < 0, \quad f(-1) < 0, \tag{12}$$

and in addition, since $a > 0$, by (6)

$$\lim_{u\to+\infty} f(u) = +\infty. \tag{13}$$

If ϑ_0 was the value of the angle ϑ for $t = 0$, and $u_0 = \cos\vartheta_0$, then by (9)

$$f(u_0) = u_0^{\cdot 2} \geqq 0. \tag{14}$$

From relations (12)—(14) it follows that the equation $f(u) = 0$ has three real roots, two of which, namely, u_1 and u_2, lie between -1 and $+1$, and the third $u_3 > 1$. In a particular case we can have $u_1 = u_2$ (a double root).

Let us assume that $u_1 < u_2$. Since $u = \cos\vartheta$ must lie between -1 and $+1$, and moreover $f(u) \geqq 0$ (by (9)), then $u_1 \leqq u \leqq u_2$ or

$$u_1 \leqq \cos\vartheta \leqq u_2. \tag{15}$$

Therefore: *during motion the angle ϑ varies between the limits ϑ_1 and ϑ_2, where $\cos\vartheta_1 = u_1$ and $\cos\vartheta_2 = u_2$. Since $l\cos\vartheta$ denotes the height of the centre of mass above the horizontal xy-plane, the centre of mass oscillates between two horizontal planes $z = l\cos\vartheta_1$ and $z = l\cos\vartheta_2$.*

The numerator of the right member of equation (10) is zero only for $u = \beta / \alpha$. Therefore, if $|\beta / \alpha| > 1$, then the sign of $\psi^{\cdot}$ is constant, because $|u| \leqq 1$. It is obvious that if

$$u_1 < \beta / \alpha < u_2, \tag{16}$$

then $\psi^{\cdot}$ changes its sign. It is easy to show that inequality (16) is equivalent to the inequalities:

$$|\beta / \alpha| < 1, \quad \beta / \alpha < b / a. \tag{17}$$

For if (16) holds, then $|\beta / \alpha| < 1$ and moreover

$$f(\beta / \alpha) = (b - a\beta / \alpha)(1 - \beta^2 / \alpha^2) > 0; \tag{18}$$

hence $b - a\beta / \alpha > 0$, whence $\beta / \alpha < b / a$ (because $a > 0$ by (6)). Conversely, if the inequalities (17) hold, then according to (18) $f(\beta / \alpha) > 0$; hence either inequality (16) or inequality $\beta / \alpha > u_3$ holds. However, the latter is impossible, since $f(b / a) = -(\beta - \alpha b / a)^2 < 0$; hence $b / a < u_3$, and consequently $\beta / \alpha < u_3$ by (17).

§ 9. Motion of a sphere on a plane. Let a heavy sphere of constant density move along a horizontal plane Π (example: the motion of a sphere along a billiard table). Let us consider friction and assume that the reaction of the plane reduces to one force acting at the point of tangency S (Fig. 302). Let us denote by R and T the absolute values of the reactions: normal $\mathbf{R}$ and tangent (friction) $\mathbf{T}$, and by μ the coefficient of friction. Consequently

$$T = \mu R. \tag{1}$$

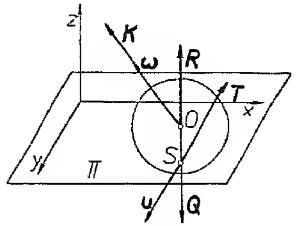

Fig. 302.

If the point of tangency S of the sphere with the plane Π has a velocity different from zero, then the friction $\mathbf{T}$ has the direction of this velocity, but an opposite sense (p. 367). Let us take the plane Π as the xy-plane of the coordinate system (x, y, z) and give the z-axis a sense vertically upwards. Denoting by $\mathbf{p}_0$ the acceleration of the centre of mass $O(x_0, y_0, z_0)$ of the sphere, by m the mass of the sphere, and by $\mathbf{Q}$ its weight, we have

$$m\mathbf{p}_0 = \mathbf{Q} + \mathbf{R} + \mathbf{T}. \tag{2}$$

Forming projections on the coordinate axes, we get:

$$mx_0^{\cdot\cdot} = T_x, \; my_0^{\cdot\cdot} = T_y, \; mz_0^{\cdot\cdot} = -mg + R. \tag{3}$$

Since $z_0 = \text{const} = r$ (where r denotes the radius of the sphere), $z_0^{\cdot\cdot} = 0$, and consequently $R = mg$, and by (1), under the assumption that S has a velocity different from zero,

$$T = \mu mg. \tag{4}$$

Let $\mathbf{K}$ be the angular momentum with respect to the centre O of the sphere, $\boldsymbol{\omega}$ the instantaneous angular velocity, and A the moment of inertia with respect to a diameter. Since by hypothesis the sphere is

homogeneous, its centre O is a spherical point. Consequently (by (3), p. 395) $\boldsymbol{K} = A\boldsymbol{\omega}$, whence

$$\boldsymbol{K}^{\cdot} = A\boldsymbol{\omega}^{\cdot}. \tag{5}$$

The moment $\boldsymbol{M}$ of the forces with respect to the point O reduces to the moment of the force $\boldsymbol{T}$. Consequently:

$$M_x = -rT_y, \quad M_y = rT_x, \quad M_z = 0. \tag{6}$$

Since $\boldsymbol{K}^{\cdot} = \boldsymbol{M}$, we get from (5):

$$A\omega_x^{\cdot} = -rT_y, \quad A\omega_y^{\cdot} = rT_x^{\cdot}, \quad A\omega_z^{\cdot} = 0. \tag{7}$$

From the last equation we obtain

$$\omega_z = \text{const}. \tag{8}$$

Let $\boldsymbol{u}$ denote the velocity of the point of tangency S. The instantaneous motion of the sphere is the composition of an advancing motion with a velocity $\boldsymbol{v}_0$ of the center of mass O and of a rotation about an axis passing through O with an angular velocity $\boldsymbol{\omega}$. Therefore $\boldsymbol{u} = \boldsymbol{v}_0 + \overline{OS} \times \boldsymbol{\omega}$, whence:

$$u_x = x_0^{\cdot} + r\omega_y, \quad u_y = y_0^{\cdot} - r\omega_x, \quad u_z = 0. \tag{9}$$

Calculating the derivatives with respect to time, we obtain:

$$u_x^{\cdot} = x_0^{\cdot\cdot} + r\omega_y^{\cdot}, \quad u_y^{\cdot} = y_0^{\cdot\cdot} - r\omega_x^{\cdot}, \quad u_z^{\cdot} = 0, \tag{10}$$

whence by (3) and (7):

$$u_x^{\cdot} = (1/m + r^2/A)\,T_x, \quad u_y^{\cdot} = (1/m + r^2/A)\,T_y, \quad u_z^{\cdot} = 0. \tag{11}$$

Multiplying both sides of the first of the equations (11) by u_x, and both sides of the second by u_y, and adding, we obtain:

$$u_x u_x^{\cdot} + u_y u_y^{\cdot} = (1/m + r^2/A)(T_x u_x + T_y u_y). \tag{12}$$

If $\boldsymbol{u} \neq 0$, then $\boldsymbol{T}$ has the direction of $\boldsymbol{u}$, but an apposite sense. Consequently $\boldsymbol{Tu} \leq 0$ constantly, i. e. $T_x u_x + T_y u_y \leq 0$, from which by (12) $u_x u_x^{\cdot} + u_y u_y^{\cdot} \leq 0$. Since $u_x u_x^{\cdot} + u_y u_y^{\cdot} = \frac{1}{2}\mathrm{d}(u_x^2 + u_y^2)/\mathrm{d}t$, it follows that $|\boldsymbol{u}|^2 = u_x^2 + u_y^2$ is a non-increasing function. Therefore, if $\boldsymbol{u} = 0$ at a certain moment, then from this moment on $\boldsymbol{u} = 0$ constantly.

Let us assume that, during a certain interval of time, $\boldsymbol{u}$ was different from zero and the friction $\boldsymbol{T}$ had the direction of $\boldsymbol{u}$ (but an opposite sense); we can therefore assume that $\boldsymbol{T} = \lambda\boldsymbol{u}$, where $\lambda < 0$ (while λ depends on the time). Hence $\lambda u_x = T_x$ and $\lambda u_y = T_y$, whence by (11)

$$u_x y_y^{\cdot} - u_x^{\cdot} u_y = 0. \tag{13}$$

From equation (13) it follows that $\boldsymbol{u}$ has a constant direction: for putting $u = |\boldsymbol{u}|$ and denoting by φ the angle between $\boldsymbol{u}$ and the x-axis, we get $u_x = u\cos\varphi$, $u_y = u\sin\varphi$, $u_x^{\cdot} = u^{\cdot}\cos\varphi - u\varphi^{\cdot}\sin\varphi$, and $u_y^{\cdot} = u^{\cdot}\sin\varphi + u\varphi^{\cdot}\cos\varphi$; hence $u_x u_y^{\cdot} - u_x^{\cdot} u_y = u^2\varphi^{\cdot}$, whence by (13) $u^2\varphi^{\cdot} = 0$, and since $u^2 \neq 0$, $\varphi^{\cdot} = 0$, i. e. $\varphi = \text{const}$.

Now, since $\boldsymbol{T}$ has the direction of the velocity $\boldsymbol{u}$, the direction of the friction $\boldsymbol{T}$ is also constant. Under the assumption that the coefficient of friction μ is constant (p. 367) we obtain by (4) $T = \text{const}$.

Therefore: *during the entire time in which* $\boldsymbol{u} \neq 0$, $\boldsymbol{T} = \text{const}$.

Since the motion of a material point under the influence of a constant force takes place along a parabola (p. 82), *the centre of mass of the sphere describes a parabola* (whose axis is parallel to the direction of $\boldsymbol{T}$) *during the entire time in which* $\boldsymbol{u} \neq 0$ (i. e. *in which the point of tangency of the sphere with the plane Π has a velocity different from zero*).

Let us assume that at $t = 0$:

$$x_0 = 0, \quad y_0 = 0, \quad z_0 = r, \quad x_0^{\cdot} = a, \quad y_0^{\cdot} = b, \quad z_0^{\cdot} = 0, \qquad \omega_x = \omega_x^0, \quad \omega_y = \omega_y^0, \quad \omega_z = \omega_z^0. \tag{14}$$

Hence by (9) the initial velocity $\boldsymbol{u}_0$ of the point of contact S has the projections:

$$u_x^0 = a + r\omega_y^0, \quad u_y^0 = b - r\omega_x^0, \quad u_z^0 = 0. \tag{15}$$

Let us assume that $\boldsymbol{u}_0 \neq 0$ and give to the y-axis a direction and sense of the velocity $\boldsymbol{u}_0$. Therefore by (15) there will be the following relations among the given initial values:

$$u_x^0 = a + r\omega_y^0 = 0, \quad u_y^0 = b - r\omega_x^0 > 0. \tag{16}$$

Now, because $\boldsymbol{u}$ and $\boldsymbol{T}$ have the same directions, but opposite senses,

$$T_x = 0, \quad T_y = -\mu m g. \tag{17}$$

After integration and consideration of the initial conditions, we obtain from equations (3) and (7):

$$x_0 = at, \quad y_0 = -\tfrac{1}{2}\mu g t^2 + bt, \quad z_0 = r, \tag{18}$$

$$\omega_x = \mu m g r t/A + \omega_x^0, \quad \omega_y = \omega_y^0, \quad \omega_z = \omega_z^0. \tag{19}$$

Substituting the values from (18) and (19) in equations (9), we obtain in view of (16):

$$u_x = 0, \quad u_y = (b - r\omega_x^0) - (1 + mr^2/A)\,gt\mu, \quad u_z = 0. \tag{20}$$

Since $b - r\omega_x^0 > 0$ it follows by (16), that after the time

$$t_1 = \frac{b - r\omega_x^0}{(1 + mr^2/A)\,\mu g} \tag{21}$$

$u_x = 0$, $u_y = 0$, and $u_z = 0$, i. e. $\boldsymbol{u} = 0$; and after the time t_1, $\boldsymbol{u} = 0$ constantly. Hence by (11) $T_x = 0$ and $T_y = 0$, i. e. $\boldsymbol{T} = 0$ constantly. From equations (3) and (7) we obtain then:

$$x_0^{\cdot\cdot} = 0, \quad y_0^{\cdot\cdot} = 0, \quad z_0^{\cdot\cdot} = 0, \quad \omega_x^{\cdot} = 0, \quad \omega_y^{\cdot} = 0, \quad \omega_z^{\cdot} = 0.$$

Therefore: *from the time* t_1, $\boldsymbol{v}_0 =$ const. *and* $\omega =$ const. *constantly, i. e. the centre of the sphere will move with a uniform motion along a straight line from the time* t_1 *on, while the instantaneous angular velocity of the sphere will be constant.*

§ 10. Foucault's gyroscope. This is the name we give to a heavy body having an axis of symmetry and suspended at the centre of mass (*the so--called Cardan's suspension*).

Since the force of gravity acts at the center of mass, in the case when no other forces act on the body, the motion of a gyroscope reduces to a rotation of the body about the centre of mass under the action of no forces.

If the body is set spinning about the centre of mass and initially the axis of symmetry is the instantaneous axis of rotation, then the axis of symmetry will maintain a constant direction in space. This follows from the theorem given on p. 401 and from the observation that the axis of symmetry is a central axis of inertia of the body.

It is true that the axis of symmetry will move relative to the earth, however, this will only be an apparent motion (induced by the rotation of the earth): for if the axis of symmetry is directed towards some fixed star, then the axis will point to it constantly.

We shall consider here the cases in which the axis of symmetry is not free, but is confined either to a meridional plane or to a horizontal plane.

Motion of the axis of symmetry in a meridional plane. Let it be possible for a body (suspended at the centre of mass) to move only in a meridional plane passing through a given point on the earth. We can assume that the forces (reactions) holding the axis in the meridional plane are perpendicular to this plane and have their points of application on the axis of symmetry.

Let us denote by $\boldsymbol{\omega}_1$ the angular velocity vector of the earth and set:

$$\omega_1 = |\boldsymbol{\omega}_1|. \tag{1}$$

Let us take the centre of mass O of the body as the origin of the coordinate system (x, y, z), giving the z-axis the direction and sense of the

angular velocity $\boldsymbol{\omega}_1$ of the earth, and the x-axis a horizontal direction with a sense towards the east (Fig. 303).

The yz-plane will consequently be a meridional plane, and at a given place the z-axis will make an angle of $90° - \varphi$ with the vertical, where φ denotes the latitude of this place.

In addition, let us select a second coordinate system (ξ, η, ζ) whose origin is at O, taking the axis of symmetry of the body as the ζ-axis, and the x-axis as the ξ-axis. The plane $\eta\zeta$ will therefore be identical with the meridional plane yz. The position of the system (ξ, η, ζ) is defined by the angle ϑ which the axes ζ and z make with each other (where the angle ϑ is defined as the angle through which it is necessary to rotate the z-axis from right to left with respect to the x-axis, in order that the positive directions of the axes z and ζ coincide with each other).

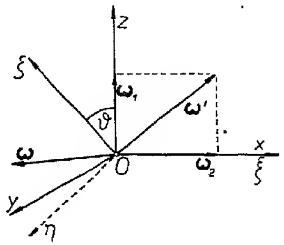

Fig. 303.

Let $\boldsymbol{\omega}'$ be the instantaneous angular velocity of the system (ξ, η, ζ) with respect to an inertial frame, which we take to be a frame attached to the sun and the fixed stars. It is easy to see that $\boldsymbol{\omega}'$ is the resultant of the instantaneous angular velocity $\boldsymbol{\omega}_2$ of the system (ξ, η, ζ) relative to (x, y, z) and of the angular velocity $\boldsymbol{\omega}_1$ of the system (x, y, z) relative to the inertial frame. Consequently

$$\boldsymbol{\omega}' = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2. \tag{2}$$

Since the vector $\boldsymbol{\omega}_1$ has by hypothesis the direction and sense of the z-axis, its projections on the axes of the system (ξ, η, ζ) are in virtue of (1):

$$\omega_{1_\xi} = 0, \quad \omega_{1_\eta} = -\omega_1 \sin\vartheta, \quad \omega_{1_\zeta} = \omega_1 \cos\vartheta. \tag{3}$$

The system (ξ, η, ζ) rotates about the ξ-axis relative to the system (x, y, z). Since the angle of rotation is ϑ, the instantaneous angular velocity has the direction of the ξ-axis and its component with respect to the ξ-axis is $\vartheta^{\cdot}$. Consequently:

$$\omega_{2_\xi} = \vartheta^{\cdot}, \quad \omega_{2_\eta} = 0, \quad \omega_{2_\zeta} = 0. \tag{4}$$

In virtue of (2) — (4):

$$\omega'_\xi = \vartheta^{\cdot}, \quad \omega'_\eta = -\omega_1 \sin\vartheta, \quad \omega'_\zeta = \omega_1 \cos\vartheta. \tag{5}$$

Let $\boldsymbol{\omega}$ denote the instantaneous angular velocity of a body relative to the inertial frame. The vector $\boldsymbol{\omega}$ can be considered as the composition of

the instantaneous angular velocity $\boldsymbol{\omega}_3$ of the body relative to the system (ξ, η, ζ) and the velocity $\boldsymbol{\omega}'$ of the system (ξ, η, ζ) relative to the inertial frame. Therefore $\boldsymbol{\omega} = \boldsymbol{\omega}_3 + \boldsymbol{\omega}'$. Since the motion of the body relative to (ξ, η, ζ) is a rotation about the ζ-axis, it follows that $\omega_{3\xi} = 0$ and $\omega_{3\eta} = 0$, whence:

$$\omega_\xi = \omega'_\xi, \quad \omega_\eta = \omega'_\eta, \quad \omega_\zeta = \omega_{3\zeta} + \omega'_\zeta \tag{6}$$

(in addition, since ω_1 is very small, by (5) ω'_ζ is also small; hence for all practical purposes $\omega_\zeta = \omega_{3\zeta}$). Putting $\omega_\zeta = \omega$, we obtain by (5) and (6):

$$\omega_\xi = \vartheta^\cdot, \quad \omega_\eta = -\omega_1 \sin\vartheta, \quad \omega_\zeta = \omega. \tag{7}$$

The axes ξ, η, ζ, are the central axes of inertia of the body, because ζ is the axis of symmetry and O the centre of mass. Denoting the angular momentum with respect to O by $\mathbf{K}$, the moments of inertia with respect to ξ and η by A, and the moment of inertia with respect to ζ by C, we obtain by (III), p. 394, and (7):

$$K_\xi = A\vartheta^\cdot, \quad K_\eta = -A\omega_1 \sin\vartheta, \quad K_\zeta = C\omega, \tag{8}$$

whence after differentiation:

$$K^\cdot_\xi = A\vartheta^{\cdot\cdot}, \quad K^\cdot_\eta = A\omega_1\vartheta^\cdot \cos\vartheta, \quad K^\cdot_\zeta = C\omega^\cdot. \tag{9}$$

The moment of the weight with respect to O is zero. The moment of the forces holding the axis of symmetry of the body in the plane of the meridian is zero with respect to the axes ξ and ζ, because these forces have their points of application on the ζ-axis and are parallel to the ξ-axis. Therefore, denoting by $\boldsymbol{M}$ the moment of the forces with respect to the centre of mass O, we obtain:

$$M_\xi = 0, \quad M_\zeta = 0. \tag{10}$$

To determine the motion of the body we apply equations (II), p. 398. From these equations, after substituting in them the values from (5), (8), (9), (10), and after reducing, we obtain:

$$\begin{aligned} &A\vartheta^{\cdot\cdot} - A\omega_1^2 \sin\vartheta\cos\vartheta + C\omega\omega_1\sin\vartheta = 0, \\ &-2A\omega_1\vartheta^\cdot\cos\vartheta + C\omega\vartheta^\cdot = M_\eta, \quad C\omega^\cdot = 0. \end{aligned} \tag{I}$$

In virtue of the last equation $\omega = \text{const}$. Dropping the term containing ω_1^2 from the first of the equations (I), because it is very small, we obtain

$$\vartheta^{\cdot\cdot} = -\frac{C\omega\omega_1}{A}\sin\vartheta. \tag{11}$$

Since $\omega = \text{const}$, we can give the ζ-axis a sense such that $\omega > 0$ constantly, i. e. such that the rotation of the body relative to the axis of

symmetry ζ is from right to left. Under this assumption $C\omega\omega_1 / A > 0$. Equation (11) therefore has the form of the differential equation for the simple pendulum (p. 130, formula (I)). The positions of equlibrium occur for $\vartheta = 0$ and $\vartheta = \pi$.

The axis of symmetry of the body will therefore oscillate about the z-axis, i. e. about a line parallel to the axis of the earth. The axis of the body can be at rest only for $\vartheta = 0$ or for $\vartheta = \pi$, i. e. only when it is parallel to the axis of the earth. Therefore, determining the position of equilibrium of the axis of the body, we obtain the direction of the axis of the earth. Since the axis of the earth makes an angle of $90° - \varphi$ with the vertical at a given place, *we can in this manner obtain the latitude φ of the given place.*

It can be shown that $\vartheta = 0$ is the position of stable equilibrium, and $\vartheta = \pi$ that of unstable equilibrium. Hence the ζ-axis tends to assume a position such that its direction and sense agree with the direction of the axis of the earth and the sense of the vector $\boldsymbol{\omega}_1$. From formula (3), p. 130, it follows that the period of oscillation of the axis of the body (when the initial angle ϑ_0 is small and $\vartheta^\cdot_0 = 0$) is

$$T = 2\pi\sqrt{A / C\omega\omega_1}. \tag{12}$$

The period of oscillation is large because ω_1 is small (ca 0.00007 sec^{-1}). However, we can decrease it by increasing ω, i. e. by spinning the body faster about its own axis of symmetry.

Motion of the axis in a horizontal plane. Let us now assume that the axis of symmetry of a body can move only in a horizontal plane. We can therefore assume that the reactions holding the axis horizontally have their points of application on this axis and have a vertical direction.

Let us choose two systems of coordinates (x, y, z) and (ξ, η, ζ) whose common origin is at the center of mass O of the body (Fig 304). Let us give the y-axis a sense vertically upwards, the x-axis a sense towards the east, and the z-axis towards the north. Let us take the axis of symmetry of the body as the ζ-axis and the y-axis as the η-axis. Therefore the $\xi\zeta$ plane will be constantly horizontal. The position of the system (ξ, η, ζ) is defined by the angle ϑ between the axes ζ and z (where ϑ is defined as the angle through which it is necessary to rotate the z-axis about the y-axis from right to left, in order that the positive directions of the axes ζ and z coincide with each other).

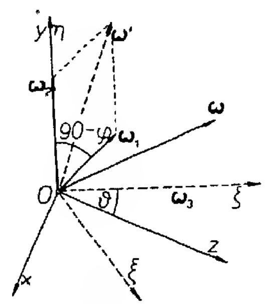

Fig. 304.

The instatanenous angular velocity of the system (x, y, z) with respect to the inertial frame is equal to $\boldsymbol{\omega}_1$ (i. e. to the angular velocity of the earth). The vector $\boldsymbol{\omega}_1$ lies in the yz plane and makes an angle of $90° - \varphi$ with the y-axis (where φ denotes the latitude of the given place). Let $\omega_1 = |\boldsymbol{\omega}_1|$. The projections of $\boldsymbol{\omega}_1$ on the axes ξ, η, ζ, are therefore:

$$\omega_{1_\xi} = \omega_1 \cos\varphi \sin\vartheta, \quad \omega_{1_\eta} = \omega_1 \sin\varphi, \quad \omega_{1_\zeta} = \omega_1 \cos\varphi \cos\vartheta. \tag{13}$$

The instantaneous angular velocity $\boldsymbol{\omega}_2$ of the system (ξ, η, ζ) with respect to the system (x, y, z) is equal to $\vartheta^{\cdot}$ and has the direction of the η-axis, because (ξ, η, ζ) rotates about η relative to (x, y, z). Consequently, the instantaneous angular velocity $\boldsymbol{\omega}'$ of the system (ξ, η, ζ) with respect to the inertial frame is the composition of the angular velocity $\boldsymbol{\omega}_2$ and the angular velocity $\boldsymbol{\omega}_1$, whence by (13):

$$\omega'_\xi = \omega_1 \cos\varphi \sin\vartheta, \quad \omega'_\eta = \vartheta^{\cdot} + \omega_1 \sin\varphi, \quad \omega'_\zeta = \omega_1 \cos\varphi \cos\vartheta. \tag{14}$$

Let $\boldsymbol{\omega}$ denote the instantaneous angular velocity of a body relative to the inertial frame. Since the instantaneous motion of the body relative to the system (ξ, η, ζ) is an instantaneous rotation about the ζ-axis, the projections of the instantaneous angular velocity $\boldsymbol{\omega}_3$ of the body relative to the system (ξ, η, ζ) on the axes of this system are: $\omega_{3_\xi} = 0$, $\omega_{3_\eta} = 0$. Since $\boldsymbol{\omega} = \boldsymbol{\omega}' + \boldsymbol{\omega}_3$, we obtain by (14) (putting $\omega_\zeta = \omega$):

$$\omega_\xi = \omega_1 \cos\varphi \sin\vartheta, \quad \omega_\eta = \vartheta^{\cdot} + \omega_1 \sin\varphi, \quad \omega_\zeta = \omega. \tag{15}$$

Denoting the angular momentum with respect to O by $\boldsymbol{K}$, the moments of inertia with respect to the axes ξ and η by A, and with respect to ζ by C, we obtain by (III), p. 394, and (15):

$$K_\xi = A\omega_1 \cos\varphi \sin\vartheta, \quad K_\eta = A(\vartheta^{\cdot} + \omega_1 \sin\varphi), \quad K_\zeta = C\omega, \tag{16}$$

whence by differentiation:

$$K^{\cdot}_\xi = A\omega_1\vartheta^{\cdot} \cos\varphi \cos\vartheta, \quad K^{\cdot}_\eta = A\vartheta^{\cdot\cdot}, \quad K^{\cdot}_\zeta = C\omega^{\cdot}. \tag{17}$$

As the reactions holding the axis of the body in the horizontal plane have their points of application on the ζ-axis of the body and are perpendicular to $\xi\zeta$, denoting by $\boldsymbol{M}$ the moment of the acting forces, we obtain:

$$M_\eta = 0, \quad M_\zeta = 0. \tag{18}$$

From formulae (II), p. 398, we obtain after substituting the values from (17), (16), (14), and (18):

$$\begin{gathered} A\omega_1\vartheta^{\cdot} \cos\varphi \cos\vartheta + A(\vartheta^{\cdot} + \omega_1 \sin\varphi)\,\omega_1 \cos\varphi \cos\vartheta - \\ - C\omega(\vartheta^{\cdot} + \omega_1 \sin\varphi) = M_\xi, \\ A\vartheta^{\cdot\cdot} + C\omega\omega_1 \cos\varphi \sin\vartheta - A\omega_1^2 \cos^2\varphi \cos\vartheta \sin\vartheta = 0, \quad C\omega^{\cdot} = 0. \end{gathered} \tag{II}$$

In virtue of the last of the equations (II) $\omega = \text{const}$. Dropping the term ω_1^2 in the second of these equations, because it is very small, we get

$$\vartheta^{\cdot\cdot} = -\frac{C\omega\omega_1 \cos\varphi}{A} \sin\vartheta. \tag{19}$$

Let us give the ζ-axis a sense so that $\omega > 0$. Then

$$C\omega\omega_1 \cos\varphi \,/\, A > 0$$

and equation (19) assumes the form of the equation of the simple pendulum ((I), p. 130). The positions of equilibrium occur for $\vartheta = 0$ and $\vartheta = \pi$.

The axis of symmetry of the body will therefore oscillate about the z-axis, i. e. about a horizontal axis running from south to north. The axis of the body can be at rest only for $\vartheta = 0$ or $\vartheta = \pi$, i. e. only when it lies in a meridional plane. Consequently, determining the position of equilibrium of the axis of a body, *we obtain the direction of the meridian: the body can therefore be used as a compass.*

Let us still note that, as before, $\vartheta = 0$ corresponds to stable equilibrium, and $\vartheta = \pi$ to unstable equilibrium.

The period of oscillation is obtained from formula (3), p. 130:

$$T = 2\pi\sqrt{A \,/\, C\omega\omega_1 \cos\varphi}. \tag{20}$$

It will be smallest on the equator (i. e. for $\varphi = 0$). On the pole, however (i. e. for $\varphi = 90°$), every position will be a position of equilibrium, as follows from formula (19). For when $\varphi = 90°$, $\vartheta^{\cdot\cdot} = 0$; as one comes closer to the pole $T \to \infty$.

The results obtained found confirmation in experiments which proves the earth's rotation about an axis. Such experiments were first performed by L. FOUCAULT.

CHAPTER IX

PRINCIPLE OF VIRTUAL WORK

§ 1. Holonomo-scleronomic systems. In the case of the equilibrium of a constrained system of material points or of rigid bodies, in which friction does not appear, the constraints are independent of the time and usually depend on the fact that only certain positions of the system are possible and others impossible.

Examples are: a material point constrained to remain on a fixed curve or surface, a system of two material points connected by a rigid massless wire, a rigid body having a fixed point or axis, a system of rigid bodies tangent to one another or joint-connected, etc.

The constraints of a system can be induced in various ways: e. g. by means of rigid bodies, supports, etc. It turns out, however, that in the case when there is no friction, the conditions of equilibrium of the acting forces do not depend on the origin of the constraints, but only on what positions are possible. Moreover, if it is a matter of investigating the equilibrium of a system, then it is sufficient to know only those positions compatible with the constraints which are near the position investigated.

We shall first consider the manner in which it is possible to represent the position of a system compatible with the constraints. We shall first study this matter by means of examples.

BILATERAL CONSTRAINTS

Example 1. Let us assume that a material point is constrained to remain on a certain surface S. The constraints can be defined by giving the equation of this surface, for example, in the form

$$F(x, y, z) = 0. \tag{1}$$

Only those positions of the point will be possible in which the coordinates x, y, z, satisfy equation (1). To investigate the equilibrium of

the point at some position $A(x, y, z)$ it is sufficient if (1) is the equation of an element of the surface on which this point lies.

Example 2. Let us assume that a material point is constrained to remain on the curve C whose equations are:

$$F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0; \tag{2}$$

then the coordinates of the point must satisfy the equations (2). To examine the conditions for the equilibrium of the point it is sufficient if the equations (2) represent only an arc of the curve C within which the material point lies.

Example 3. If a system consisting of two points A_1 and A_2 is a rigid system (i. e. the distance of the points $A_1A_2 = \text{const.} = d$), then the coordinates x_1, y_1, z_1, and x_2, y_2, z_2, of these points must satisfy the equation

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - d^2 = 0. \tag{3}$$

If in addition, for example, the sum of the distances of the points from the origin O of the coordinate system is constant and equal to h, then the coordinates of the points must also satisfy the equation

$$\sqrt{x_1^2 + y_1^2 + z_1^2} + \sqrt{x_2^2 + y_2^2 + z_2^2} - h = 0. \tag{4}$$

Example 4. If a system composed of n points:

$$A_1(x_1, y_1, z_1),\ A_2(x_2, y_2, z_2),\ \ldots,\ A_n(x_n, y_n, z_n)$$

is a rigid system (i. e. such that the mutual distances of its points are constant), then the coordinates of each pair of points A_i, A_j must satisfy the equation

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - r_{ij}^2 = 0, \tag{5}$$

where $r_{ij} = A_iA_j$. There are as many equations (5) as there are pairs of points, i. e. $\frac{1}{2}n(n-1)$.

In examples 1—4 the constraints were expressed by equations of the form

$$F(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) = 0, \tag{6}$$

where F is a function defined in a certain region of the variables $x_1, \ldots, z_n$ and is independent of the time t.

The constraints represented by equations (6) are called *holonomic bilateral constraints independent of the time.*

UNILATERAL CONSTRAINTS

Example 5. A point is to remain within the sphere $x^2+y^2+z^2-r^2=0$ or on its surface (as e. g. a material point attached to an inextensible string of length r and suspended from the origin of the system). Hence the coordinates of the point must satisfy the inequality

$$x^2+y^2+z^2-r^2\leqq 0, \tag{7}$$

which states that the distance of the material point from the origin of the coordinate system is not greater than r.

Example 6. Two material points A_1, A_2 are connected by an inextensible string of length h passing through the origin O of the coordinate system. Consequently $OA_1+OA_2\leqq h$. The coordinates of the points therefore satisfy the inequality

$$\sqrt{x_1^2+y_1^2+z_1^2}+\sqrt{x_2^2+y_2^2+z_2^2}-h\leqq 0. \tag{8}$$

In examples 5 and 6 it was possible to express the constraints by inequalities of the form

$$\Phi(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n)\leqq 0. \tag{9}$$

Constraints defined by inequalities of the form (9) are called *holonomic unilateral constraints independent of the time.*

Remark. If the constraints are given by the inequality

$$\Psi(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n)\geqq 0,$$

then putting $\Psi=-\Phi$, we obtain the form (9) after changing the sign.

The position of a system in which the relation (9) assumes the form of an equality (i. e. in which $\Phi=0$) is called a *boundary position.*

In examples 5 and 6 the boundary position occurs when the string is in tension.

The constraints of a system can consist simultaneously of bilateral and unilateral constraints of the form (6) and (9). For example, if a material point is constrained to remain on the upper half of the sphere $x^2+y^2+z^2-r^2=0$, then the coordinates of the point must satisfy the relations:

$$x^2+y^2+z^2-r^2=0,\ z\geqq 0\ \text{(or } -z\leqq 0). \tag{10}$$

A system whose constraints can be represented by means of relations of the form (6) or (9) is called a *holonomo-scleronomic* system. We say that the relations (6) and (9) represent the *constraints in a finite form.*

In this chapter we shall deal exclusively with holonomo-scleronomic systems.

Degrees of freedom of a system. When we investigate the equilibrium of a system, we assume in general that for positions close to those investigated, the bilateral constraints consist in the fact that the coordinates of the points of the system must satisfy the equations:

$$\begin{gathered}F_1(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n)=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ F_m(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n)=0,\end{gathered} \tag{11}$$

which we write compactly as

$$F_j(x_1, \ldots, z_n)=0 \qquad (j=1, 2, \ldots, m). \tag{I}$$

In general, we assume that the functions F_j are continuous and have continuous partial derivatives of the first and second order in a certain region of the values $x_1, \ldots, z_n$, defining the given position of the system.

Moreover, we assume that these functions are *independent* in this region, i. e. that no one of them is a function of those remaining.

We shall finally assume that $m<3n$. For were $m=3n$, then the system of equations (I) would have in general only one solution and hence only one position of the body would be possible. On the other hand, were $m>3n$, then the system of equations (I) would not have in general any solution, since the number of unknowns would be smaller than the number of equations.

The number $k=3n-m$ is called the *number of degrees of freedom.*

Knowing k variables from among the variables $x_1, \ldots, z_n$, we can calculate from equations (I) the remaining variables: their number is $3n-k=m$.

If a system is free, then $m=0$. i.e. $k=3n$. In particular, a free material point therefore has three degrees of freedom. We can then choose all three of its coordinates arbitrarily.

On the other hand, if a point is constrained to remain on a surface (as in example 1), then its coordinates have to satisfy only the equation (1). Consequently $m=1$, i. e. $k=3\cdot 1-1=2$. The point constrained to the surface therefore has two degrees of freedom. Then one of its coordinates depends on the remaining two.

Finally, if the point is constrained to a curve (as in example 2), then its coordinates must satisfy the two equations (2). Consequently $k=3\cdot 1-2=1$ and the point has only one degree of freedom: knowing one of its coordinates, we can determine the remaining two.

A rigid system of two points (example 3) has $3 \cdot 2 - 1 = 5$ degrees of freedom. Nevertheless, if this system is to satisfy both equations (3) and (4), then $k = 3 \cdot 2 - 2 = 4$.

Generally, let a rigid system consist of n material points (as in example 4). The coordinates of these points must satisfy $\frac{1}{2}n(n-1)$ equations (5). However, these equations are not independent of each other for $n > 3$.

Let us note that the position of a rigid system is determined by giving the position of three of its non-collinear points (e. g. A_1, A_1, A_3), (p. 313). Hence, knowing the coordinates $x_1, y_1, z_1, x_2, y_2, z_2$, and x_3, y_3, z_3, of the points A_1, A_2, A_3, we shall be able to calculate the coordinates of the points $A_4, A_5, \ldots, A_n$, from equations (5).

Among the coordinates of the points A_1, A_2, A_3, there are three equations of the form (5), expressing the fact that the distances A_1A_2, A_1A_3, and A_2A_3, are constant magnitudes. From these equations we can in general determine three unknown coordinates, if we know the six remaining ones. We see, therefore, that if we know a certain six of tho $3n$ coordinates $x_1, \ldots, z_n$, we can calculate the remaining ones from the equations (5). Consequently the number of degrees of freedom is $k = 6$.

Therefore: *a rigid system of points has six degrees of freedom.*

The number of independent equations is $m = 3n - k$; hence $m = 3n - 6$. From among $\frac{1}{2}n(n-1)$ equations (5) there are therefore only $3n - 6$ independent ones.

§ 2. Virtual displacements. Point on a surface. Let us assume that a material point is to remain constantly on a certain surface S and that it is at the point A of this surface.

Let us displace the point from position A to position B. The displacement $\overline{AB}$ is said to be *possible* if B also lies on the surface S. In the contrary case the displacement $\overline{AB}$ is called an *impossible displacement*.

If a material point at A is given a velocity $\mathbf{v}$, then this velocity is said to be *possible* or *compatible with the constraints* when the point can possess it while moving on the surface. In the contrary case this velocity is said to be *impossible* or *incompatible with the constraints*.

It is easy to see that every vector tangent to a surface at the point A represents a possible velocity. Conversely, possible velocities are tangent to the surface.

An important role is played by displacements proportional to possible velocities, i. e. those that can be represented by vectors equal to possible velocity vectors.

Displacements proportional to velocities possible at the point A are called *virtual displacements* at this point.

A virtual displacement therefore has a direction tangent to the surface, but an arbitrary sense and magnitude. In general, virtual displacements are not possible displacements. However, they are possible displacements when the surface S is a plane, for instance.

Let the surface S have the equation

$$F(x, y, z) = 0. \tag{1}$$

If a material point is at the point A of the surface S, then its coordinates x, y, z, satisfy equation (1). Let us suppose that the material point moves on the surface S in anentirely arbitrary manner. Equation (1) is therefore satisfied constantly. Differentiating (1) with respect to the time t, we obtain

$$\frac{\partial F}{\partial x}x^{\cdot} + \frac{\partial F}{\partial y}y^{\cdot} + \frac{\partial F}{\partial z}z^{\cdot} = 0. \tag{2}$$

Denoting by $\mathbf{v}$ the velocity of the point, we have $v_x = x^{\cdot}$, $v_y = y^{\cdot}$, $v_z = z^{\cdot}$; consequently

$$\frac{\partial F}{\partial x}v_x + \frac{\partial F}{\partial y}v_y + \frac{\partial F}{\partial z}v_z = 0. \tag{3}$$

Hence we see that the possible velocities must satisfy equation (3). The partial derivatives appearing in this equation are proportional to the direction numbers of the normal to the surface at the point A. Equation (3) therefore expresses the fact that the velocity $\mathbf{v}$ is perpendicular to the normal, i. e. that it lies in the tangent plane.

Conversely, if some velocity satisfies equation (3), then it is a possible velocity.

Let us denote by $\overline{\delta s}$ an arbitrary displacement of the point A, and by $\delta x, \delta y, \delta z$, the projections of this displacement on the coordinate axes (Fig. 305). According to the definition, the virtual displacement is proportional to a possible velocity $\mathbf{v}$. The displacement $\overline{\delta s} = \mathbf{v}$ will consequently be a virtual displacement.

In virtue of (3) the projections of the virtual displacement therefore satisfy the equation

$$\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z = 0. \tag{4}$$

Conversely, if the projections of some vector $\overline{\delta s}$ satisfy equation (4), then $\overline{\delta s}$ is a virtual displacement.

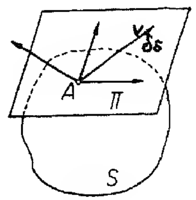

Fig. 305.

We can therefore say that *a virtual displacement is a vector whose projections satisfy equation* (4).

Example 1. A point is constrained to lie on the sphere $x^2 + y^2 + z^2 - r^2 = 0$. Let $A(x, y, z)$ be an arbitrary point on this sphere. The virtual displacements at the point A satisfy the equation

$$2x\,\delta x + 2y\,\delta y + 2z\,\delta z = 0, \quad \text{whence} \quad x\,\delta x + y\,\delta y + z\,\delta z = 0.$$

Assuming e. g. that $z \neq 0$, we obtain

$$\delta z = -(x\,\delta x + y\,\delta y)\,/\,z. \tag{5}$$

Choosing arbitrary δx, δy, and taking the value δz from (5), we get a set of numbers δx, δy, δz, representing the projections of the virtual displacement at the point A.

Point on a curve. Suppose that a material point is constrained to remain on a fixed curve L defined by the equations:

$$F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0. \tag{6}$$

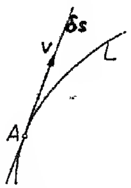

Fig. 306.

If the material point is at the point A, then the coordinates x, y, z, of this point satisfy equations (6). It is easy to see that the material point can have only those velocities whose directions are tangent to the curve L at the point A (Fig. 306). By definition, therefore, the virtual displacements have directions tangent to the curve, but arbitrary senses and lengths.

If the point moves along the curve L, equations (6) are satisfied constantly. Differentiating them, we get:

$$\frac{\partial F_1}{\partial x} x^{\cdot} + \frac{\partial F_1}{\partial y} y^{\cdot} + \frac{\partial F_1}{\partial z} z^{\cdot} = 0, \quad \frac{\partial F_2}{\partial x} x^{\cdot} + \frac{\partial F_2}{\partial y} y^{\cdot} + \frac{\partial F_2}{\partial z} z^{\cdot} = 0. \tag{7}$$

Denoting by $\overline{\delta s}$ the virtual displacement, and by δx, δy, δz, its projections, we can assume according to the definition $\overline{\delta s} = \mathbf{v}$, whence $\delta x = v_x = x^{\cdot}$, etc. Hence in virtue of (7):

$$\frac{\partial F_1}{\partial x}\delta x + \frac{\partial F_1}{\partial y}\delta y + \frac{\partial F_1}{\partial z}\delta z = 0, \quad \frac{\partial F_2}{\partial x}\delta x + \frac{\partial F_2}{\partial y}\delta y + \frac{\partial F_2}{\partial z}\delta z = 0. \tag{8}$$

Conversely, if some displacement $\overline{\delta s}$ satisfies equations (8), then it is a virtual displacement.

Example 2. A point is constrained to a curve defined by equations:

$$x^2 + y^2 + z^2 - r^2 = 0, \quad x^2 + 2y^2 - z = 0. \tag{9}$$

Let the point $A(x, y, z)$ lie on this line. The virtual displacement $\overline{\delta s}$ at the point A therefore satisfies the equations:

$$x\,\delta x + y\,\delta y + z\,\delta z = 0, \quad 2x\,\delta x + 4y\,\delta y - \delta z = 0. \tag{10}$$

If $x \neq 0$ and $y \neq 0$, then we get from (10):

$$\delta x = -(1 + 4z)\,\delta z\,/\,2x, \quad \delta y = (1 + 2z)\,\delta z\,/\,2y.$$

Hence, selecting δz arbitrarily and then determining δx, δy, from the last equations, we obtain a set of numbers δx, δy, δz, defining the virtual displacement at A.

On the other hand, if $x = 0$, for example, then because $z \geqq 0$ (which follows from the second of the equations (9)), we obtain by (10) $\delta y = 0$ and $\delta z = 0$. In this case the virtual displacement will consequently have the projections δx, 0, 0, where δx is an arbitrary number. Therefore: the virtual displacement has the direction of the x-axis.

Holonomo-scleronomic systems. Let us now define virtual displacements in the general case.

Let there be given a holonomo-scleronomic system of n material points $A_1, A_2, \ldots, A_n$.

The system of vectors $\overline{A_1B_1}, \overline{A_2B_2}, \ldots, \overline{A_nB_n}$ (representing the displacements of the individual points), is called briefly a *displacement of the system*. A displacement of the system of points $A_1, A_2, \ldots, A_n$ is said to be *possible*, if the final positions $B_1, B_2, \ldots, B_n$ are compatible with the constraints. In the contrary case the displacement of the system is said to be *impossible*.

Let us give a system of points in the position $A_1, A_2, \ldots, A_n$ the arbitrary velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. A system of these velocities is said to be a *system of possible velocities* if the points can have these velocities and move compatible with the constraints. In the contrary case the system of velocities is said to be *impossible*.

A *virtual displacement* of a system of material points in a certain position is said to be a displacement in which the separate points experience displacements proportional to the system of possible velocities.

Therefore, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, is a system of possible velocities, then the virtual displacement of a system is obtained by giving the points of the system the displacements:

$$\overline{\delta s_1} = \mathbf{v}_1, \quad \overline{\delta s_2} = \mathbf{v}_2, \ldots, \overline{\delta s_n} = \mathbf{v}_n. \tag{11}$$

Let us note that if a system is free, then every displacement of the system is a virtual displacement, because every system of velocities is a possible system.

We shall now consider the determination of the virtual displacements. We shall first discuss the case of bilateral constraints and then that of the unilateral constraints.

Bilateral constraints. Let us assume that the constraints are defined by the equations:

$$F_j(x_1, \ldots, z_n) = 0 \qquad (j = 1, 2, \ldots, m), \qquad (12)$$

which must be satisfied by the coordinates $x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n$, of the points of the system. Let us give the system an arbitrary motion compatible with the constraints. Differentiating equations (12), we get:

$$\frac{\partial F_j}{\partial x_1}\dot{x}_1 + \ldots + \frac{\partial F_j}{\partial z_n}\dot{z}_n = 0 \qquad (j = 1, 2, \ldots, m). \qquad (13)$$

Denoting by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ the velocities of the points of the system we therefore obtain $v_{1_x} = \dot{x}_1, \ldots, v_{n_z} = \dot{z}_n$, whence

$$\frac{\partial F_j}{\partial x_1} v_{1_x} + \ldots + \frac{\partial F_j}{\partial z_n} v_{n_z} = 0 \qquad (j = 1, 2, \ldots, m). \qquad (14)$$

Consequently every possible system of velocities must satisfy equations (14). Conversely, it is possible to show that if a system of velocities satisfies equations (14), then it is a possible system of velocities.

Let us assume that the displacement of a system, in which the displacements of the successive points $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$, is a virtual displacement. The velocities $\mathbf{v}_1, \ldots, \mathbf{v}_n$, defined by equations (11) therefore form a possible system of velocities, in view of which the equations (14) are satisfied. Denoting the projections of the displacements by $\delta x_1, \ldots, \delta z_n$, respectively, we obtain from (14)

$$\frac{\partial F_j}{\partial x_1}\delta x_1 + \ldots + \frac{\partial F_j}{\partial z_n}\delta z_n = 0 \qquad (j = 1, 2, \ldots, m) \quad (15)$$

or, written differently,

$$\sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i}\delta x_i + \frac{\partial F_j}{\partial y_i}\delta y_i + \frac{\partial F_j}{\partial z_i}\delta z_i\right) = 0 \qquad (j = 1, 2, \ldots, m). \qquad (\mathrm{I})$$

Therefore: every set of numbers $\delta x_1, \ldots, \delta z_n$, defining a virtual displacement of a system of points, satisfies the system of equations (I). Conversely, every set of numbers $\delta x_1, \ldots, \delta z_n$, satisfying the system of equations (I), defines a virtual displacement.

Making use of this fact, we can give the following definition of virtual displacements (equivalent to the preceding):

A virtual displacement of a system whose constraints are given by equations (12) *is one in which every displacement* $\delta x_1, \ldots, \delta z_n$, *satisfies equations* (I).

The system of equations (I) (or (15)) is a system of m equations with $3n$ unknowns $\delta x_1, \ldots, \delta z_n$.

We usually assume that equations (I) are independent of one another. Because of this we can choose arbitrarily $k = 3n - m$ unknowns from among the unknowns $\delta x_1, \ldots, \delta z_n$, and we can calculate those remaining from equations (I).

If the set of numbers $\delta x_1, \ldots, \delta z_n$, satisfies equations (I), then the set of numbers $-\delta x_1, \ldots, -\delta z_n$, obviously satisfies equations (I) also.

A virtual displacement of a system of points is said to be *reversible* if upon changing in it the senses of the displacements of all its points we again obtain a virtual displacement of the system.

We see, therefore, that *in the case of bilateral constraints the virtual displacements are reversible.*

Remark 1. The differential of the function $F_j(x_1, \ldots, z_n)$ is

$$\mathrm{d}F_j = \frac{\partial F_j}{\partial x_1}\mathrm{d}x_1 + \ldots + \frac{\partial F_j}{\partial z_n}\mathrm{d}z_n = \sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i}\mathrm{d}x_i + \frac{\partial F_j}{\partial y_i}\mathrm{d}y_i + \frac{\partial F_j}{\partial z_i}\mathrm{d}z_i\right).$$

We see from this that the left side of equation (I) is obtained by forming the differential of the function F_j formally and then writing $\delta x_1, \ldots, \delta z_n$, instead of $\mathrm{d}x_1, \ldots, \mathrm{d}z_n$.

Remark 2. Let $V(x_1, \ldots, x_n)$ be a given function defined in a certain region of the variables $x_1, \ldots, z_n$, and continuous together with its partial derivatives in this region. Let us choose an arbitrary set of values of the variables $x_1, \ldots, z_n$, and denote by $\delta x_1, \ldots, \delta z_n$ the arbitrary increments of these variables.

We do not denote here the increments by the symbols $\Delta x_1, \ldots, \Delta z_n$, because when the variables $x_1, \ldots, z_n$, are functions of the time t the symbols $\Delta x_1, \ldots, \Delta z_n$ usually denote the increments of these variables in the time Δt. The symbols $\delta x_1, \ldots, \delta z_n$, serve to indicate, instead, that the increments are entirely arbitrary and have nothing in common with the increments of the independent variables on which $x_1, \ldots, z_n$, depend (in this instance on the time t).

By Taylor's theorem we have:

$$V(x_1 + \delta x_1, \ldots, z_n + \delta z_n) - V(x_1, \ldots, z_n) =$$
$$= \frac{\partial V}{\partial x_1}\delta x_1 + \ldots + \frac{\partial V}{\partial z_n}\delta z_n + R, \qquad (16)$$

where the remainder R can be written in the form

$$R = \varepsilon(|\delta x_1| + \ldots + |\delta z_n|), \tag{17}$$

where ε depends on $\delta x_1, \ldots, \delta z_n$ and tends to zero together with $\delta x_1, \ldots, \delta z_n$.

Let us put

$$\delta V = \frac{\partial V}{\partial x_1}\,\delta x_1 + \ldots + \frac{\partial V}{\partial z_n}\,\delta z_n,$$

i. e.

$$\delta V = \sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\,\delta x_i + \frac{\partial V}{\partial y_i}\,\delta y_i + \frac{\partial V}{\partial z_i}\,\delta z_i\right). \tag{18}$$

By (16) we have

$$V(x_1 + \delta x_1, \ldots, z_n + \delta z_n) - V(x_1, \ldots, z_n) = \delta V + R. \tag{19}$$

If the number $|\delta x_1| + \ldots + |\delta z_n|$ is sufficiently small, then $|\varepsilon|$ is also a small number, and consequently by (17) $|R|$ is evanescent as compared with $|\delta x_1| + \ldots + |\delta z_n|$. In this case, therefore, δV represents approximately the increment of the function V. We express this usually by saying that δV denotes the increment of the function V corresponding to the "infinitesimal" increments $\delta x_1, \ldots, \delta z_n$, of the variables $x_1, \ldots, z_n$, or that for "infinitesimal" increments we have

$$V(x_1 + \delta x_1, \ldots, z_n + \delta z_n) - V(x_1, \ldots, z_n) = \delta V. \tag{20}$$

The preceding statement is not altogether exact, but it is convenient. We give it because physicists use it frequently.

By (18) equations (I), p. 426, defining the virtual displacements, can be written in the form

$$\delta F_j = 0 \qquad (j = 1, 2, \ldots, m). \tag{21}$$

In a position of a system compatible with the constraints we have $F_j = 0$ $(j = 1, 2, \ldots, m)$. Hence by (21) we have $F_j + \delta F_j = 0$ $(j = 1, 2, \ldots, m)$ for the virtual displacements. In virtue of (20) we can then say that after an "inifinitesimal" virtual displacement the system is likewise in a position compatible with the constraints. This gives rise to the definition of a virtual displacement as an "infinitesimal displacement compatible with the constraints". This definition (not exact, but rather intuitive) is to be understood in the above given sense.

Example 3. A system consisting of two material points A_1, A_2, has to maintain the constant distance $A_1A_2 = r$. The coordinates x_1, y_1, z_1, and x_2, y_2, z_2, of these points consequently satisfy the equation

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - r^2 = 0. \tag{22}$$

The virtual displacement of the system is threfore defined by the equation

$$\begin{aligned}(x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2) +\\ + (z_1 - z_2)\cdot(\delta z_1 - \delta z_2) = 0.\end{aligned} \tag{23}$$

Of the numbers $\delta x_1, \delta y_1, \delta z_1, \delta x_2, \delta y_2, \delta z_2$, we can hence choose five arbitrarily and determine the sixth from (23).

Let us assume, for example, that $x_1 = y_1 = z_1 = 0$, $x_2 = r$, $y_2 = z_2 = 0$. Then equation (23) assumes the form

$$-r(\delta x_1 - \delta x_2) + 0\cdot(\delta y_1 - \delta y_2) + 0\cdot(\delta z_1 - \delta z_2) = 0. \tag{24}$$

If we take $\delta x_1 = \delta x_2$, equation (24) will be satisfied for arbitrary values of $\delta x_2, \delta y_1, \delta y_2, \delta z_1, \delta z_2$. The virtual displacements of the points A_1, A_2, therefore have equal projections on the direction A_1A_2. This follows easily from the theorem given on p. 321, according to which the projections of the velocities of the points A_1, A_2, on the line joining these points are equal.

Example 4. **Unconstrained rigid body.** Let us assume that a rigid body is a rigid system of material points (p. 190).

Let us give the body an arbitrary advancing motion of velocity $\overline{\delta u}$. Since the points of the body will have this same velocity $\overline{\delta u}$, the virtual displacement of the body is obtained by assuming that the displacements of the points of the body were equal:

$$\overline{\delta s} = \overline{\delta u}. \tag{25}$$

It follows from this that *a translation of a body is a virtual displacement.*

Let us give a body an arbitrary rotation with an angular velocity $\overline{\delta\omega}$ about an axis passing through an (arbitrary) point O (Fig. 307). The velocity of an arbitrary point A of the body is equal to $\overline{\delta w} = \overline{OA} \times \overline{\delta\omega}$ (p. 46). The virtual velocity of the body is therefore obtained by giving the arbitrary point A a displacement $\overline{\delta s} = \overline{\delta w}$, i. e.

$$\overline{\delta s} = \overline{OA} \times \overline{\delta\omega}. \tag{26}$$

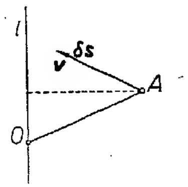

Fig. 307.

It follows from this that *the virtual displacement of a body is obtained by giving the points of the body displacements proportional to the velocity which they would have if the body were rotating about an arbitrary axis with an arbitrary angular velocity.*

Let us note that in this case the virtual displacement is not a possible displacement.

The most general instantaneous motion of a rigid body is the composition of an advancing motion and a rotation (p. 332).

Therefore: *the most general virtual displacement of a body is the composition of two virtual displacements of which one is a translation, and in the other the displacements of the points are proportional to their velocities during a rotation of the body about an axis.*

By (25) and (26) the most general virtual displacement of a body is obtained by choosing an arbitrary point O as well as the vectors $\overline{\delta u}$, $\overline{\delta\omega}$ and giving every point A of the body a displacement

$$\overline{\delta s} = \overline{\delta u} + \overline{OA} \times \overline{\delta\omega}. \tag{27}$$

So far we have assumed that the rigid body is free. Let us now consider several cases of a constrained body.

Fixed point. If a rigid body has one fixed point, e. g. the point O, then it can only rotate about this point. The instantaneous motion of the body is consequently an instantaneous rotation about a certain axis passing through O (p. 331). The most general virtual displacement of the body is obtained by giving the points of the body displacements defined by formula (26), in which $\overline{\delta\omega}$ can be chosen arbitrarily.

Fixed axis. If a rigid body has a fixed axis, then the motion of the body can only be a rotation about this axis. Therefore in a virtual displacement of a body the points have displacements defined by formula (26), where O is an arbitrary point of the axis and $\overline{\delta\omega}$ is an arbitrary vector having the direction of the axis.

Motion of a figure in a plane. The instantaneous motion of a plane figure in its plane is either an advancing motion or a rotation about the instantaneous centre of rotation (p. 326).

In the most general case, therefore, the virtual displacement of a plane figure is either a translation or a displacement, in which the displacements of the points of the figure are proportional to the velocities of these points in a rotation about a certain point lying in the plane of the figure.

Example 5. Two material points A_1 and A_2, joined by a rigid (massless) rod of length d, are constrained to remain on the curves C_1 and C_2 lying in a horizontal plane and given by the equations:

$$f_1(x, y) = 0, \quad f_2(x, y) = 0. \tag{28}$$

The following relations among the coordinates of the points A_1 and A_2 therefore hold:

$$f_1(x_1, y_1) = 0, \quad f_2(x_2, y_2) = 0, \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 - d^2 = 0. \tag{29}$$

Equations (29) define the constraints of the system. The virtual displacement consequently satisfies the equations:

$$\begin{gathered} \frac{\partial f_1}{\partial x_1}\,\delta x_1 + \frac{\partial f_1}{\partial y_1}\,\delta y_1 = 0, \quad \frac{\partial f_2}{\partial x_2}\,\delta x_2 + \frac{\partial f_2}{\partial y_2}\,\delta y_2 = 0, \\ (x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2) = 0. \end{gathered} \tag{30}$$

Hence we can select one of the numbers $\delta x_1, \delta y_1, \delta x_2, \delta y_2$, arbitrarily and obtain those remaining from equations (30). Let us note that the velocities of the points A_1, A_2, are tangent to the curves C_1, C_2. The instantaneous centre of rotation O of the rod A_1A_2 is therefore the point of intersection of the normals at the points A_1 and A_2 to the curves C_1 and C_2 (cf. example 1, p. 327). Since the instantaneous motion of the rod can only be a rotation about O, the points A_1 and A_2 can only have velocities whose directions are tangent to C_1 and C_2 and whose magnitudes are proportional to OA_1 and OA_2. The virtual displacement of the system of points A_1, A_2 is therefore obtained by giving these points displacements tangent to the curves C_1, C_2 whose magnitudes are proportional to the distances OA_1, OA_2, and whose senses are as in the rotation about O.

Unilateral constraints. Let us suppose that among the relations that the coordinates of the points of a system must satisfy, there appears the unequality

$$\Phi(x_1, \ldots, z_n) \leqq 0. \tag{31}$$

Let us assume that $\Phi < 0$ in a certain position of the system. If at a certain instant the system is given an arbitrary motion compatible with all the relations except (31), then — as it is easy to see — in a small interval of time $\Phi < 0$ constantly (on account of continuity). Therefore the motion will satisfy relation (31). It follows from this that in a position in which the inequality $\Phi < 0$ holds, relation (31) does not constitute any limitation on the possible velocities and (as a consequence of this) on the virtual displacements. In determining the virtual displacements in this case, therefore, we need not take inequality (31) into account at all.

Let us now assume that the system occupies a boundary position, i. e. that the equality $\Phi = 0$ holds. At the instant t let us give the system an arbitrary motion compatible with the constraints. The function Φ will have the value $\Phi' = \Phi + \Delta\Phi$ at the time $t + \Delta t$ (where $\Delta t > 0$). Since $\Phi' \leqq 0$ and $\Phi = 0$, $\Delta\Phi \leqq 0$. Consequently $\lim\limits_{\Delta t \to 0} \Delta\Phi \,/\, \Delta t \leqq 0$, whence

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial x_1} \dot{x}_1 + \ldots + \frac{\partial \Phi}{\partial z_n} \dot{z}_n \leqq 0. \tag{32}$$

The possible velocities must therefore satisfy inequality (32). If $\delta x_1, \ldots, \delta z_n$ is a virtual displacement of the system, then putting $\dot{x}_1 = \delta x_1, \ldots, \dot{z}_n = \delta z_n$, we obtain a system of possible velocities. Consequently $\dot{x}_1, \ldots, \dot{z}_n$, satisfy inequality (32), whence

$$\frac{\partial \Phi}{\partial x_1} \delta x_1 + \ldots + \frac{\partial \Phi}{\partial z_n} \delta z_n \leqq 0. \tag{33}$$

Therefore, if relation (31) becomes an equality in a certain position of the system, then the virtual displacement must satisfy relation (33) and conversely: a displacement satisfying relation (33) is a virtual displacement.

If the sign „$<$" appears in relation (33) for a certain virtual displacement $\delta x_1, \ldots, \delta z_n$, then the displacement $-\delta x_1, \ldots, -\delta z_n$, is not a virtual displacement; hence the given virtual displacement is irreversible (p. 427). On the other hand, if the displacement $\delta x_1, \ldots, \delta z_n$, is reveresible, the sign „$=$" must appear in (33).

Collecting the results obtained, we can therefore say:

If the constraints of a system are defined by the relations:

$$F_j(x_1, \ldots, z_n) = 0 \qquad (j = 1, 2, \ldots, m),$$
$$\Phi_r(x_1, \ldots, z_n) \leqq 0 \qquad (r = 1, 2, \ldots, s),$$

then the virtual displacement $\delta x_1, \ldots, \delta z_n$ in a given position of the system satisfies the equations:

$$\sum_{i=1}^{n} \left(\frac{\partial F_j}{\partial x_i} \delta x_i + \frac{\partial F_j}{\partial y_i} \delta y_i + \frac{\partial F_j}{\partial z_i} \delta z_i \right) = 0 \qquad (j = 1, 2, \ldots, m) \quad \text{(I)}$$

as well as those relations from among

$$\sum_{i=1}^{n} \left(\frac{\partial \Phi_r}{\partial x_i} \delta x_i + \frac{\partial \Phi_r}{\partial y_i} \delta y_i + \frac{\partial \Phi_r}{\partial z_i} \delta z_i \right) \leqq 0 \qquad (r = 1, 2, \ldots, s) \quad \text{(II)}$$

for which the equality $\Phi_r = 0$ holds in this position of the system.

Example 6. Let us assume that a material point is constrained to remain within the sphere $x^2 + y^2 + z^2 - r^2 = 0$ or on its surface. The coordinates x, y, z, of this point must consequently satisfy the inequality

$$x^2 + y^2 + z^2 - r^2 \leqq 0. \tag{34}$$

If the point lies inside the sphere then the inequality

$$x^2 + y^2 + z^2 - r^2 < 0 \tag{35}$$

holds; in this position the point can have an arbitrary velocity and therefore every velocity is a possible velocity. It follows from this that every displacement is then a virtual displacement.

If the point is on the surface of the sphere, then

$$x^2 + y^2 + z^2 - r^2 = 0, \tag{36}$$

and possible velocities are velocities tangent to the sphere or velocities having senses towards the interior of the sphere. In that case the virtual displacements are therefore displacements whose directions are tangent to the sphere, as well as those whose directions are not tangent, but have a sense towards the interior of the sphere (Fig. 308).

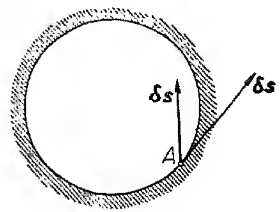

Fig. 308.

By (II) the virtual displacement $\delta x, \delta y, \delta z$, satisfies the inequality which we obtain by differentiating (34):

$$2x\,\delta x + 2y\,\delta y + 2z\,\delta z \leqq 0, \quad \text{i. e.} \quad x\,\delta x + y\,\delta y + z\,\delta z \leqq 0.$$

Example 7. A material point, tied to a string of length l attached to the origin of a coordinate system, is constrained to remain on the surface $z = x^2 + y^2$. The coordinates of the point therefore satisfy the relations:

$$z - x^2 - y^2 = 0, \quad x^2 + y^2 + z^2 - l^2 \leqq 0. \tag{37}$$

If the string is not in tension, i. e. if $x^2 + y^2 + z^2 - l^2 < 0$, then the virtual displacements satisfy only the equality

$$\delta z - 2x\,\delta x - 2y\,\delta y = 0. \tag{38}$$

If the string is in tension, then the point occupies a boundary position; hence $x^2 + y^2 + z^2 - l^2 = 0$; consequently in this case the inequality

$$x\,\delta x + y\,\delta y + z\,\delta z \leqq 0 \tag{39}$$

must hold in addition to the equality (38).

From (38) we get

$$\delta z = 2x\,\delta x + 2y\,\delta y, \tag{40}$$

whence after substituting in (39)

$$x(1 + 2z)\,\delta x + y(1 + 2z)\,\delta y \leqq 0. \tag{41}$$

Let us put

$$w = x(1 + 2z)\,\delta x + y(1 + 2z)\,\delta y. \tag{42}$$

Hence, if $y \neq 0$, then

$$\delta y = [w - x(1 + 2z)\,\delta x] / y(1 + 2z). \tag{43}$$

Therefore, for every value δx and every non-positive value of w we obtain from (40) and (43) the virtual displacement δx, δy, δz, satisfying relations (38) and (39).

§ 3. Principle of virtual work. Virtual work. Let a holonomo-scleronomic system consist of n material points $A_1, \ldots, A_n$, at which the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, are applied. Let us give the system an arbitrary virtual displacement $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$. The work of the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$ on this displacement is

$$\delta' L = \mathbf{P}_1\,\overline{\delta s_1} + \ldots + \mathbf{P}_n\,\overline{\delta s_n} = \sum_{i=1}^{n} \mathbf{P}_i\,\overline{\delta s_i}. \tag{I}$$

If $\delta x_1, \delta y_1, \delta z_1, \ldots, \delta x_n, \delta y_n, \delta z_n$ are the projections of the vectors $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$, then

$$\delta' L = \sum_{i=1}^{n} (P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i). \tag{I'}$$

The work defined by formulae (I) and (I′) is called the *virtual work* of the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$ on the virtual displacement $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$ (having the projections $\delta x_1, \ldots, \delta z_n$).

Remark. The virtual work is denoted by $\delta' L$ (with the prime) because the symbol δL could suggest the supposition that L is a function and δL an expression defined by formula (18), p. 428.

If the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, have a potential V, then by (III), p. 211, we have $P_{i_x} = \partial V / \partial x_i$, $P_{i_y} = \partial V / \partial y_i$, $P_{i_z} = \partial V / \partial z_i$.

From (I′) we get

$$\delta' L = \sum_{i=1}^{n} \left(\frac{\partial V}{\partial x_i}\,\delta x_i + \frac{\partial V}{\partial y_i}\,\delta y_i + \frac{\partial V}{\partial z_i}\,\delta z_i \right), \tag{1}$$

whence by (18), p. 428,

$$\delta' L = \delta V. \tag{I''}$$

Example I. A point is constrained to the sphere $x^2 + y^2 + z^2 - r^2 = 0$. The virtual displacement is defined by the equation

$$x\,\delta x + y\,\delta y + z\,\delta z = 0.$$

If the force $\mathbf{P}$ acts on the point, then its virtual work is

$$\delta' L = P_x\,\delta x + P_y\,\delta y + P_z\,\delta z. \tag{2}$$

Let us assume that $z \neq 0$. Consequently

$$\delta z = -(x\,\delta x + y\,\delta y) / z, \tag{3}$$

whence after substituting in (2)

$$\delta' L = [(P_x z - P_z x)\,\delta x + (P_y z - P_z y)\,\delta y] / z. \tag{4}$$

The values δx, δy, in formula (4) are arbitrary.

If

$$P_x z - P_z x = 0, \quad P_y z - P_z y = 0, \tag{5}$$

then we obtain as the virtual work $\delta' L = 0$ for every virtual displacement. Conversely, if $\delta' L = 0$ constantly, then taking in formula (4) first $\delta x = 1$, $\delta y = 0$, and then $\delta x = 0$, $\delta y = 1$, we obtain equations (5) which express the fact that the direction of the force $\mathbf{P}$ passes through the origin of the coordinate system (or through the centre of the sphere), i. e. that the force is normal to the surface of the sphere.

The results obtained can be verified in the following way. Let us note that the virtual displacement at an arbitrary point of the sphere is every vector tangent to the sphere at this point (p. 422). The virtual work of the force $\mathbf{P}$ will therefore be constantly zero then, and only then, when the force $\mathbf{P}$ is perpendicular to every virtual displacement, and hence to a plane tangent to the sphere, i. e. when the direction of the force is normal to the surface of the sphere.

Principle of virtual work. Let a holonomo-scleronomic system of material points $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$, be acted upon by forces. The forces that cause the system to maintain the constraints are called reactions, and the forces that act at the points of the system and are not reactions are called, in order to distinguish them from the former, *acting forces*. When the system is at rest, i. e. in equilibrium, the acting forces are said to *balance one another*.

It is obvious that a system does not have to be at rest even when the acting forces balance one another: e. g. a material point on which no forces act can move with a uniform motion.

Let us assume that a system of points is in equilibrium. At each separate point the forces acting on this point are therefore annulled by the reactions. Denoting by $\mathbf{P}_1, \ldots, \mathbf{P}_n$, the forces acting on the individual points, and by $\mathbf{R}_1, \ldots, \mathbf{R}_n$, the reactions, we hence obtain:

$$\mathbf{P}_1 + \mathbf{R}_1 = 0, \quad \mathbf{P}_2 + \mathbf{R}_2 = 0, \quad \ldots, \quad \mathbf{P}_n + \mathbf{R}_n = 0. \tag{6}$$

Let us consider an arbitrary virtual displacement $\delta s_1, \ldots, \delta s_n$. The work of the acting forces and reactions on this displacement, in view of (6), is

$$(\mathbf{P}_1 + \mathbf{R}_1)\,\overline{\delta s_1} + \ldots + (\mathbf{P}_n + \mathbf{R}_n)\,\overline{\delta s_n} = 0, \tag{7}$$

i. e.

$$(\mathbf{P}_1\,\overline{\delta s_1} + \ldots + \mathbf{P}_n\,\overline{\delta s_n}) + (\mathbf{R}_1\,\overline{\delta s_1} + \ldots + \mathbf{R}_n\,\overline{\delta s_n}) = 0. \tag{8}$$

Experience shows that if there is no friction, the work of the reactions on every virtual displacement is non-negative.

For example, if a point is constrained to remain within a certain smooth sphere, or on its surface, then, in the case when the point is on the sphere, the reaction is directed towards the centre of the sphere. The virtual displacement is either tangent to the sphere or it has a sense towards the interior (p. 433). In the first case the work of the reaction is zero, but in the second case it is positive.

Therefore, under the assumption that there is no friction, we have

$$\mathbf{R}_1 \overline{\delta s_1} + \ldots + \mathbf{R}_n \overline{\delta s_n} \geqq 0. \tag{9}$$

By (8)

$$\mathbf{P}_1 \overline{\delta s_1} + \ldots + \mathbf{P}_n \overline{\delta s_n} = -(\mathbf{R}_1 \overline{\delta s_1} + \ldots + \mathbf{R}_n \overline{\delta s_n}); \tag{10}$$

hence from (9) we obtain

$$\mathbf{P}_1 \overline{\delta s_1} + \ldots + \mathbf{P}_n \overline{\delta s_n} \leqq 0. \tag{11}$$

The expression on the left side of the inequality (11) represents, according to (I), p. 434, the virtual work of the acting forces.

Therefore: *if there is no friction and a system is in equilibrium, the work of the acting forces for every virtual displacement is either zero or a negative number.*

If the virtual displacement $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$, is reversible (p. 427), then $-\overline{\delta s_1}, \ldots, -\overline{\delta s_n}$, is also a virtual displacement. In the case of equilibrium, (11) as well as

$$-\mathbf{P}_1 \overline{\delta s_1} - \ldots - \mathbf{P}_n \overline{\delta s_n} \leqq 0 \tag{12}$$

hold.

From (11) and (12) it follows that

$$\mathbf{P}_1 \overline{\delta s_1} + \ldots + \mathbf{P}_n \overline{\delta s_n} = 0. \tag{13}$$

Therefore: in the case of the equilibrium of a system the virtual work of the acting forces is equal to zero for every reversible virtual displacement.

In particular, if the constraints are bilateral, every virtual displacement is reversible and consequently the virtual work of the acting forces is then zero for every virtual displacement.

The condition of equilibrium obtained is a necessary condition. Experience teaches that it is also sufficient.

This condition is known as the *principle of virtual work.*

We can state it as follows:

Principle of virtual work. *If a system of n material points $A_1, \ldots, A_n$, is holonomo-scleronomic and there is no friction, then the necessary and sufficient condition for the equilibrium of the acting forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, is that for every virtual displacement $\delta x_1, \ldots, \delta z_n$ the virtual work of the acting forces be zero or a negative number*, i. e. *that the relation*

$$\delta' L = \sum_{i=1}^{n} (P_{i_x}\, \delta x_i + P_{i_y}\, \delta y_i + P_{i_z}\, \delta z_i) \leqq 0 \tag{II}$$

hold.

If the constraints are bilateral, condition (II) *assumes the form*

$$\delta' L = \sum_{i=1}^{n} (P_{i_x}\, \delta x_i + P_{i_y}\, \delta y_i + P_{i_z}\, \delta z_i) = 0. \tag{III}$$

In many cases the principle of virtual work can be proved. We accept it as a law verified by experience in all those cases in which the concept of friction is defined. In the general case it can be said that friction does not appear in a system if the principle of virtual work applies to the system.

The importance of the principle of virtual work consists in the fact that it gives the condition for the equilibrium of the acting forces without the aid of the reactions.

Example 2. Let A be a free material point. Let us denote by $\mathbf{P}$ the sum of the forces acting on A. The virtual work is $\delta' L = \mathbf{P}\, \overline{\delta s}$, where $\overline{\delta s}$ is a virtual displacement. In the case of equilibrium $\delta' L = 0$, i. e.

$$\mathbf{P}\, \overline{\delta s} = 0 \tag{14}$$

for every virtual displacement. Since the point A is free, $\overline{\delta s}$ is arbitrary. It follows from this, in view of (14), that $\mathbf{P} = 0$. For were $\mathbf{P} \neq 0$, then assuming that $\overline{\delta s}$ has the direction and sense of the force $\mathbf{P}$, we should have $\mathbf{P}\, \overline{\delta s} = |\mathbf{P}| \cdot |\overline{\delta s}| \neq 0$, contrary to (14).

We have thus verified the principle of virtual work in the case of a free point.

Example 3. A material point A, subjected to the action of the force $\mathbf{P}$, is constrained to remain on the surface S. In the position of equilibrium the virtual work is $\delta' L = \mathbf{P}\, \overline{\delta s} = 0$, where $\overline{\delta s}$ is a virtual displacement and hence an arbitrary vector lying in the plane Π tangent to the surface S at the point A (p. 422). It follows from this that $\mathbf{P} \perp \Pi$. Conversely, if $\mathbf{P} \perp \Pi$, then obviously $\delta' L = \mathbf{P}\, \overline{\delta s} = 0$, and hence the point A is in the position of equilibrium.

Let us assume now that the point A is constrained to lie on one side of the surface S. The constraints are consequently unilateral. In the case of equilibrium we therefore have $\delta' L = \mathbf{P}\, \overline{\delta s} \leqq 0$ for every virtual displacement. If $\overline{\delta s}$ lies in the tangent plane Π, then it is a reversible displacement (p. 427), and hence $\mathbf{P}\, \overline{\delta s} = 0$. It follows from this that $\mathbf{P} \perp \Pi$.

The most general virtual displacement is any vect or (whose origin is at A) directed towards that side of the surface S on which the point A lies. Since $\mathbf{P}\,\overline{\delta s} \leq 0$, $\mathbf{P}$ has a sense in the direction of the surface S (i. e. it presses the point A to the surface S). Conversely, if $\mathbf{P} \perp \Pi$ and $\mathbf{P}$ has a sense in the direction of the surface S, then, as is easily seen, $\delta' L = \mathbf{P}\,\overline{\delta s} \leq 0$ for every virtual displacement. The point is consequently in the position of equilibrium.

Example 4. The material point A, subjected to the action of the force $\mathbf{P}$, is constrained to remain on the curve C. In the position of equilibrium the virtual work for every virtual displacement δs is $\delta' L = \mathbf{P}\,\overline{\delta s} = 0$. Since $\overline{\delta s}$ is an arbitrary vector tangent to C at the point A, $\mathbf{P}$ is perpendicular to C.

Conversely, if $\mathbf{P}$ is perpendicular to C, then obviously $\mathbf{P}\,\overline{\delta s} = 0$, and hence the point is in the position of equilibrium.

Example 5. A lever AB is acted upon by weights $\mathbf{Q}_1$, $\mathbf{Q}_2$, suspended from the points A, B, and as the weight $\mathbf{Q}$ acting at the centre S of its mass. The acting forces lie in a vertical plane perpendicular to tho axis of rotation at the point O, while $OS \perp AB$. Determine in the position of equilibrium the angle φ which OS makes with the vertical (cf. example 1, p. 274).

Let us denote by $\overline{\delta s_1}$, $\overline{\delta s_2}$, $\overline{\delta s}$, the virtual displacements of the points of application A, B, S. The lever can only rotate about its axis. The possible velocities, and — as a consequence — the virtual displacements of the points A, B, S are perpendicular to OA, OB, OS, (Fig. 309). Denoting by $\delta\omega$ an arbitrary angular velocity, we consequently have:

$$|\overline{\delta s_1}| = OA\ \delta\omega, \quad |\delta s_2| = OB\ \delta\omega, \quad |\overline{\delta s}| = OS\ \delta\omega. \tag{15}$$

In the position of equilibrium the virtual work is zero, e. i. $\mathbf{Q}_1\,\overline{\delta s_1} + \mathbf{Q}_2\,\overline{\delta s_2} + \mathbf{Q}\,\overline{\delta s} = 0$. Calculating the scalar products and denoting the absolute values of the forces by Q_1, Q_2, Q, we obtain by (15) $(Q_1 \cdot OA \cos\varphi + Q \cdot OS \sin\varphi - Q_2 \cdot OB \cos\varphi)\ \delta\omega = 0$, whence after dividing by $\delta\omega$

$$\tan\varphi = (Q_2 \cdot OB - Q_1 \cdot OA) \,/\, Q \cdot OS. \tag{16}$$

Formula (16) was obtained before in another way (cf. formula (6), p. 275).

Example 6. On an inclined plane making an angle α with the horizontal there lies a heavy point A which remains in equilibrium under action of a force $\mathbf{P}$ having a horizontal direction (Fig. 310). Determine the force $\mathbf{P}$ under the assumption that there is no friction.

Let us denote by $\mathbf{Q}$ the weight of the body and by $\overline{\delta s}$ an arbitrary virtual displacement. The virtual work is $\delta' L = \mathbf{Q}\,\overline{\delta s} + \mathbf{P}\,\overline{\delta s}$. In order that equilibrium occur, we must have for every virtual displacement $\delta' L \leq 0$, i. e.

$$\mathbf{Q}\,\overline{\delta s} + \mathbf{P}\,\overline{\delta s} \leq 0. \tag{17}$$

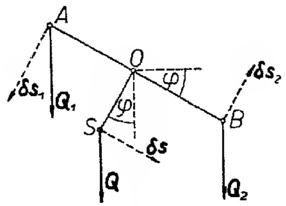

Fig. 309.

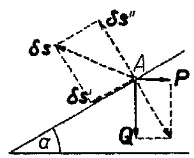

Fig. 310.

Let us first consider the virtual displacement $\overline{\delta s'}$ having the direction of the inclined plane. Under this assumption $\overline{\delta s'}$ is a reversible displacement; consequently (17) assumes the form of the equality

$$\mathbf{Q}\,\overline{\delta s'} + \mathbf{P}\,\overline{\delta s'} = 0. \tag{18}$$

Let Π be a vertical plane passing through A and perpendicular to the inclined plane, and let $\overline{\delta s'} \perp \Pi$. Therefore $\mathbf{Q}\,\overline{\delta s'} = 0$, whence by (2) $P \cdot \overline{\delta s'} = 0$, i. e. $\mathbf{P} \perp \overline{\delta s'}$. Hence $\mathbf{P}$ lies in the plane Π.

Let us now assume that $\overline{\delta s'}$ lies on the inclined plane and in the plane Π (Fig. 310); giving the displacement $\overline{\delta s'}$ a downward sense and putting $Q = |\mathbf{Q}|$, $P = |\mathbf{P}|$, we obtain from (18)

$$Q|\overline{\delta s'}| \sin\alpha \pm P|\overline{\delta s'}| \cos\alpha = 0. \tag{19}$$

The sign „$\pm$" depends on the sense of the force $\mathbf{P}$. From the equality (19) it follows that it is necessary to take the sign „$-$". Hence the force $\mathbf{P}$ must press the point to the plane. Using the sign „$-$" we obtain from (19)

$$P = Q \tan\alpha. \tag{20}$$

We have thus determined the direction, sense, and magnitude, of the force $\mathbf{P}$ under the assumption of equilibrium. From (20) it follows easily that the sum $\mathbf{Q} + \mathbf{P}$ is perpendicular to the inclined plane.

In order to show that equilibrium really occurs, it is necessary to prove that condition (17) holds for every virtual displacement. In order to demonstrate this, let us resolve the arbitrary displacement $\overline{\delta s}$ into two displacements: $\overline{\delta s''}$ perpendicular to the inclined plane and $\overline{\delta s'}$ lying on the inclined plane. The work of the forces $\mathbf{P}$ and $\mathbf{Q}$ on the displacement $\overline{\delta s'}$ is zero, because $\mathbf{P} + \mathbf{Q} \perp \overline{\delta s'}$. The displacement $\overline{\delta s''}$ and the sum

$\mathbf{P} + \mathbf{Q}$ have the same direction, but opposite senses; hence the work of the forces $\mathbf{P}$ and $\mathbf{Q}$ on $\overline{\delta s''}$ is negative. It follows from this that the work of the forces $\mathbf{P}$ and $\mathbf{Q}$ on the displacement $\overline{\delta s}$ is negative. Relation (17) therefore holds for every virtual displacement. Consequently the forces $\mathbf{Q}$ and $\mathbf{P}$ balance each other.

Example 7. **Rigid body.** By appealing to the principle of virtual work we shall now derive the conditions for the equilibrium of forces acting on a rigid body.

Unconstrained body. Let the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, having origins $A_1, \ldots, A_n$, act on a free rigid body. Let us give the body an arbitrary virtual displacement and denote by $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$, the displacements of the points $A_1, \ldots, A_n$. Hence by (I), p. 434, the virtual displacement is

$$\delta' L = \mathbf{P}_1\, \overline{\delta s_1} + \ldots + \mathbf{P}_n\, \overline{\delta s_n}. \tag{21}$$

In example 4, p. 429, we considered the virtual displacement of a rigid body.

Let the virtual displacement of the body be a translation (p. 429); the displacements of the points are therefore equal $\overline{\delta u}$. By (21) we have $\delta' L = \mathbf{P}_1\, \overline{\delta u} + \ldots + \mathbf{P}_n\, \overline{\delta u} = (\mathbf{P}_1 + \ldots + \mathbf{P}_n)\, \overline{\delta u}$. Putting $\mathbf{P} = \mathbf{P}_1 + \ldots + \mathbf{P}_n$, we obtain

$$\delta' L = \mathbf{P}\, \overline{\delta u}. \tag{22}$$

Let O be an arbitrary point of the body and l an axis passing through O. Let us give the body a virtual displacement, in which displacements of the points are proportional to the velocities during a rotation of the body about the axis l with an angular velocity $\overline{\delta\omega}$. By (26), p. 429, we obtain

$$\overline{\delta s_i} = \overline{OA_i} \times \overline{\delta\omega} \qquad (i = 1, 2, \ldots, n),$$

whence by (21) $\delta' L = \mathbf{P}_1(\overline{OA_1} \times \overline{\delta\omega}) + \ldots + \mathbf{P}_n(\overline{OA_n} \times \overline{\delta\omega})$. Since $\mathbf{a}(\mathbf{b} \times \mathbf{c}) = \mathbf{c}(\mathbf{a} \times \mathbf{b})$ (formula (II), p. 13),

$$\delta' L = \overline{\delta\omega}(\mathbf{P}_1 \times \overline{OA_1}) + \ldots + \overline{\delta\omega}(\mathbf{P}_n \times \overline{OA_n}).$$

But $\mathbf{P}_1 \times \overline{OA_1} = \mathrm{Mom}_O \mathbf{P}_1$, etc. Consequently

$$\delta' L = \overline{\delta\omega}(\mathrm{Mom}_O \mathbf{P}_1 + \ldots + \mathrm{Mom}_O \mathbf{P}_n) = \overline{\delta\omega} \cdot \mathbf{M}, \tag{23}$$

where $\mathbf{M}$ is the total moment of the forces with respect to O.

Let us denote by $\delta\omega$ the component of $\overline{\delta\omega}$ with respect to the axis l, and by φ the angle which $\mathbf{M}$ makes with the axis l. Then $\overline{\delta\omega} \cdot \mathbf{M} = \delta\omega \cdot |\mathbf{M}| \cos\varphi$. Since the moment of the acting forces with respect to the axis l is $M_l = |\mathbf{M}| \cos\varphi$, by (23)

$$\delta' L = M_l\, \delta\omega. \tag{24}$$

The most general virtual displacement of a rigid body is a composition of the displacements defined by formulae (25) and (26), p. 429. Therefore for the most general virtual displacement

$$\delta' L = \mathbf{P} \cdot \overline{\delta u} + M_l\, \delta\omega, \tag{25}$$

where $\mathbf{P}$ denotes the sum of all the acting forces, M_l the moment with respect to an arbitrary axis l, $\overline{\delta u}$ an arbitrary vector, and $\delta\omega$ an arbitrary number.

We now proceed to determine the conditions for equilibrium. Let us assume that the system of acting forces is in equilibrium. Hence the virtual work $\delta' L$ on every virtual displacement is zero. Giving the body an arbitrary displacement $\overline{\delta u}$, we have by (22) $\mathbf{P}\, \overline{\delta u} = 0$, from which it follows that

$$\mathbf{P} = 0, \tag{26}$$

for were $\mathbf{P} \neq 0$, then choosing $\overline{\delta u}$ in the direction of $\mathbf{P}$, we should have $\mathbf{P}\, \overline{\delta u} \neq 0$.

Let us now select an arbitrary point O and an axis l passing through O. By (24) $M_l\, \delta\omega = 0$ for every $\delta\omega$; hence $M_l = 0$. Since l is an arbitrary axis passing through O, the total moment with respect to O is

$$\mathbf{M} = 0. \tag{27}$$

In this way we have proved that the equalities (26) and (27) are n e c e s s a r y conditions for the equilibrium of the acting forces. We shall now show that they are likewise s u f f i c i e n t conditions.

For if the equalities (26) and (27) hold, then the virtual work given in the most general case by formula (25) is obviously zero.

We have therefore obtained the conditions of equilibrium for a free rigid body, which were derived in another way on p. 244.

We shall now consider several cases of equilibrium of a constrained body.

Body having a fixed point. Let a body have the point O fixed. The instantaneous motion of the body can only be a rotation about an axis passing through O. Consequently the virtual work is expressed by formula (24).

If the acting forces balance one another, then $\delta' L = 0$, whence by (24) $M_l\, \delta\omega = 0$. Since $\delta\omega$ is arbitrary, $M_l = 0$, where l is an arbitrary axis passing through O. It follows from this that the total moment with respect to O of the forces acting on the body is $\mathbf{M} = 0$.

Conversely, if $\mathbf{M} = 0$, then $M_l = 0$ with respect to every axis l passing through O. Hence by (24) $\delta' L = 0$ constantly.

Therefore: a system of forces acting on a body having one fixed point O is in equilibrium then, and only then, when the total moment of the forces with respect to O is zero.

This condition was obtained before in another way (p. 271).

Plane motion of a body (p. 272). Let Π be the directional plane of a body in plane motion. The instantaneous motion of the body is either an advancing motion with a velocity parallel to Π, or a rotation about an axis l perpendicular to Π. Consequently the virtual work is expressed by formula (22), where $\overline{\delta u} \parallel \Pi$, or by formula (24), where $l \perp \Pi$. In the case of equilibrium we therefore obtain $\mathbf{P}\, \overline{\delta u} = 0$ for $\overline{\delta u} \parallel \Pi$, and $M_l\, \delta\omega = 0$ for $l \perp \Pi$. It follows from this that

$$\mathbf{P} \perp \Pi \text{ and } M_l = 0 \text{ for } l \perp \Pi. \tag{28}$$

It is easy to prove that the condition obtained is equivalent to the condition that the projections of the forces on the directional plane Π form a system equipollent to zero. If condition (28) holds, then from (22) and (24) it follows that the virtual work is zero. Condition (28) is therefore necessary and sufficient for the equilibrium of the acting forces.

Body having a fixed axis. Let us assume that a body has a fixed axis l. In this case the body can only rotate about the axis l. The virtual work is therefore expressed by formula (24). Hence by (24) the necessary and sufficient condition for the equilibrium of the acting forces is that $\delta' L = M_l\, \delta\omega = 0$, whence $M_l = 0$ (for $\delta\omega$ is arbitrary).

We obtained this condition before on p. 272.

Body having a fixed axis of twist. Let us assume that a body can only rotate about a certain axis l as well as to move along it, which is the case e. g. with a sphere strung on a straight rigid rod. Hence the instantaneous motion of the body is the composition of an advancing motion whose velocity has the direction of the axis l and a rotation about this axis, and consequently the motion is a twist about the axis l. In the most general case the virtual work is therefore defined by formula (25), in which $\overline{\delta u}$ has the direction of the axis l and $\delta\omega$ is arbitrary.

If equilibrium occurs, then by (25)

$$\delta' L = \mathbf{P}\, \overline{\delta u} + M_l\, \delta\omega = 0. \tag{29}$$

Assuming $\overline{\delta u} = 0$ and $\delta\omega \neq 0$, we get $M_l = 0$; and if we assume $\delta\omega = 0$, we obtain from (29) $\mathbf{P}\, \overline{\delta u} = 0$. Since $\overline{\delta u}$ has the direction of the axis l, $\mathbf{P} \perp l$.

Conversely, if $M_l = 0$ and $\mathbf{P} \perp l$, then by (29) obviously $\delta' L = 0$.

Therefore: *a necessary and sufficient condition for the equilibrium of a body which can rotate about a fixed axis and slide along it, is that the sum of the forces be perpendicular to the axis and the moment of the forces with respect to the axis be zero.*

Screw. A rigid body which can only move so that a certain helix in the body slides along itself is called a *screw.*

The axis of the screw can only slide along itself, and consequently the velocities of the points on the axis have the direction of the axis. It follows from this (p. 334) that the instantaneous motion is an instantaneous twist about the axis of the screw.

Denoting by $\mathbf{u}$ the velocity of the instantaneous advancing motion, by $\boldsymbol{\omega}$ the instantaneous angular velocity, and by h tho lead of the screw, we have by (15), p. 337,

$$|\mathbf{u}| \,/\, |\boldsymbol{\omega}| = h \,/\, 2\pi. \tag{30}$$

Since $\mathbf{u}$ and $\boldsymbol{\omega}$ have the direction of the axis l of the screw, denoting by u and ω the components with respect to the axis l of the vectors $\mathbf{u}$ and $\boldsymbol{\omega}$, we obtain from (30)

$$u = \varepsilon h \omega \,/\, 2\pi, \tag{31}$$

where $\varepsilon = +1$, if the screw is left-handed, and $\varepsilon = -1$, if it is right-handed.

The virtual work is expressed by formula (25), in which $\delta\omega$ is arbitrary, and $\overline{\delta u}$ has the direction of the axis l of the screw, while by (31)

$$\delta u = \varepsilon h\, \delta\omega \,/\, 2\pi, \tag{32}$$

where δu is the component of $\overline{\delta u}$ with respect to the axis l.

Denoting by P_l the projection of $\mathbf{P}$ on the axis l, we have $\mathbf{P}\, \overline{\delta u} = P_l\, \delta u$. Hence by (25) and (32)

$$\delta' L = (\varepsilon P_l h \,/\, 2\pi + M_l)\, \delta\omega. \tag{33}$$

From the principle of virtual work it follows by (33) that *a necessary and sufficient condition for the equilibrium of the forces acting on a screw is that the forces satisfy the equation*

$$M_l \,/\, P_l = -\varepsilon h \,/\, 2\pi. \tag{34}$$

Example 8. Determination of stresses in the bars of a frame. Kinematical method. A certain method of determining the stresses in the bars of a frame rests on the principle of virtual work. First we shall illustrate this method by means of an example.

Forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, act at the joints of a plane frame and are in equilibrium. In order to determine the stress in the bar BD, for example, we

remove this bar. The remaining system of bars will again be in equilibrium, if at the points B and D we apply the forces $\mathbf{S}$ and $-\mathbf{S}$, equal to the stresses at these two points in the bar removed (Fig. 311).

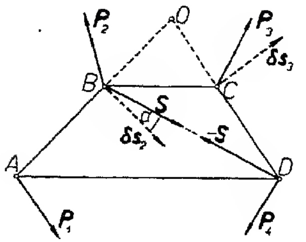

Fig. 311.

Let us give the system of bars AB, BC, CD, DA, an arbitrary virtual displacement and denote by $\delta s_1, \ldots, \delta s_4$, the displacements of the points $A, \ldots, D$. The virtual work will therefore be zero, i. e.

$$\mathbf{P}_1 \overline{\delta s_1} + \mathbf{P}_2 \overline{\delta s_2} + \mathbf{P}_3 \overline{\delta s_3} + $$
$$+ \mathbf{P}_4 \overline{\delta s_4} + \mathbf{S}\, \overline{\delta s_2} - \mathbf{S}\, \overline{\delta s_4} = 0,$$

whence

$$\mathbf{S}(\overline{\delta s_2} - \overline{\delta s_4}) = -(\mathbf{P}_1 \overline{\delta s_1} + \mathbf{P}_2 \overline{\delta s_2} + \mathbf{P}_3 \overline{\delta s_3} + \mathbf{P}_4 \overline{\delta s_4}). \tag{35}$$

Let us put $S = \pm |\mathbf{S}|$, where the sign depends on whether the stress $\mathbf{S}$ is a tension or a compression, and let us denote by δr the projection of the difference $\overline{\delta s_2} - \overline{\delta s_4}$ on the direction of $\overline{BD}$. With these notations the left side of the equality (35) is $S\,\delta r$. Therefore, if we choose the virtual displacement in such a way that $\delta r \neq 0$, then we obtain from (35)

$$S = -(\mathbf{P}_1 \overline{\delta s_1} + \mathbf{P}_2 \overline{\delta s_2} + \mathbf{P}_3 \overline{\delta s_3} + \mathbf{P}_4 \overline{\delta s_4}) \,/\, \delta r. \tag{36}$$

The sought for virtual displacement is obtained by assuming that the points A and D are fixed; consequently:

$$\overline{\delta s_1} = 0, \quad \overline{\delta s_4} = 0. \tag{37}$$

The instantaneous motion of the bar BD (under the assumption that A and D are fixed) is an instantaneous rotation about the centre O, which is the point of intersection of the lines AB and DC (cf. example 4, p. 328).

The displacements $\overline{\delta s_2}$ and $\overline{\delta s_3}$ are proportional to the velocities of the points B and C during a rotation of the rod BC about O. Consequently $\overline{\delta s_2}$ and $\overline{\delta s_3}$ are perpendicular to AB and DC, respectively, and

$$|\overline{\delta s_3}| \,/\, |\overline{\delta s_2}| = OC \,/\, OB. \tag{38}$$

Let P_2' and P_3' denote the projections of the forces $\mathbf{P}_2$, $\mathbf{P}_3$, on the directions of $\overline{\delta s_2}$, $\overline{\delta s_3}$, and α the angle between $\overline{\delta s_2}$ and $\overline{BD}$. Consequently:

$$\mathbf{P}_2 \overline{\delta s_2} = P_2' |\overline{\delta s_2}|, \quad \mathbf{P}_3 \overline{\delta s_3} = P_3' |\overline{\delta s_3}|, \quad \delta r = |\overline{\delta s_2}| \cos\alpha. \tag{39}$$

From (36) we obtain by (37)—(39)

$$S = -(P_2' \cdot OB + P_3' \cdot OC) \,/\, OB \cos\alpha. \tag{40}$$

Let us now proceed the general case of a frame with joints $A_1, \ldots, A_n$, at which the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, act and are in equilibrium. In order to determine the stresses in the bar of the frame connecting the joints A_r and A_s, we remove the bar A_rA_s and apply to the joints A_r and A_s the forces $\mathbf{S}$ and $-\mathbf{S}$, equal to the stresses at A_r and A_s in the bar removed. Since the system of remaining bars is in equilibrium, the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, $\mathbf{S}$, $-\mathbf{S}$, balance one another. The virtual work is therefore zero. Denoting by $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$, the virtual displacements of the joints $A_1, \ldots, A_n$ (under the assumption that the bar A_rA_s was removed), we obtain

$$\mathbf{P}_1 \overline{\delta s_1} + \ldots + \mathbf{P}_n \overline{\delta s_n} + \mathbf{S}\, \overline{\delta s_r} - \mathbf{S}\, \overline{\delta s_s} = 0. \tag{41}$$

Hence, putting $S = \pm |\mathbf{S}|$ (where the sign depends on whether S is a tension or a compression) and denoting by δr the projection of $\overline{\delta s_r} - \overline{\delta s_s}$ on the direction of $\overline{A_rA_s}$, we obtain from (41)

$$S\,\delta r = -(\mathbf{P}_1 \overline{\delta s_1} + \ldots + \mathbf{P}_n \overline{\delta s_n}). \tag{42}$$

If the virtual displacements can be so chosen that $\delta r \neq 0$ (i. e. so that $\overline{\delta s_r} \neq \overline{\delta s_s}$ and the difference $\overline{\delta s_r} - \overline{\delta s_s}$ is not perpendicular to A_rA_s), then from (42) we shall be able to determine S.

Now, it is possible to show that if a frame is statically determinate (p. 297), then the virtual displacement having the required properties always exists.

Let us denote by $x_1, y_1, \ldots, x_n, y_n$, the coordinates of the joints $A_1, \ldots, A_n$, and by d_{ij} the lengths of the bars A_iA_j. Consequently

$$(x_i - x_j)^2 + (y_i - y_j)^2 - d_{ij}^2 = 0. \tag{43}$$

Let δx_i, δy_i be the projections of the virtual displacement of the point A_i.

Then by (43)

$$(x_i - x_j)(\delta x_i - \delta x_j) + (y_i - y_j)(\delta y_i - \delta y_j) = 0. \tag{44}$$

If the bar A_rA_s is removed, then the virtual displacements of the joints are defined by equations (44) (among which the equation corresponding to the bar A_rA_s does not appear); from these we can calculate the displacements.

The given method of determining the stresses in the bars of a frame is known as the *kinematical method*.

The kinematical method can be applied to plane frames as well as to space frames.

There also exist graphical methods of determining the possible velocities (and, as a consequence, the virtual displacements $\overline{\delta s_1}, \ldots, \overline{\delta s_n}$) of

the joints of the frame by means of the so-called *diagram of velocities* (*virtual displacements*).

§ 4. Determination of the position of equilibrium in a force field. One of the principal problems which we shall encounter in the investigation of the equilibrium of a system of material points is the determination of the position of equilibrium of the system in a given force field.

We shall here give the solution of this problem for bilateral constraints. For unilateral constraints the solution of the problem is considerably more complex and we shall therefore confine ourselves to an example (p. 450, example 2).

Let the constraints of the system of points $A_1, \ldots, A_n$, be defined by the equations

$$F_j(x_1, \ldots, z_n) = 0 \qquad (j = 1, 2, \ldots, m). \tag{1}$$

Let us assume that the system is in a force field, i. e. that the forces $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n$, acting on the points $A_1, \ldots, A_n$, are functions of the variables $x_1, \ldots, z_n$. Therefore:

$$P_{i_x} = \Phi_i(x_1, \ldots, z_n),\ P_{i_y} = \Psi_i(x_1, \ldots, z_n),\ P_{i_z} = X_i(x_1, \ldots, z_n). \tag{2}$$

The virtual displacements satisfy the equations (I), p. 426:

$$\sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i}\,\delta x_i + \frac{\partial F_j}{\partial y_i}\,\delta y_i + \frac{\partial F_j}{\partial z_i}\,\delta z_i\right) = 0 \qquad (j = 1, 2, \ldots, m). \tag{3}$$

In the position of equilibrium we have in virtue of the principle of virtual work

$$\sum_{i=1}^{n}(P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i) = 0. \tag{4}$$

Since (3) consists of m equations, we can determine m of the n unknowns $\delta x_1, \ldots, \delta z_n$, giving the remaining $k = 3n - m$ unknowns the arbitrarily chosen values $\delta h_1, \ldots, \delta h_k$. Determining the unknowns $\delta x_1, \ldots, \delta z_n$, from equations (3) in terms of $\delta h_1, \ldots, \delta h_k$, and substituting in (4), we obtain after simplifying the equation

$$a_1\,\delta h_1 + a_2\,\delta h_2 + \ldots + a_k\,\delta h_k = 0, \tag{5}$$

where $a_1, \ldots, a_k$, are certain numbers depending on the position of the system, i. e. on the coordinates $x_1, \ldots, z_n$.

In a position of equilibrium equality (5) must hold for every set of numbers $\delta h_1, \ldots, \delta h_k$. Assuming $\delta h_1 = 1, \delta h_2 = 0, \ldots, \delta h_k = 0$, we get $a_1 = 0$. Proceeding similarly, we obtain:

$$a_1 = 0,\quad a_2 = 0,\quad \ldots,\quad a_k = 0. \tag{6}$$

Conversely, if the equalities (6) are satisfied in a certain position of the system, then obviously equation (5) holds and consequently the given position of the system is a position of equilibrium. Equalities (6) therefore define the position of equilibrium of the system.

From equations (6) and (1), whose total number is $k + m = 3n$, we can determine in general $3n$ unknown coordinates $x_1, \ldots, z_n$, corresponding to the position of equilibrium of the system.

Lagrange's multipliers. We shall give still another method of determining the positions of equilibrium of a system, called the *method of Lagrange's multipliers*.

Let us denote the left sides of equations (3) by $W_1, \ldots, W_m$, and the left side of equation (4) by W.

Regarding $\delta x_1, \ldots, \delta z_n$, as unknowns, we can say that in the position of equilibrium every solution of the equations $W_1 = 0, \ldots, W_m = 0$ satisfies the equation $W = 0$. Hence, by a well-known theorem from the theory of linear equations it follows that W can be represented as a linear combination of $W_1, \ldots, W_m$, i. e. that there exist numbers $a_1, \ldots, a_m$, such that for arbitrary $\delta x_1, \ldots, \delta z_n$, we have the identity

$$W = a_1 W_1 + \ldots + a_m W_m.$$

Putting $\lambda_1 = -a_1, \ldots, \lambda_m = -a_m$, we can write the above identity in the form

$$W + \lambda_1 W_1 + \ldots + \lambda_m W_m = 0 \quad \text{or} \quad W + \sum_{j=1}^{m}\lambda_j W_j = 0. \tag{7}$$

Writing the left sides of equations (3) and (4) instead of $W_1, \ldots, W_m$, and W, we obtain

$$\begin{gathered}\sum_{i=1}^{n}(P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i) + \\ + \sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i}\,\delta x_i + \frac{\partial F_j}{\partial y_i}\,\delta y_i + \frac{\partial F_j}{\partial z_i}\,\delta z_i\right) = 0.\end{gathered} \tag{8}$$

Arranging the left side of equation (8) according to $\delta x_1, \ldots, \delta z_n$, we get

$$\begin{gathered}\sum_{i=1}^{n}\left[\left(P_{i_x} + \sum_{j=1}^{m}\lambda_j\frac{\partial F_j}{\partial x_i}\right)\delta x_i + \left(P_{i_y} + \sum_{j=1}^{m}\lambda_j\frac{\partial F_j}{\partial y_i}\right)\delta y_i + \right. \\ \left. + \left(P_{i_z} + \sum_{j=1}^{m}\lambda_j\frac{\partial F_j}{\partial z_i}\right)\delta z_i\right] = 0.\end{gathered} \tag{9}$$

Equality (7), and hence also (9), holds for arbitrary $\delta x_1, \ldots, \delta z_n$; consequently the coefficients of $\delta x_1, \ldots, \delta z_n$, must be equal to zero:

$$P_{i_x} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial x_i} = 0, \quad P_{i_y} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial y_i} = 0, \quad P_{i_z} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial z_i} = 0 \qquad \text{(I)}$$
$$(i = 1, 2, \ldots, n).$$

We have thus proved that in a position of equilibrium it is possible to choose numbers $\lambda_1, \ldots, \lambda_m$, such that equations (I) hold.

Conversely, if in a certain position of a system it is possible to choose numbers $\lambda_1, \ldots, \lambda_m$, satisfying equations (I), then equality (9) holds, and hence also (8), i. e. (7). Since, in virtue of (3), for virtual displacements $W_1 = 0, \ldots, W_m = 0$, from (7) we get $W = 0$, i. e. (4). The given position is consequently a position of equilibrium.

Therefore: *a necessary and sufficient condition for the equilibrium of forces in a certain position of a system is that there exist a set of numbers $\lambda_1, \ldots, \lambda_m$, satisfying equations* (I).

From equations (1) and (I), whose total number is $3n + m$, we can determine in general $3n + m$ unknowns, i. e. $\lambda_1, \ldots, \lambda_m$, and as the coordinates $x_1, \ldots, z_n$, defining the position of equilibrium.

The numbers $\lambda_1, \ldots, \lambda_m$, are called *Lagrange's multipliers*.

Remark. Denoting by $\mathbf{R}_1, \ldots, \mathbf{R}_n$, the forces of reaction in the position of equilibrium, we obviously have $\mathbf{P}_i + \mathbf{R}_i = 0$, i. e.

$$P_{i_x} + R_{i_x} = 0, \; P_{i_y} + R_{i_y} = 0, \; P_{i_z} + R_{i_z} = 0 \qquad (i = 1, 2, \ldots n).$$

Comparing with (I), we get:

$$R_{i_x} = \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial x_i}, \; R_{i_y} = \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial y_i}, \; R_{i_z} = \sum_{j=1}^{m} \lambda_j \frac{\partial F_j}{\partial z_i} \; (i = 1, 2, \ldots, n). \qquad \text{(II)}$$

Example I. Two heavy material points A_1, A_2, of masses m_1, m_2, are connected by a rigid rod of length d (massless) and are constrained to remain on two lines l_1 and l_2. The line l_1 is vertical and the line l_2 cuts l_1 and makes with it an angle $\varphi = 45°$ (Fig. 312). Determine the position of equilibrium, assuming that there is no friction.

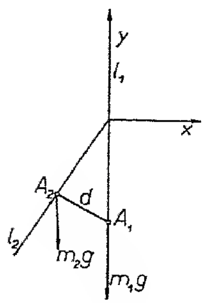

Fig. 312.

Let us choose the point of intersection of the lines l_1, l_2, as the origin of the coordinate system (x, y), taking the plane in which these lines lie as the xy-plane and the line l_1 as the y-axis (with an upward sense).

The equations of the lines l_1 and l_2 are $x = 0$ and $y = x$. Consequently the coordinates x_1, y_1, and x_2, y_2, satisfy the equations:

$$x_1 = 0, \quad y_2 - x_2 = 0. \qquad (10)$$

Since $A_1A_2 = d$,

$$x_2^2 + (y_1 - y_2)^2 - d^2 = 0. \qquad (11)$$

Equations (10) and (11) define the constraints of the system.

The virtual displacements satisfy the equations which we obtain from (10) and (11):

$$\delta x_1 = 0, \; \delta y_2 - \delta x_2 = 0, \; x_2\, \delta x_2 + (y_1 - y_2)(\delta y_1 - \delta y_2) = 0. \qquad (12)$$

The acting forces are the weights of the points. The projections of the weights on the axes of the coordinate system are respectively $0, -m_1 g$, and $0, -m_2 g$. The virtual work of the acting forces is equal to $\delta' L = -m_1 g\, \delta y_1 - m_2 g\, \delta y_2$. In the position of equilibrium $\delta' L = 0$, and hence

$$m_1\, \delta y_1 + m_2\, \delta y_2 = 0. \qquad (13)$$

Let us assume that $y_1 - y_2 \neq 0$. Selecting δy_2 arbitrarily we get from (12):

$$\delta x_1 = 0, \; \delta x_2 = \delta y_2, \; \delta y_1 = (y_1 - y_2 - x_2)\, \delta y_2 \,/\, (y_1 - y_2). \qquad (14)$$

Substituting in (13) we obtain after getting rid of the denominator

$$[m_1(y_1 - y_2 - x_2) + m_2(y_1 - y_2)]\, \delta y_2 = 0. \qquad (15)$$

Since δy_2 is arbitrary, equality (15) will hold only in the case when

$$m_1(y_1 - y_2 - x_2) + m_2(y_1 - y_2) = 0. \qquad (16)$$

Solving the system of equations (10), (11), (16), we obtain the coordinates:

$$x_1 = 0, \; y_1 = -(2m_1 + m_2)\, d \,/\, a, \; x_2 = -(m_1 + m_2)\, d \,/\, a = y_2,$$

in the position of equilibrium, where $a = \sqrt{(m_1 m_2)^2 + m_1^2}$.

Let us assume now that $y_1 - y_2 = 0$. By (11) we have $x_2^2 - d^2 = 0$, whence $x_2 \neq 0$. In view of this the last one of the equations (12) gives $\delta x_2 = 0$, and hence the second one of the equations (12) gives $\delta y_2 = 0$. Consequently by (12) the virtual displacement is the displacement:

$$\delta x_1 = 0, \quad \delta x_2 = 0, \quad \delta y_2 = 0, \quad \delta y_1 \; \textit{arbitrary}.$$

The condition of equilibrium (13) will therefore assume the form $m_1\, \delta y_1 = 0$. However, this equality is not satisfied, because δy_1 is arbitrary.

Therefore: the position for which $y_1 - y_2 = 0$ is not a position of equilibrium.

Example 2. A heavy point of mass m, subjected to the action of the force $\boldsymbol{P}$, is constrained to remain on the surface of the sphere

$$x^2 + y^2 + z^2 - r^2 = 0. \tag{17}$$

We assume that the z-axis has a vertical direction and an upward sense. Determine the position of equilibrium, assuming that friction does not appear.

The virtual displacement $\delta x, \delta y, \delta z$, satisfies the equation which we get by differentiating (17):

$$x\,\delta x + y\,\delta y + z\,\delta z = 0. \tag{18}$$

The virtual work of the force $\boldsymbol{P}$ is $P_x\,\delta x + P_y\,\delta y + P_z\,\delta z$, and that of the force of gravity $-mg\,\delta z$. In the position of equilibrium we consequently have

$$\delta' L = P_x\,\delta x + P_y\,\delta y + (P_z - mg)\,\delta z = 0. \tag{19}$$

Applying the method of Lagrange's multipliers (formula (I), p. 448) and replacing 2λ by λ we obtain the following equations:

$$P_x + \lambda x = 0, \quad P_y + \lambda y = 0, \quad P_z - mg + \lambda z = 0. \tag{20}$$

From equations (17) and (20) we can determine λ as well as the coordinates x, y, z, of the position of equilibrium.

Calculating x, y, z, from equations (20) and substituting in (17), we got:

$$\lambda = \pm \sqrt{P_x^2 + P_y^2 + (P_z - mg)^2}\,/\,r. \tag{21}$$

Knowing λ we obtain from (20):

$$x = -P_x/\lambda, \quad y = -P_y/\lambda, \quad z = -(P_z - mg)/\lambda. \tag{22}$$

Since we have obtained two values (21) for λ, there will exist two positions of equilibrium.

Let us now assume that the point is constrained to remain within the sphere (17) or on its surface. The constraints are therefore unilateral and the coordinates of the point must satisfy the relation

$$x^2 + y^2 + z^2 - r^2 \leqq 0. \tag{23}$$

In the position of equilibrium on the surface of the sphere the virtual displacements are defined by the inequality

$$x\,\delta x + y\,\delta y + z\,\delta z \leqq 0. \tag{24}$$

The virtual work consequently satisfies the inequality

$$P_x\,\delta x + P_y\,\delta y + (P_z - mg)\,\delta z \leqq 0. \tag{25}$$

For reversible virtual displacements, i. e. those satisfying equation (18), the virtual work is zero and hence equality (19) holds. It follows from this that the position of equilibrium on the surface of the sphere can only be one of the positions given by formulae (21) and (22). In order to prove which one of them is a position of equilibrium, it is necessary to examine for which one of them the relation (24) implies (25).

Assuming that the virtual displacement satisfies condition (24) we obtain by (22) after substituting in (24)

$$-[P_x\,\delta x + P_y\,\delta y + (P_z - mg)\,\delta z]\,/\,\lambda \leqq 0. \tag{26}$$

We see from this that formula (25) will be satisfied only then when $-1/\lambda > 0$, i. e. when $\lambda < 0$.

Therefore formulae (21) and (22) define the position of equilibrium; in formula (21) it is necessary to choose the sign „—".

§ 5. Lagrange's generalized coordinates. Parameters of a system. The position of a system of points or of a rigid body is defined by means of certain numbers. These numbers can be, in particular, the coordinates of the points with respect to a certain rectangular coordinate system; however, in many cases they can have another meaning.

For example, the coordinates of a point in a plane can be given by means of the rectangular coordinates x, y, as well as by the polar coordinates r, φ, etc. In particular, the position of a system consisting of two points A_1, A_2, whose distance d is constant and which are constrained to lie in the xy-plane, can be defined by giving either the coordinates x_1, y_1 and x_2, y_2, of these points or e. g. by the coordinates x_0, y_0, of the centre of the segment A_1A_2 and the angle φ which this segment makes with the x-axis. Knowing x_0, y_0, and φ, we determine the coordinates of the points A_1, A_2, from the formulae:

$$x_1 = x_0 - \tfrac{1}{2}d\cos\varphi, \quad y_1 = y_0 - \tfrac{1}{2}d\sin\varphi,$$
$$x_2 = x_0 + \tfrac{1}{2}d\cos\varphi, \quad y_2 = y_0 + \tfrac{1}{2}d\sin\varphi.$$

We can define the position of a rigid body in space similarly by choosing an arbitrary system of coordinates (ξ, η, ζ) attached rigidly to the body, and giving the coordinates x_0, y_0, z_0, of the origin of the system (ξ, η, ζ) with respect to a fixed system (x, y, z) as well as the angles $\alpha_1, \ldots, \gamma_3$, which the axes ξ, η, ζ, make with the axes x, y, z, of the fixed system. The coordinates of the points of the body are then determined by formulae (I), p. 53.

The position of a rigid body having a fixed axis is defined by one number φ denoting the angle through which it would be necessary to ro-

tate the body about the axis in order that it pass from its initial position to the given position. Choosing the axis of rotation as the axis of z and denoting by ξ, η, ζ, the coordinates of an arbitrary point in the initial position, and by x, y, z, the coordinates of this point after a rotation through the angle φ, we have:

$$x = \xi \cos\varphi - \eta \sin\varphi, \quad y = \xi \sin\varphi + \eta \cos\varphi, \quad z = \zeta.$$

Let a holonomo-scleronomic system consisting of n material points be given.

Let us assume that to every position of a system of points which is compatible with the constraints and near the position investigated there corresponds a set of k numbers $q_1, \ldots, q_k$, in such a way that to different positions of the system there correspond different sets of numbers $q_1, \ldots, q_k$. From this assumption it follows that the coordinates $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, of the points of the system are functions of the variables $q_1, \ldots, q_k$:

$$\begin{gathered} x_1 = f_1(q_1, \ldots, q_k), \quad y_1 = \varphi_1(q_1, \ldots, q_k), \quad z_1 = \psi_1(q_1, \ldots, q_k), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = f_n(q_1, \ldots, q_k), \quad y_n = \varphi_n(q_1, \ldots, q_k), \quad z_n = \psi_n(q_1, \ldots, q_k). \end{gathered}$$

We write the above equations more briefly in the form:

$$x_i = f_i(q_1, \ldots, q_k), \quad y_i = \varphi_i(q_1, \ldots, q_k), \quad z_i = \psi_i(q_1, \ldots, q_k), \quad (i = 1, 2, \ldots, n). \tag{I}$$

In general, we assume that the functions f_i, φ_i, ψ_i, are together with their partial derivatives continuous in a certain region of the variables $q_1, \ldots, q_k$, and moreover that to different sets of numbers $q_1, \ldots, q_k$, there correspond in this region different sets of numbers $x_1, \ldots, z_n$.

The numbers $q_1, \ldots, q_k$, are called the *parameters of the system* or *Lagrange's generalized coordinates*.

The coordinates $x_1, \ldots, z_n$, with respect to a certain inertial frame are called *natural coordinates* in order to distinguish them from Lagrange's generalized coordinates.

They are obviously a particular case of Lagrange's coordinates.

Parameters are said to be *independent* if to every set of the variables $q_1, \ldots, q_k$, the functions (I) correspond to positions of the system compatible with the constraints.

In the case of bilateral constraints we can in general choose independent parameters. For let the constraints of the system be defined by the functions:

$$F_j(x_1, \ldots, z_n) = 0 \qquad (j = 1, 2, \ldots, m). \tag{1}$$

From equations (1) we can in general determine m unknowns knowing the remaining $k = 3n - m$. Therefore choosing arbitrarily k variables from among $x_1, \ldots, z_n$, and denoting them by $q_1, \ldots, q_k$, we shall be able to represent the variables $x_1, \ldots, z_n$, as functions of the variables $q_1, \ldots, q_k$, in the form (I). Since to the arbitrarily chosen values $q_1, \ldots, q_k$, there correspond the variables $x_1, \ldots, z_n$, satisfying the system of equations (1), the parameters are independent.

The number $k = 3n - m$ was called the number of degrees of freedom of the system (p. 421).

Therefore: *the number of independent parameters is equal to the number of degrees of freedom of a system.*

If the parameters are dependent, then in the case of bilateral constraints certain relations

$$\Phi_j(q_1, \ldots, q_k) = 0 \qquad (j = 1, 2, \ldots, \varrho) \tag{2}$$

must hold among those parameters $q_1, \ldots, q_k$, which define the position of the system compatible with the constraints.

Choosing certain $k - \varrho$ parameters arbitrarily, with their aid we can determine the remaining parameters from relations (2). The parameters chosen will be independent parameters.

In the case of unilateral constraints the parameters defining the position of the system compatible with the constraints must satisfy, in addition to relations of the form (2), certain inequalities of the form

$$\Psi_r(q_1, \ldots, q_k) \leqq 0 \qquad (r = 1, 2, \ldots, s). \tag{3}$$

If the parameters are independent, then the functions (I) define the constraints of the system, because they give all of its positions compatible with the constraints. If the parameters are dependent, then in addition to functions (I) it is necessary to give relations (2) and (3), which the parameters corresponding to the positions of the system compatible with the constraints must satisfy.

Example 1. A point is constrained to lie on the surface of the sphere $x^2 + y^2 + z^2 = r^2$. Choosing arbitrary x and y, we have for the upper hemisphere $z = \sqrt{r^2 - x^2 - y^2}$. Hence if we put

$$x = q_1, \quad y = q_2, \quad \text{then} \quad z = \sqrt{r^2 - q_1^2 - q_2^2}. \tag{4}$$

The numbers q_1 and q_2 are independent parameters.

Example 2. A material point A is constrained to remain on the surface of the sphere $x^2 + y^2 + z^2 - r^2 = 0$. Denoting by q_1, q_2, q_3, the co-

sines of the angles made by the vector $\overline{OA}$ (where O denotes the origin of the coordinate system) with the coordinate axes, we have:

$$x = rq_1, \quad y = rq_2, \quad z = rq_3. \tag{5}$$

The parameters q_1, q_2, q_3, are dependent, for they must obviously satisfy the equation

$$q_1^2 + q_2^2 + q_3^2 - 1 = 0; \tag{6}$$

from this equation (when $z \geqq 0$) we obtain $q_3 = \sqrt{1 - q_1^2 - q_2^2}$, whence by substituting in (5):

$$x_1 = rq_1, \quad y = rq_2, \quad z = r\sqrt{1 - q_1^2 - q_2^2}. \tag{7}$$

The parameters q_1 and q_2 in representation (7) are independent.

Virtual displacements. Let the constraints be defined parametrically by the functions:

$$x_i = f_i(q_1, \ldots, q_k), \quad y_i = \varphi_i(q_1, \ldots, q_k), \quad z_i = \psi_i(q_1, \ldots, q_k), \quad (i = 1, 2, \ldots, n). \tag{II}$$

Let us assume that the parameters are independent.

If we give the system an arbitrary motion compatible with the constraints, then $q_1, \ldots, q_k$ will be functions of the time. Differentiating (II), we obtain:

$$\dot{x}_i = \frac{\partial f_i}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial f_i}{\partial q_k}\dot{q}_k, \quad \dot{y}_i = \frac{\partial \varphi_i}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial \varphi_i}{\partial q_k}\dot{q}_k,$$

$$\dot{z}_i = \frac{\partial \psi_i}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial \psi_i}{\partial q_k}\dot{q}_k, \quad (i = 1, 2, \ldots, n). \tag{8}$$

Conversely, if we assume that $q_1, \ldots, q_k$, are arbitrary functions of time, the formulae (II) obviously define the motion of the system compatible with the constraints; hence equations (8) give the velocities of the points of the system in this motion. It follows from this that if the system is in a certain position defined by the parameters $q_1, \ldots, q_k$, then all the systems of possible velocities in this position are obtained from (8) by substituting arbitrary values for $\dot{q}_1, \ldots, \dot{q}_k$. Assuming:

$$\delta x_1 = \dot{x}_1, \quad \ldots, \quad \delta z_n = \dot{z}_n,$$
$$\delta q_1 = \dot{q}_1, \quad \ldots, \quad \delta q_k = \dot{q}_k,$$

and writing $\frac{\partial x_i}{\partial q_1}$ instead of $\frac{\partial f_i}{\partial q_1}$, etc., we obtain from (8):

$$\delta x_i = \frac{\partial x_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial x_i}{\partial q_k}\delta q_k, \quad \delta y_i = \frac{\partial y_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial y_i}{\partial q_k}\delta q_k,$$

$$\delta z_i = \frac{\partial z_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial z_i}{\partial q_k}\delta q_k \quad (i = 1, 2, \ldots n). \tag{III}$$

Substituting arbitrary values for $\delta q_1, \ldots, \delta q_k$, in formulae (III), we obtain the virtual displacements of a system. Conversely, every virtual displacement of a system is obtained by the substitution of suitable values of $\delta q_1, \ldots, \delta q_k$.

Let us note that the system of formulae (III) can be obtained by forming the derivatives of the system of formulae (II) formally, and then writing $\delta x_i, \delta y_i, \delta z_i$, instead of dx_i, dy_i, dz_i, and $\delta q_1, \ldots, \delta q_k$, instead of $dq_1, \ldots, dq_k$.

In example 1, p. 453, we have from formulae (4):

$$\delta x = \delta q_1, \quad \delta y = \delta q_2, \quad dz = -(q_1\,\delta q_1 + q_2\,\delta q_2) / \sqrt{1 - q_1^2 - q_2^2}.$$

Choosing the values of δq_1 and δq_2 arbitrarily, we obtain the virtual displacements $\delta x, \delta y, \delta z$, in the position corresponding to the parameters q_1 and q_2.

If $q_1, \ldots, q_k$, are dependent parameters and relations of the form (2), p. 453, hold among them, then $\delta q_1, \ldots, \delta q_k$, are not arbitrary numbers in formulae (III), but — as can be shown (cf. the proof of formula (15), p. 426) — they must satisfy the system of equations

$$\frac{\partial \Phi_j}{\partial q_1}\delta q_1 + \ldots + \frac{\partial \Phi_j}{\partial q_k}\delta q_k = 0 \quad (j = 1, 2, \ldots, \varrho). \tag{IV}$$

In example 2, p. 453, we have from formulae (5):

$$\delta x = r\,\delta q_1, \quad \delta y = r\,\delta q_2, \quad \delta z = r\,\delta q_3,$$

where by (6), p. 454, the relation

$$q_1\,\delta q_1 + q_2\,\delta q_2 + q_3\,\delta q_3 = 0 \text{ holds among } \delta q_1, \delta q_2, \delta q_3.$$

It can be shown (cf. the proof of formula (II), p. 432) that if the constraints are unilateral and in addition to relations (2), p. 453, relations (3) hold, (p. 453), then $\delta q_1, \ldots, \delta q_k$, must satisfy besides (IV) those relations from among

$$\frac{\partial \Psi_r}{\partial q_1}\delta q_1 + \ldots + \frac{\partial \Psi_r}{\partial q_k}\delta q_k \leqq 0 \quad (r = 1, 2, \ldots, s), \tag{V}$$

for which the equality $\Phi_r = 0$ holds in a given position of the system.

Virtual work. Generalized forces. Let the constraints of a system be defined parametrically by the equations:

$$x_i = f_i(q_1, \ldots, q_k), \quad y_i = \varphi_i(q_1, \ldots, q_k), \quad z_i = \psi_i(q_1, \ldots, q_k), \quad (i = 1, 2, \ldots, n). \tag{9}$$

The virtual displacements are expressed by formulae (III), p. 454.

If the parameters are independent, then $\delta q_1, \ldots, \delta q_k$, are arbitrary numbers. In the contrary case certain relations (IV) and (V) hold among them.

Substituting in (I′), p. 434, the expressions (III), p. 454, for δx_i, δy_i, δz_i we obtain

$$\delta' L = \sum_{i=1}^{n}\Big[P_{i_x}\Big(\frac{\partial x_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial x_i}{\partial q_k}\delta q_k\Big) + \\ + P_{i_y}\Big(\frac{\partial y_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial y_i}{\partial q_k}\delta q_k\Big) + P_{i_z}\Big(\frac{\partial z_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial z_i}{\partial q_k}\delta q_k\Big)\Big]. \quad (10)$$

Arranging the terms according to $\delta q_1, \ldots, \delta q_k$, we get

$$\delta q_1 \sum_{i=1}^{n}\Big(P_{i_x}\frac{\partial x_i}{\partial q_1} + P_{i_y}\frac{\partial y_i}{\partial q_1} + P_{i_z}\frac{\partial z_i}{\partial q_1}\Big) + \ldots + \\ + \delta q_k \sum_{i=1}^{n}\Big(P_{i_x}\frac{\partial x_i}{\partial q_n} + P_{i_y}\frac{\partial y_i}{\partial q_k} + P_{i_z}\frac{\partial z_i}{\partial q_k}\Big).$$

Denoting by $Q_1, \ldots, Q_k$, the sums appearing as coefficients of $\delta q_1, \ldots, \delta q_k$:

$$Q_1 = \sum_{i=1}^{n}\Big(P_{i_x}\frac{\partial x_i}{\partial q_1} + P_{i_y}\frac{\partial y_i}{\partial q_1} + P_{i_z}\frac{\partial z_i}{\partial q_1}\Big), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ Q_k = \sum_{i=1}^{n}\Big(P_{i_x}\frac{\partial x_i}{\partial q_k} + P_{i_y}\frac{\partial y_i}{\partial q_k} + P_{i_z}\frac{\partial z_i}{\partial q_k}\Big), \quad \text{(VI)}$$

which we write more briefly as

$$Q_j = \sum_{i=1}^{n}\Big(P_{i_x}\frac{\partial x_i}{\partial q_j} + P_{i_y}\frac{\partial y_i}{\partial q_j} + P_{i_z}\frac{\partial z_i}{\partial q_j}\Big) \qquad (j = 1, 2, \ldots, k). \quad \text{(VI′)}$$

We obtain from this in virtue of (10) $\delta' L = Q_1\,\delta q_1 + \ldots + Q_k\,\delta q_k$, i.e.

$$\delta' L = \sum_{j=1}^{k} Q_j\,\delta q_j. \quad \text{(VII)}$$

The expressions $Q_1, \ldots, Q_k$, defined by formulae (VI) and (VI′) are called the *components of the generalized force* or briefly *generalized forces*.

Comparing formula (VII) with formula (I′), p. 434, we see that the virtual work is expressed by means of the components Q_j of the generalized forces and by means of the displacements δq_j in a similar manner as by means of the components $P_{i_x}, P_{i_y}, P_{i_z}$, and the displacements δx_i, δy_i, δz_i.

Conditions of equilibrium. Let the constraints of a holonomo-scleronomic system be defined be means of the parameters $q_1, \ldots, q_k$. By (VII) condition (II), p. 437, of the equilibrium of a system assumes the form

$$\delta' L = \sum_{j=1}^{k} Q_j\,\delta q_j \leq 0. \quad \text{(VIII)}$$

If the constraints are bilateral, then ((III), p. 437)

$$\delta' L = \sum_{j=1}^{k} Q_j\,\delta q_j = 0. \quad \text{(IX)}$$

The generalized forces $Q_1, \ldots, Q_k$, appearing in formulae (VIII) and (IX) are defined by formulae (VI) and (VI′).

If the parameters $q_1, \ldots, q_k$, are independent, then $\delta q_1, \ldots, \delta q_k$, are arbitrary. Hence by (IX), taking $\delta q_1 = 1$ and $\delta q_2 = \delta q_3 = \ldots = \delta q_k = 0$, we get $Q_1 = 0$ and similarly $Q_2 = 0, \ldots, Q_k = 0$.

Therefore: *if the parameters defining the position of a holonomo-scleronomic system are independent (and there is no friction), then the necessary and sufficient condition for the equilibrium of the acting forces is that the generalized forces be equal to zero*:

$$Q_j = 0 \qquad (j = 1, 2, \ldots, k). \quad \text{(X)}$$

If the parameters $q_1, \ldots, q_k$, are dependent and satisfy relations (2) or (3), p. 453, then $\delta q_1, \ldots, \delta q_k$, are not arbitrary numbers: they must satisfy relations (IV) or (V), p. 455. The position of equilibrium is then determined in the same way as for the natural coordinates, i. e. in the manner given on p. 445.

Remark. Natural coordinates are a particular case of generalized coordinates. Therefore, if the natural coordinates of the points of a system are $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, then putting:

$$x_1 = q_1, \quad y_1 = q_2, \quad z_1 = q_3, \quad \ldots, \quad z_n = q_{3n},$$

we can apply the formulae (I)—(X) of the present §. The results obtained in this § are hence a generalization of the corresponding results for the natural coordinates.

Example 3. Four rods of equal length a, lying in a vertical plane, are pin-connected at A, B, C, and D. The joint A is fixed and C can move only along a vertical line l passing through A. Horizontal forces $\mathbf{P}$ and $-\mathbf{P}$ act at the joints B and D, while a vertical force $\mathbf{Q}$ (having a downward sense) acts at C. Determine the position of equilibrium neglecting the weights of the rods and assuming that there is no friction and that the rods can move only in the vertical plane in which the forces $\mathbf{P}$ and $\mathbf{Q}$ lie (Fig. 313).

Let us take A as the origin of the coordinate system (x, y) of the vertical plane in which the rods lie and the line l as the y-axis (with a downward sense). Then the position of the system will be defined by giving

the angle α between the rod AB and the y-axis. Consequently α is a parameter of the system.

From Fig. 313 it is seen that the coordinates x_1, y_1, x_2, y_2, and x_3, y_3, of the joints B, C, and D, are the following:

$$x_1 = -a \sin\alpha, \quad x_2 = 0, \quad x_3 = a \sin\alpha, \qquad y_1 = a\cos\alpha, \quad y_2 = 2a\cos\alpha, \quad y_3 = a\cos\alpha. \tag{11}$$

The virtual work is

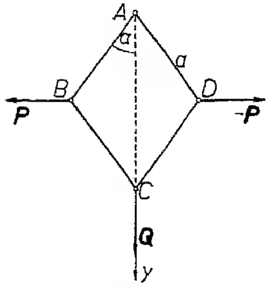

Fig. 313.

$$\delta' L = -P\,\delta x_1 + Q\,\delta y_2 + P\,\delta x_3, \tag{12}$$

where $P = |\mathbf{P}|$ and $Q = |\mathbf{Q}|$. By (11) we have:

$$\delta x_1 = -a\cos\alpha\,\delta\alpha,$$
$$\delta y_2 = -2a\sin\alpha\,\delta\alpha,$$
$$\delta x_3 = a\cos\alpha\,\delta\alpha.$$

Substituting these values in (12), we get

$$\delta' L = 2a(P\cos\alpha - Q\sin\alpha)\,\delta\alpha. \tag{13}$$

In the position of equilibrium $\delta' L = 0$, hence $2a(P\cos\alpha - Q\sin\alpha)\,\delta\alpha = 0$; consequently $P\cos\alpha - Q\sin\alpha = 0$; whence

$$\tan\alpha = P / Q.$$

Example 4. A system of rods $A_0A_1, A_1A_2, \ldots, A_{n-1}A_n$, pin-connected at $A_1, A_2, \ldots, A_{n-1}$, is given. Forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, with origins at $A_1, A_2, \ldots, A_n$, act on the system. The point A_0 is fixed. Determine the position of equilibrium, assuming that the rods and the forces lie in one plane.

Let us take the point A_0 as the origin of the coordinate system (x, y) of this plane and denote by $a_1, a_2, \ldots, a_n$, the lengths of the rods, finally by $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$, the coordinates of the points $A_1, A_2, \ldots, A_n$ (Fig. 314). We have:

$$x_1 = a_1\cos\alpha_1, \quad x_2 = a_1\cos\alpha_1 + a_2\cos\alpha_2, \quad \ldots, \qquad x_n = a_1\cos\alpha_1 + \ldots + a_n\cos\alpha_n, \tag{14}$$

$$y_1 = a_1\sin\alpha_1, \quad y_2 = a_1\sin\alpha_1 + a_2\sin\alpha_2, \quad \ldots, \qquad y_n = a_1\sin\alpha_1 + \ldots + a_n\sin\alpha_n. \tag{15}$$

From (14) and (15) it follows that the angles $\alpha_1, \alpha_2, \ldots, \alpha_n$, define the position of the system of rods. Consequently the variables $\alpha_1, \ldots, \alpha_n$, are the parameters of the system, and since they are not related in any way, they are independent parameters.

Differentiating (14) and (15), we obtain:

$$\delta x_1 = -a_1\sin\alpha_1\,\delta\alpha, \ \ldots, \ \delta x_n = -a_1\sin\alpha_1\,\delta x_1 - \ldots - a_n\sin\alpha_n\,\delta\alpha_n, \tag{16}$$

$$\delta y_1 = a_1\cos\alpha_1\,\delta x_1, \ \ldots, \ \delta y_n = a_1\cos\alpha_1\,\delta x_1 + \ldots + a_n\cos\alpha_n\,\delta\alpha_n. \tag{17}$$

In the position of equilibrium the virtual work is

$$\delta' L = (P_{1_x}\,\delta x_1 + P_{1_y}\,\delta y_1) + \ldots + (P_{n_x}\,\delta x_n + P_{n_y}\,\delta y_n) = 0. \tag{18}$$

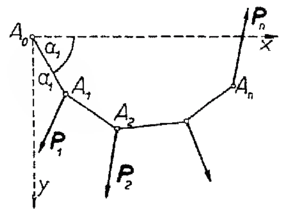

Fig. 314.

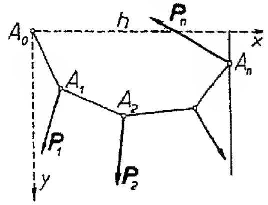

Fig. 315.

Substituting the values (16) and (17) in (18), we get after arranging terms

$$a_1[-(P_{1_x} + \ldots + P_{n_x})\sin\alpha_1 + (P_{1_y} + \ldots + P_{n_y})\cos\alpha_1]\,\delta\alpha_1 +$$
$$+ a_2[-(P_{2_x} + \ldots + P_{n_x})\sin\alpha_2 + (P_{2_y} + \ldots + P_{n_y})\cos\alpha_2]\,\delta\alpha_2 + \tag{19}$$
$$+ \ldots + a_n[-P_{n_x}\sin\alpha_n + P_{n_y}\cos\alpha_n]\,\delta\alpha_n = 0.$$

The coefficients of $\delta\alpha_1, \ldots, \delta\alpha_n$, are the generalized forces $Q_1, \ldots, Q_n$. Since equality (19) holds for every set of numbers $\delta\alpha_1, \ldots, \delta\alpha_n$, the generalized forces are zero. Consequently:

$$-(P_{1_x} + \ldots + P_{n_x})\sin\alpha_1 + (P_{1_y} + \ldots + P_{n_y})\cos\alpha_1 = 0,$$
$$-(P_{2_x} + \ldots + P_{n_x})\sin\alpha_2 + (P_{2y} + \ldots + P_{n_y})\cos\alpha_2 = 0, \tag{20}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$-P_{n_x}\sin\alpha_n + P_{n_y}\cos\alpha_n = 0.$$

From equations (20) it is easy to calculate the tangents of the angles $\alpha_1, \ldots, \alpha_n$. Since the tangents are equal for angles differing by 180°, we obtain many solutions.

If the coefficients of the sines and cosines in one of the equations (20) are zero, then the corresponding angle can be chosen arbitrarily.

Let us now assume that the point A_n is to remain on a line l having the equation $x = h$, which is the case e. g. when the end A_n of the rod $A_{n-1}A_n$ is a ring that slides on a rigid wire having the position of the line l (Fig. 315). Under this assumption the parameters $\alpha_1, \ldots, \alpha_n$, will

not be independent, since the relation $x_n = h$ will have to hold, i. e. in view of (14)

$$a_1 \cos\alpha_1 + \ldots + a_n \cos\alpha_n - h = 0. \tag{21}$$

Differentiating (21), we obtain

$$-a_1 \sin\alpha_1\, \delta x_1 + \ldots - a_n \sin\alpha_n\, \delta\alpha_n = 0; \tag{22}$$

by substituting arbitrary values for $\delta x_1, \ldots, \delta\alpha_{n-1}$, in (22) we obtain

$$\delta x_n = -(a_1 \sin\alpha_1\, \delta x_1 + \ldots + a_{n-1} \sin\alpha_{n-1}\, \delta\alpha_{n-1}) / a_n \sin\alpha_n. \tag{23}$$

In view of (22) equation (19) assumes the form

$$\begin{aligned} &a_1[-(P_{1_n} + \ldots + P_{n-1,x}) \sin\alpha_1 + (P_{1_y} + \cdots + P_{n_y}) \cos\alpha_1]\, \delta\alpha_1 + \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ a_{n-1}[-P_{n-1,x} \sin\alpha_{n-1} + (P_{n-1,y} + P_{n_y}) \cos\alpha_{n-1}]\, \delta x_{n-1} + \\ &+ a_n P_{n_y} \cos\alpha_n\, \delta\alpha_n = 0. \end{aligned} \tag{24}$$

Substituting the value from (23) in (24), we get

$$\begin{aligned} &-a_1(P_{1_x} + \ldots + P_{n-1,x} + P_{n_y} \cot\alpha_n) \sin\alpha_1\, \delta x_1 + \\ &\quad + a_1 (P_{1_y} + \ldots + P_{n_y}) \cos\alpha_1\, \delta\alpha_1 + \ldots + \\ &+ a_{n-1}[-(P_{n-1,x} + P_{n_y} \cot\alpha_n) \sin\alpha_{n-1} + (P_{n-1,y} + P_{n_y}) \cos\alpha_{n-1}]\, \delta\alpha_{n-1} = 0. \end{aligned}$$

Since $\delta x_1, \ldots, \delta x_{n-1}$ are arbitrary numbers, their coefficients are zero. We therefore obtain a system of $n - 1$ equations:

$$\begin{aligned} &-(P_{1_x} + \ldots + P_{n-1,x} + P_{n_y} \cot\alpha_n) \sin\alpha_1 + (P_{1_y} + \ldots + P_{n_y}) \cos\alpha_1 = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &-(P_{n-1,x} + P_{n_y} \cot\alpha_n) \sin\alpha_{n-1} + (P_{n-1,y} + P_{n_y}) \cos\alpha_{n-1} = 0. \end{aligned}$$

These equations together with equations (21) constitute a system of n equations from which we can determine n unknowns $\alpha_1, \ldots, \alpha_n$.

Example 5. A heavy ring K of mass m_1 slides on a curve C lying in the vertical plane xy. A string (massless and inextensible) passes through the ring; one of the ends of the string is tied at the origin O of the coordinate system; a heavy point A of mass m_2 is carried at the other end (Fig. 316). Determine the position of equilibrium, assuming that there is no friction.

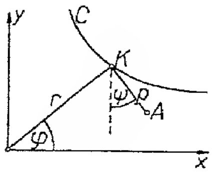

Fig. 316.

Let tho equation of the curve C in polar coordinates be

$$r = f(\varphi). \tag{25}$$

The coordinates x_1, y_1, of the point K will therefore be:

$$x_1 = r \cos\varphi, \quad y_1 = r \sin\varphi. \tag{26}$$

Let us put $\varrho = AK$. Denoting the length of the string by l we have

$$r + \varrho - l \leqq 0. \tag{27}$$

Let ψ be the angle between AK and the vertical, and x_2, y_2, the coordinates of the point A. Consequently $x_2 = x_1 + \varrho \sin\psi$, $y_2 = y_1 - \varrho \cos\psi$, whence by (26):

$$x_2 = r \cos\varphi + \varrho \sin\psi, \quad y_2 = r \sin\varphi - \varrho \cos\psi. \tag{28}$$

Equations (26) and (28) define the constraints of the system in terms of the parameters $r, \varrho, \varphi, \psi$, among which the relations (25) and (27) hold.

The virtual work is $\delta' L = -m_1 g\, \delta y_1 - m_2 g\, \delta y_2$. In the position of equilibrium $\delta' L \leqq 0$; hence

$$m_1\, \delta y_1 + m_2\, \delta y_2 \geqq 0. \tag{29}$$

When the string is not in tension the point A is free; hence δy_2 can be arbitrary. Taking $\delta y_1 = 0$ and $\delta y_2 < 0$ in this case, we should obtain $m_1\, \delta y_1 + m_2\, \delta y_2 < 0$ contrary to (29), which proves that the system cannot be in equilibrium when the string is not in tension.

Let us assume, therefore, that the string is in tension (i. e. that the equality sign holds in (27)) as well as that $r > 0$ and $\varrho > 0$; from (26) and (28) we obtain:

$$\begin{aligned} \delta y_1 &= \delta r \sin\varphi + r \cos\varphi\, \delta\varphi, \\ \delta y_2 &= \delta r \sin\varphi + r \cos\varphi\, \delta\varphi - \delta\varrho \cos\psi + \varrho\, \delta\psi \sin\psi. \end{aligned} \tag{30}$$

By (25) and (27) the following relations hold among δr, $\delta\varphi$, and $\delta\varrho$:

$$\delta r = f'(\varphi)\, \delta\varphi, \quad \delta r + \delta\varrho \leqq 0, \tag{31}$$

while $\delta\psi$ is arbitrary.

Substituting for δy_1, δy_2, in (29) the expressions from (30), we obtain

$$\begin{aligned} &(m_1 + m_2) \sin\varphi\, \delta r + (m_1 + m_2)\, r \cos\varphi\, \delta\varphi - m_2\, \delta\varrho \cos\psi + \\ &\quad + m_2 \varrho\, \delta\psi \sin\psi \geqq 0. \end{aligned} \tag{32}$$

In virtue of (31) we can assume that $\delta r = 0$, $\delta\varphi = 0$, and $\delta\varrho = 0$, whence by (32)

$$m_2 \varrho\, \delta\psi \sin\psi \geqq 0. \tag{33}$$

Since $\delta\psi$ is arbitrary, inequality (33) will hold only when $m_2 \varrho \sin\psi = 0$ or when $\sin\psi = 0$, and therefore for $\psi = 0$ and $\psi = \pi$. It is intuitively evident that in the position of equilibrium we can only have $\psi = 0$, for the equality $\psi = \pi$ means that A is above K, which is obviously

impossible in the position of equilibrium. This also follows from relation (32), for by (31) assuming $\delta r = 0$, $\delta\varphi = 0$, $\delta\varrho < 0$, and $\delta\psi = 0$, we should have from (32) $-m_2\,\delta\varrho\cos\psi \geqq 0$, and since $\delta\varrho < 0$, $\cos\psi \geqq 0$, whence $\psi \neq \pi$.

Consequently we have proved that $\psi = 0$ in the position of equilibrium, i. e. that A is below K. Substituting $\psi = 0$ in (32), we obtain by (25) and (31)

$$(m_1 + m_2)(f'(\varphi)\sin\varphi + f(\varphi)\cos\varphi)\,\delta\varphi - m_2\,\delta\varrho \geqq 0. \tag{34}$$

By (31) we can put

$$\delta r + \delta\varrho = 0, \quad \text{whence} \quad \delta\varrho = -\delta r = -f'(\varphi)\,\delta\varphi,$$

where $\delta\varphi$ is arbitrary. Substituting in (34), we get then

$$[(m_1 + m_2)(f'(\varphi)\sin\varphi + f(\varphi)\cos\varphi) + m_2 f'(\varphi)]\,\delta\varphi \geqq 0.$$

Since $\delta\varphi$ is arbitrary, this relation holds only when

$$(m_1 + m_2)(f'(\varphi)\sin\varphi + f(\varphi)\cos\varphi) + m_2 f'(\varphi) = 0. \tag{35}$$

Equation (35) enables one to determine the angle φ in the position of equilibrium.

Conversely, if equation (35) holds, then inequality (34) must be satisfied. For let us denote by W the left side of this inequality. By (35) $W = -m_2 f'(\varphi)\,\delta\varphi - m_2\,\delta\varrho$, whence by (31) $W = -m_2(\delta r + \delta\varrho)$. Since, because of (31), $\delta r + \delta\varrho \leqq 0$, whence $W \geqq 0$. Therefore, if the angle φ satisfies equation (35), then equilibrium occurs.

We shall yet investigate for what curve C equilibrium occurs for every value of φ, i. e. the case when equation (35) becomes an identity.

Let us note in this connection that the left side of equation (35) is the derivative of the function $f(\varphi)[(m_1 + m_2)\sin\varphi + m_2]$; consequently

$$f(\varphi)[(m_1 + m_2)\sin\varphi + m_2] = c = \text{const.}$$

Since $r = f(\varphi)$ by (1),

$$r = \frac{c}{(m_1 + m_2)\sin\varphi + m_2}. \tag{36}$$

If $c \neq 0$, $m_1 > 0$ and $m_2 > 0$, then the above equation is the equation of the lower branch of the hyperbola whose real axis is the y-axis.

Equilibrium in a potential field. Let us assume that the forces $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n$, have the potential V. Consequently ((III), p. 211):

$$P_{i_x} = \frac{\partial V}{\partial x_i}, \quad P_{i_y} = \frac{\partial V}{\partial y_i}, \quad P_{i_z} = \frac{\partial V}{\partial z_i}. \tag{37}$$

Substituting these values in formula (VI′), p. 456, we get

$$Q_j = \sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_j}\right). \tag{38}$$

The potential V is a function of the variables $x_1, \ldots, z_n$. Expressing them in terms of the parameters $q_1, \ldots, q_k$, we can therefore assume that V is a function of the parameters $q_1, \ldots, q_k$. From a well-known formula in differential calculus on the partial derivative of a compound function, it follows that the right side of equation (38) is the partial derivative $\partial V / \partial q_j$. Consequently

$$Q_j = \partial V / \partial q_j \qquad (j = 1, 2, \ldots, k). \tag{XI}$$

Comparing (37) and (XI) we see that the components of the generalized forces are expressed similarly as the components of forces relative to the natural coordinates.

Therefore: *if a force field is a potential field, then the components of the generalized forces are the partial derivatives of the potential with respect to the generalized coordinates.*

From formulae (VII), p. 456, and (XI) we obtain

$$\delta' L = \sum_{j=1}^{k}\frac{\partial V}{\partial q_j}\,\delta q_j. \tag{39}$$

By (18), p. 428, we can therefore write

$$\delta' L = \delta V, \tag{XII}$$

where V is considered as a function of the variables $q_1, \ldots, q_k$.

Formula (XII) has the same form as formula (I″), p. 434. The difference consists in the fact that in formula (I″) we consider V as a function of the natural coordinates $x_1, \ldots, z_n$, while in formula (XII) we consider V as a function of the parameters $q_1, \ldots, q_k$.

Remark. The meaning of the expression δV is illustrated as follows. At the moment t let us give the system in a given position an arbitrary motion compatible with the constraints. The parameters $q_1, \ldots, q_k$, as well as the potential V, will therefore be functions of the time. We have

$$\frac{dV}{dt} = \sum_{j=1}^{k}\frac{\partial V}{\partial q_j}\dot{q}_j. \tag{40}$$

Since we can assume that $\dot{q}_j = \delta q_j$ for $j = 1, 2, \ldots$, by (40) we have $\delta V = dV / dt$. In this way δV represents the rate of change (i. e. the derivative) of the potential for an arbitrarily given motion of the system compatible with the constraints.

From the principle of virtual work it follows by (XII) that the necessary and sufficient condition for the equilibrium of a system is that

$$\delta V \leqq 0. \tag{XIII}$$

In particular, therefore, *a position of equilibrium of a system is a position in which the potential attains a maximum, and in the case of bilateral constraints a position in which the potential attains a minimum.*

For if we give a system an arbitrary motion compatible with the constraints at the moment when V attains a maximum, then after a time Δt the increase in the potential will be $\Delta V \leqq 0$, from which $\mathrm{d}V / \mathrm{d}t \leqq 0$ (where the inequality $\mathrm{d}V / \mathrm{d}t < 0$ may hold in the boundary position for unilateral constraints), and consequently $\delta V \leqq 0$ (in view of the remark on the meaning of δV). If the constraints are bilateral and V has a minimum value, then $\mathrm{d}V / \mathrm{d}t = 0$, whence $\delta V = 0$.

If the only forces acting on a system of material points are the gravitational forces, then the potential is ((13), p. 211)

$$V = -mgz_0, \tag{41}$$

where m denotes the total mass of the system, and z_0 the coordinate of the centre of mass, under the assumption that the axis of z has a sense vertically upwards.

The position of equilibrium will therefore be every position at which V is a maximum or z_0 a minimum. If the constraints are bilateral, then the position of equilibrium will be in addition to this every position in which V is a minimum or z_0 a maximum.

If a heavy point A hangs on an inextensible string tied at the point O, then the extrema of the potential V occur when the string is in tension and has a vertical direction. A maximum occurs when A is below O, and a minimum occurs when A is above O. It is obvious that the position of equilibrium occurs only when V has a maximum value (i. e. when A is under O).

Example 6. Two heavy material points A_1 and A_2 of masses m_1 and m_2 are connected by an inextensible string passing over a pulley. The point m_2 is constrained to remain on a vertical line l. What angle does the string make with the line l in the position of equilibrium, if there is no friction (Fig. 130)?

Let us take the line l as the axis of z,, giving it an upward sense, and the point of the axis which is at the top of the pulley as the origin of the coordinate system. Let us denote by s the length of the string, by a the distance of the pulley from the axis l, by z_1 and z_2 the coordinates of the points A_1 and A_2, and by z_0 the coordinate of the centre of mass. We have:

$$z_1 = -(s - a / \sin\varphi), \quad z_2 = -a\cot\varphi.$$

Since $mz_0 = m_1z_1 + m_2z_2$, where $m = m_1 + m_2$,

$$z_0 = [-m_0(s - a / \sin\varphi) - m_2 a \cot\varphi] / m.$$

In order to determine an extremum of z_0, let us set the derivative $\mathrm{d}z_0 / \mathrm{d}\varphi$ equal to zero:

$$-m_1 a \cos\varphi / \sin^2\varphi + m_2 a / \sin^2\varphi = 0,$$

whence

$$\cos\varphi = m_2 / m_1. \tag{42}$$

It is easy to show that z_0 is a minimum for the value of φ satisfying equation (42). Therefore equation (42) defines the position of equilibrium (when $m_2 < m_1$).

Another way of solving this problem is given in example 2, p. 191.

CHAPTER X

DYNAMICS OF HOLONOMIC SYSTEMS

§ 1. Holonomic systems. In this chapter we shall consider the dynamics of certain constrained systems. We shall derive for them equations of motion in which the reactions will not appear.

Let a system of n material points be given. Let the constraints of the system be such that only certain positions of the system are possible at each moment. We do not assume (as in chap. IX) that the same positions are possible at each moment: the set of all possible positions of the system can change together with time.

An example is a point which can remain on a moving plane or a moving curve, e. g. a bead strung on a wire which moves or alters its shape.

If the constraints are bilateral (p. 419) at the moment t, then the coordinates $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, of the points of the system must satisfy certain equations (p. 421) at the time t:

$$F_1(x_1, \ldots, z_n, t) = 0, \ldots, F_m(x_1, \ldots, z_n, t) = 0; \tag{1}$$

we write them briefly as:

$$F_j(x_1, \ldots, z_n, t) = 0 \qquad (j = 1, 2, \ldots, m). \tag{I}$$

If the constraints are unilateral at the time t, then, in addition to (I), the relations ((9), p. 420)

$$\Phi_r(x_1, \ldots, z_n, t) \leqq 0 \qquad (r = 1, 2, \ldots, s) \tag{II}$$

hold.

A system whose constraints can be represented by means of relations of the form (I) and (II) is called *holonomic*.

If the functions F_j and Φ_r do not depend on the time t, we say that the *constraints* are *independent of the time* and the system is called *scleronomic*.

In chap. IX we investigated the conditions of equilibrium of holonomo-scleronomic systems. The constraints of such systems can be defined by relations of the form (I), p. 421, and (9), p. 420:

$$F_j(x_1, y_1, z_1, \ldots, z_n) = 0 \qquad (j = 1, 2, \ldots, m), \tag{I'}$$

$$\Phi_r(x_1, y_1, z_1, \ldots, z_n) \leqq 0 \qquad (r = 1, 2, \ldots, s). \tag{II'}$$

If at least one of the functions in (I) or (II) depends on the time t, we say that the constraints *depend on the time* and the system is called *rheonomic*.

It is easy to see that a scleronomic system is a particular example of a rheonomic system; in other words, equations (I') and (II') are a particular case of equations (I) and (II).

In general, the functions F_j, Φ_r, are assumed to be continuous and to have continuous partial derivatives in a certain region of the variables

$$x_1, y_1, z_1, \ldots, x_n, y_n, z_n, t.$$

The equations (I), (I'), (II), (II'), are said to represent the *constraints in a finite form.*

Just as in scleronomic systems (p. 421), we assume that the functions (I) are independent of each other and that $m < 3n$. The number $k = 3n - m$ is called the *number of degrees of freedom* of the given system.

Example. Let a material point $A(x, y, z)$ be constrained to remain on the surface of a certain sphere which moves with a uniform advancing motion.

Let us denote by r the radius of the sphere, by ξ_0, η_0, ζ_0, the coordinates of the centre of the sphere at the time $t_0 = 0$, by ξ, η, ζ, the coordinates at the time t, and by a, b, c, the projections of the velocity of the advancing motion on the coordinate axes. At the time t we have $\xi = \xi_0 + at$, $\eta = \eta_0 + bt$, and $\zeta = \zeta_0 + ct$. The sphere therefore has the equation $(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 - r^2 = 0$ at the time t; hence

$$(x - \xi_0 - at)^2 + (y - \eta_0 - bt)^2 + (z - \zeta_0 - ct)^2 - r^2 = 0. \tag{2}$$

Hence the coordinates of the point A must satisfy equation (2) at each moment; as it is of the form $F(x, y, z, t) = 0$, the constraints are bilateral, dependent on the time, and therefore the system is holonomo-rheonomic.

If we assume that the point A has to remain within the sphere or on its surface, then the constraints are expressed by the inequality

$$(x - \xi_0 - at)^2 + (y - \eta_0 - bt)^2 + (z - \zeta_0 - ct)^2 - r^2 \leqq 0, \tag{3}$$

and hence they will be unilateral in this case.

§ 2. Non-holonomic systems. Not always can the bilateral constraints of a system be represented in the finite form (I) or (I').

Let us suppose, for example, that to each point A of space there corresponds a vector $\boldsymbol{H}$ whose projections depend on the coordinates x, y, z, of the point A. Consequently:

$$H_x = \alpha(x, y, z),\ H_y = \beta(x, y, z),\ H_z = \gamma(x, y, z), \tag{1}$$

where α, β, γ, are given functions.

Let us assume that a material point can move only in such a way that its velocity in every position is perpendicular to $\boldsymbol{H}$. Let us denote by $\dot{x}, \dot{y}, \dot{z}$ the projections of the velocity $\mathbf{v}$ of the material point. Therefore at each moment the relation $\boldsymbol{H}\mathbf{v} = 0$ must hold, whence

$$\alpha(x, y, z)\, x^{\cdot} + \beta(x, y, z)\, y^{\cdot} + \gamma(x, y, z)\, z^{\cdot} = 0. \tag{2}$$

If there exists a function $F(x, y, z)$ such that its partial derivatives are equal to the corresponding functions α, β, γ, then equation (2) can be written in the form $\mathrm{d}F / \mathrm{d}t = 0$, whence $F = \text{const} = c$, i. e.

$$F(x, y, z) - c = 0. \tag{3}$$

Conversely, if equation (3) holds, then differentiating it, we obtain (2). Equations (2) and (3) are therefore equivalent in this case, and consequently the constraints are holonomic, since they can be represented in the finite form (3).

However, if the functions α, β, γ, are not the partial derivatives of a function, then equation (2) may be not equivalent to any equation of the form (3). In this case, therefore, the constraints cannot be represented in a finite form and the system is said to be *non-holonomic*.

Equation (2) is usually written in the form

$$\alpha(x, y, z)\, \mathrm{d}x + \beta(x, y, z)\, \mathrm{d}y + \gamma(x, y, z)\, \mathrm{d}z = 0. \tag{4}$$

An equation of a more general form is

$$\alpha(x, y, z, t)\, \mathrm{d}x + \beta(x, y, z, t)\, \mathrm{d}y + \gamma(x, y, z, t)\, \mathrm{d}z + \varepsilon(x, y, z, t)\, \mathrm{d}t = 0. \tag{5}$$

Equation (5) is equivalent to the equation

$$\alpha x^{\cdot} + \beta y^{\cdot} + \gamma z^{\cdot} + \varepsilon = 0, \tag{6}$$

where $\alpha, \beta, \gamma, \varepsilon$, are given functions of the variables x, y, z, and t. It constitutes the necessary condition which the velocities of the points of the system must satisfy. We shall not examine non-holonomic systems more closely.

§ 3. Virtual displacements. On p. 426 we defined the virtual displacement of holonomo-scleronomic systems. We shall now consider rheonomic systems.

Point on a surface. Let the point $A(x, y, z)$ be constrained to remain on a moving surface S whose equation at the moment t is

$$F(x, y, z, t) = 0. \tag{1}$$

The coordinates of the point A therefore satisfy equation (1) at the moment t.

A *virtual displacement* is said to be every displacement $\overline{\delta s}$ of the point A with projections $\delta x, \delta y, \delta z$, satisfying the equation

$$\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z = 0. \tag{2}$$

We see from this that a virtual displacement is such as if the surface S were fixed and had the position it occupies at the moment t. Consequently a virtual displacement is an arbitrary vector tangent at the moment t to the surface S at the point A (p. 423).

Let us give the point A an arbitrary motion compatible with the constraints. The coordinates of the point will therefore satisfy equation (1). Forming the derivative with respect to the time t, we obtain from (1)

$$\frac{\partial F}{\partial x}x^{\cdot} + \frac{\partial F}{\partial y}y^{\cdot} + \frac{\partial F}{\partial z}z^{\cdot} + \frac{\partial F}{\partial t} = 0. \tag{3}$$

Denoting by $\mathbf{v}$ the velocity of the point A, we obtain from (3)

$$\frac{\partial F}{\partial x}v_x + \frac{\partial F}{\partial y}v_y + \frac{\partial F}{\partial z}v_z + \frac{\partial F}{\partial t} = 0. \tag{4}$$

Comparing (2) and (4), we see that we cannot take $\delta x = v_x$, $\delta y = v_y$, and $\delta z = v_z$, i. e. $\overline{\delta s} = \mathbf{v}$, unless $\partial F / \partial t = 0$.

Therefore, in rheonomic systems the virtual displacements in general are not proportional to possible velocities (as in scleronomic systems), i. e. they are expressed by vectors other than possible velocities.

In particular, the displacement $\overline{\delta s} = 0$ is by (2) a virtual displacement (i. e. $\delta x = 0$, $\delta y = 0$, $\delta z = 0$), and from (4) it follows that if $\partial F / \partial t \neq 0$, then $\mathbf{v} = 0$ is not a possible velocity.

Remark. The total differential of function (1) is

$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z + \frac{\partial F}{\partial t}\mathrm{d}t.$$

If we take the differential under the assumption that $t = \text{const}$, then $\mathrm{d}t = 0$; consequently

$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z.$$

Hence equation (2) is obtained by forming the differential of both sides of (1) under the assumption that the time $t = \text{const}$, and then writing δx, δy, δz, for dx, dy, dz, respectively.

In the example on p. 467 the virtual displacement satisfies the equation which one obtains by differentiating equation (2), p. 467, under the assumption that $t = \text{const}$. We get:

$$(x - \xi_0 - at)\,\delta x + (y - \eta_0 - bt)\,\delta y + (z - \zeta_0 - ct)\,\delta z = 0.$$

Choosing δy, δz, arbitrarily, we can determine δx from this equation.

Point on a curve. Let a material point A be constrained to remain on the moving curve C whose equations at the time t are:

$$F_1(x, y, z, t) = 0, \quad F_2(x, y, z, t) = 0. \tag{7}$$

A *virtual displacement* of the point A at the moment t is said to be a displacement $\overline{\delta s}$ (having the projections δx, δy, δz) which satisfies the equations:

$$\frac{\partial F_1}{\partial x}\delta x + \frac{\partial F_1}{\partial y}\delta y + \frac{\partial F_1}{\partial z}\delta z = 0, \quad \frac{\partial F_2}{\partial x}\delta x + \frac{\partial F_2}{\partial y}\delta y + \frac{\partial F_2}{\partial z}\delta z = 0. \tag{8}$$

Consequently the virtual displacement is such as if the curve C were fixed and had that position which it occupies at the time t. The virtual displacement is therefore an arbitrary vector tangent at the time t to the curve C at the point A (p. 424).

In this case also the virtual displacement is generally not proportional to a possible velocity. For by (7) the possible velocity $\mathbf{v}$ satisfies the equations (which are obtained by forming the derivatives of equations (7)):

$$\begin{aligned} \frac{\partial F_1}{\partial x}v_x + \frac{\partial F_1}{\partial y}v_y + \frac{\partial F_1}{\partial z}v_z + \frac{\partial F_1}{\partial t} = 0, \\ \frac{\partial F_2}{\partial x}v_x + \frac{\partial F_2}{\partial y}v_y + \frac{\partial F_2}{\partial z}v_z + \frac{\partial F_2}{\partial t} = 0. \end{aligned} \tag{9}$$

If $\partial F_1 / \partial t \neq 0$ or $\partial F_2 / \partial t \neq 0$, then by (8) and (9) we cannot take $\delta x = v_x$, $\delta y = v_y$, $\delta z = v_z$, i. e. $\overline{\delta s} = \mathbf{v}$.

Let us still note that equations (8) are obtained by forming the differentials of equations (7), under the assumption that $t = \text{const}$, and writing δx, δy, δz, instead of dx, dy, dz.

Example I. A material point A is constrained to remain on a parabola rotating about the z-axis with a constant angular velocity ω (positive, if the rotation takes place from right to left). At $t_0 = 0$ the parabola lies in the xz-plane and has the equation

$$z = x^2. \tag{10}$$

The parabola generates a paraboloid of revolution $z = x^2 + y^2$. The position of the parabola at the time t is obtained as the intersection of the paraboloid with the plane $x \sin \omega t + y \cos \omega t = 0$. The coordinates of the point A consequently satisfy the equations:

$$x^2 + y^2 - z = 0, \quad x \sin \omega t + y \cos \omega t = 0. \tag{11}$$

The virtual displacement δx, δy, δz, satisfies the equations obtained by differentiating (11) under the assumption that $t = \text{const}$. Therefore:

$$2x\,\delta x + 2y\,\delta y - \delta z = 0, \quad \delta x \sin \omega t + \delta y \cos \omega t = 0.$$

If $\omega t \neq \frac{1}{2}\pi$ and $\omega t \neq \frac{3}{2}\pi$, then:

$$\delta y = -\,\delta x \tan \omega t, \quad \delta z = 2(x - y \tan \omega t)\,\delta x,$$

where δx is arbitrary.

Systems of points. Let a holonomic system whose constraints are defined by the equations

$$F_j(x_1, y_1, z_1, \ldots, x_n, y_n, z_n, t) = 0 \qquad (j = 1, 2, \ldots, m) \tag{12}$$

be given.

A *virtual displacement* of a system at the moment t in the position $(x_1, \ldots, z_n)$ compatible with the constraints is defined to be every displacement $\delta x_1, \ldots, \delta z_n$, satisfying the equations:

$$\sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i}\delta x_i + \frac{\partial F_j}{\partial y_i}\delta y_i + \frac{\partial F_j}{\partial z_i}\delta z_i\right) = 0 \qquad (j = 1, 2, \ldots, m). \tag{I}$$

Equations (I) are assumed to be linearly independent at each moment t; in other words, we assume that we can choose from among the unknowns δx_i, δy_i, δz_i $k = 3n - m$ unknowns arbitrarily and determine the remaining m unknowns from equations (I).

Equations (I) have a form similar to those for a scleronomic system (cf. (I), p. 426). The virtual displacements of a system at the moment t are therefore such as if the constraints did not depend on the time and were constantly such as at the time t.

Equations (I) are obtained by forming the differentials of equations (12), under the assumption that $t = \text{const}$, and then writing $\delta x_1, \ldots, \delta z_n$, instead of $dx_1, \ldots, dz_n$, respectively.

In the case of rheonomic systems we cannot say that the virtual displacements are proportional to the possible velocities. For let us give the system an arbitrary motion compatible with the constraints. Differentiating (12), we obtain

$$\frac{\partial F_j}{\partial x_1}\dot{x}_1 + \ldots + \frac{\partial F_j}{\partial z_n}\dot{z}_n + \frac{\partial F_j}{\partial t} = 0 \qquad (j = 1, 2, \ldots, m). \tag{13}$$

Denoting by $\mathbf{v}_1, \ldots, \mathbf{v}_n$, the velocities of the points, we can write (13) in the form

$$\sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i} v_{i_x} + \frac{\partial F_j}{\partial y_i} v_{i_y} + \frac{\partial F_j}{\partial z_i} v_{i_z}\right) + \frac{\partial F_j}{\partial t} = 0 \qquad (j = 1, 2, \ldots, m). \quad (14)$$

Comparing (14) with (I) we see that we cannot in general take $\delta x_1 = v_{i_x}, \ldots, \delta z_n = v_{n_z}$, which can be done in the case of scleronomic systems.

If the relations ((II), p. 466)

$$\Phi_r(x_1, \ldots, z_n, t) \leqq 0 \qquad (r = 1, 2, \ldots, s) \quad (15)$$

hold in addition to equations (12), then, besides (I), the virtual displacement must satisfy those of the relations

$$\sum_{i=1}^{n}\left(\frac{\partial \Phi_r}{\partial x_i} \delta x_i + \frac{\partial \Phi_r}{\partial y_i} \delta y_i + \frac{\partial \Phi_r}{\partial z_i} \delta z_i\right) \leqq 0 \qquad (r = 1, 2, \ldots, s), \quad \text{(II)}$$

for which the equalities $\Phi_r = 0$ (cf. (II), p. 432) hold in a given position of the system at the moment t.

Generalized coordinates. Let the position of a holonomic system be defined by means of the parameters $q_1, \ldots, q_k$ (p. 451).

If the system is rheonomic, then the functions which define the natural coordinates $x_1, \ldots, z_n$, corresponding to the parameters $q_1, \ldots, q_k$, depend on the time. Consequently ((I), p. 452):

$$x_i = f_i(q_1, \ldots, q_k, t), \; y_i = \varphi_i(q_1, \ldots, q_k, t), \; z_i = \psi_i(q_1, \ldots, q_k, t), \quad (i = 1, 2, \ldots, n). \quad (16)$$

If the parameters are independent, then to every set of the variables $q_1, \ldots, q_k$, in a certain region of these variables (the region can depend on the time t) there corresponds a position of the system compatible with the constraints. If the parameters are dependent, then in the case of bilateral constraints certain equations (2), p. 453:

$$\Phi_r(q_1, \ldots, q_k, t) = 0 \qquad (r = 1, 2, \ldots, s) \quad (17)$$

must be satisfied, and in the case of unilateral constraints the parameters must satisfy the inequalities (3), p. 453:

$$\Psi_r(q_1, \ldots, q_k, t) \leqq 0 \qquad (r = 1, 2, \ldots, \varrho). \quad (18)$$

In particular, when the functions (16)—(18) do not depend on the time t, the system is scleronomic.

If a system is moving, then the parameters $q_1, \ldots, q_k$, depend on the

time t. The motion of the system will therefore be determined by giving the functions:

$$q_1 = q_1(t), \quad \ldots, \quad q_k = q_k(t) \quad (19)$$

defining the values of the parameters of the system at each moment t. The natural coordinates are obtained by substituting functions (19) in functions (16). If the parameters are dependent and satisfy equations (17) and possibly inequalities (18) too, then functions (19) must likewise satisfy these relations.

Let the positions of a holonomic system be defined parametrically by means of functions (16). The virtual displacement of the system at the moment t in a certain position compatible with the constraints is obtained by assuming that the constraints are independent of the time and such as they were at the moment t. Hence in virtue of (III), p. 454, we get:

$$\delta x_i = \sum_{j=1}^{k} \frac{\partial x_i}{\partial q_j} \delta q_j, \;\; \delta y_i = \sum_{j=1}^{k} \frac{\partial y_i}{\partial q_j} \delta q_j, \;\; \delta z_i = \sum_{j=1}^{k} \frac{\partial z_i}{\partial q_j} \delta q_j \; (i = 1, 2, \ldots, n). \; \textbf{(III)}$$

Formulae (III) are obtained by forming the differentials of (16), under the assumption that $t =$ const, and then writing $\delta x_i, \delta y_i, \delta z_i, \delta q_j$, instead of dx_i, dy_i, dz_i, dq_j.

If the parameters are independent, then δq_j in (III) are arbitrary. If the parameters defining the position of the system compatible with the constraints satisfy relations (17), then δq_j in (III) are not arbitrary: they must satisfy the equations ((IV), p. 455)

$$\sum_{j=1}^{k} \frac{\partial \Phi_r}{\partial q_j} \delta q_j = 0 \qquad (r = 1, 2, \ldots, s). \quad \text{(IV)}$$

Finally, if the parameters must satisfy certain inequalities (18) in addition to equations (17), then δq_j must satisfy, besides (IV), those of the relations

$$\sum_{j=1}^{k} \frac{\partial \Psi_r}{\partial q_j} q_j \leqq 0 \qquad (r = 1, 2, \ldots, \varrho), \quad \text{(V)}$$

for which the equations $\Psi_r = 0$ ((V), p. 455) hold in a given position of the system at the moment t.

Example 2. A material point is constrained to a line l lying in the xy-plane and passing through the origin O of the system. The line l rotates about O with a constant angular velocity ω.

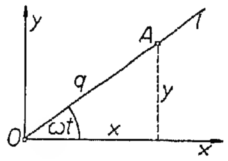

Fig. 317.

Let us take the point O as the origin of the coordinate system and give the line l an arbitrary sense (Fig. 317). Let us denote by q the coordi-

nate of the point A with respect to the axis l; finally, let us assume that the axis l coincided with the axis of x at the moment t. For the coordinates x, y, of the point A we then obtain the formulae:

$$x = q \cos \omega t, \quad y = q \sin \omega t. \tag{20}$$

The variable q defines the position of the point at the moment t; it is therefore a parameter. Differentiating equations (20), under the assumption that $t = \text{const}$, we get:

$$\delta x = \delta q \cos \omega t, \quad \delta y = \delta q \sin \omega t. \tag{21}$$

§ 4. D'Alembert's principle. Equilibrium of forces. So far we have defined the concept of the equilibrium of acting forces for scleronomic systems. According to the definition given (p. 435), the acting forces are in equilibrium if the system of points can remain at rest despite the action of these forces.

This definition of equilibrium, however, is not suitable for rheonomic systems.

For example, if a system of material points is constrained to remain constantly in a horizontal plane moving vertically upwards with a uniform motion, then obviously the system can at no time remain at rest. According to the preceding definition, therefore, we could not say that any system of forces is in equilibrium.

The principle of virtual work (p. 436) gives the necessary and sufficient condition of the equilibrium of forces for scleronomic systems (if there is no friction). Now, for rheonomic systems (when there is no friction) we take the principle of virtual work as the definition of the equilibrium of the acting forces: we therefore say that *the forces acting on a holonomo-rheonomic system (in which there is no friction) are in equilibrium at a certain time t, if for every virtual displacement at the time t the virtual work of the forces is zero or a negative number.*

According to this definition the principle of virtual work applies to holonomic systems whether they are scleronomic or rheonomic.

D'Alembert's principle. Let the forces $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n$, act on a holonomic system consisting of n material points $A(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$. Let us denote by $m_1, \ldots, m_n$, the masses, and by $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$, the accelerations of these points.

The vectors $-m_1\boldsymbol{p}_1, \ldots, -m_n\boldsymbol{p}_n$ were called *forces of inertia* (p. 73). If the system is free, then according to d'Alembert's principle (p. 188) the acting forces balance the forces of inertia. Now, experience shows that d'Alembert's principle is also true for constrained holonomic systems, in which there is no friction. Therefore we can state it as follows:

The forces acting on the points of a holonomic system (in which there is no friction) *balance the forces of inertia at each moment.*

Hence the forces $\boldsymbol{P}_i - m_i\boldsymbol{p}_i$ (where $i = 1, 2, \ldots, n$) are in equilibrium. Denoting by $\overline{\delta s_i}$ the virtual displacements, we obtain ((I), p. 434 and (II), p. 437)

$$\sum_{i=1}^{n}(\boldsymbol{P}_i - m_i\boldsymbol{p}_i)\, \overline{\delta s_i} \leq 0. \tag{I}$$

In the case of bilateral constraints (or reversible displacements) we have

$$\sum_{i=1}^{n}(\boldsymbol{P}_i - m_i\boldsymbol{p}_i)\, \overline{\delta s_i} = 0. \tag{I'}$$

Denoting by $x_i^{\cdot\cdot}, y_i^{\cdot\cdot}, z_i^{\cdot\cdot}$, the projections of the acceleration $\boldsymbol{p}_i$, and by $\delta x_i, \delta y_i, \delta z_i$, the projections of the displacements $\overline{\delta s_i}$, we can write formulae (I) and (I') in the form ((II), p. 437)

$$\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\, \delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\, \delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\, \delta z_i] \leq 0, \tag{II}$$

and in the case of bilateral constraints we have ((III), p. 437)

$$\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\, \delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\, \delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\, \delta z_i] = 0. \tag{II'}$$

Therefore d'Alembert's principle reduces the problems of dynamics to problems of statics. This principle can be proved in many instances. In the cases when friction is defined, we accept it as a law verified by experience. In the general case we say that there is no friction if d'Alembert's principle applies to a given system.

Remark. Let us assume that a system is free. Consequently $\delta x_i, \delta y_i, \delta z_i$, are arbitrary numbers. Since equation (II') has to hold for every set of numbers $\delta x_i, \delta y_i, \delta z_i$, the coefficients of these numbers must be zero. Consequently:

$$P_{i_x} - m_i x_i^{\cdot\cdot} = 0, \quad P_{i_y} - m_i y_i^{\cdot\cdot} = 0, \quad P_{i_z} - m_i z_i^{\cdot\cdot} = 0,$$

whence

$$m_i x_i^{\cdot\cdot} = P_{i_x}, \; m_i y_i^{\cdot\cdot} = P_{i_y}, \; m_i z_i^{\cdot\cdot} = P_{iz} \quad (i = 1, 2, \ldots, n).$$

The above equations are obviously Newton's equations of motion ((II), p. 186).

Example I. A heavy material point A of mass m falls (without friction) along an inclined plane making an angle α with the horizontal. Determine the motion of the point.

Denoting by $\mathbf{p}$ the acceleration, by $\mathbf{Q}$ the weight of the point A, and by $\overline{\delta s}$ the virtual displacement (Fig. 318), we obtain from d'Alembert's principle

$$(\mathbf{Q} - m\mathbf{p})\,\overline{\delta s} \leqq 0. \tag{1}$$

Let us take as the z-axis the line of the greatest fall on the inclined plane giving it a downward sense. Let $\overline{\delta s}$ have the direction of the z-axis. Denoting the projections of $\overline{\delta s}$ and $\mathbf{p}$ on the z-axis by δs and p, as well as noting that $\overline{\delta s}$ is a reversible displacement, we obtain

$$(\mathbf{Q} - m\mathbf{p})\,\overline{\delta s} = 0, \text{ i.e. } \mathbf{Q}\,\overline{\delta s} - m\mathbf{p}\,\overline{\delta s} = 0,$$

from which $mg\,\delta s \sin\alpha - mp\,\delta s = 0$, and therefore

$$m(g \sin\alpha - p)\,\delta s = 0. \tag{2}$$

Since equation (2) holds for every δs, we have $g\sin\alpha - p = 0$, whence

$$p = g \sin\alpha. \tag{3}$$

The equation (3) determines the acceleration of the point. It is easy to show that at this acceleration formula (1) holds for every $\overline{\delta s}$ (lying in the plane or not).

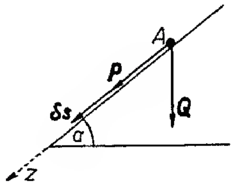

Fig. 318.

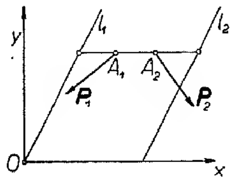

Fig. 319.

Example 2. Two material points A_1, A_2, of masses m_1, m_2, are strung on a massless rigid wire whose ends are constrained to remain on two parallel lines l_1, l_2. The forces $\mathbf{P}_1, \mathbf{P}_2$, lying in the plane of the lines l_1, l_2, act on the points (Fig. 319). Determine the motion of the points, assuming that there is no friction.

It is easy to see that the wire will have a constant direction. Let us choose the axes x and y in the plane of the lines l_1, l_2, giving the x axis the direction of the wire, and let us denote the coordinates of the points by x_1, y_1, and x_2, y_2. The constraints will therefore be defined by the equation

$$y_1 - y_2 = 0. \tag{4}$$

In virtue of d'Alembert's principle, ((II') p. 475), we obtain

$$(P_{1_x} - m_1\ddot{x}_1)\,\delta x_1 + (P_{1_y} - m_1\ddot{y}_1)\,\delta y_1 + (P_{2_x} - m_2\ddot{x}_2)\,\delta x_2 + \\ + (P_{2_y} - m_2\ddot{y}_2)\,\delta y_2 = 0. \tag{5}$$

From equation (4) we have $\delta y_1 - \delta y_2 = 0$, i.e. $\delta y_1 = \delta y_2$. Substituting this value in (5), we get

$$(P_{1_x} - m_1\ddot{x}_1)\,\delta x_1 + (P_{2_x} - m_2\ddot{x}_2)\,\delta x_2 + (P_{1_y} - m_1\ddot{y}_1 + \\ + P_{2_y} - m_2\ddot{y}_2)\,\delta y_1 = 0. \tag{6}$$

Since $\delta x_1, \delta x_2, \delta y_1$, are arbitrary numbers, their coefficients in equations (6) must be zero. Consequently:

$$m_1\ddot{x}_1 = P_{1_x}, \quad m_2\ddot{x}_2 = P_{2_x}, \tag{7}$$

$$m_1\ddot{y}_1 + m_2\ddot{y}_2 = P_{1_y} + P_{2_y}. \tag{8}$$

From (4) we have $\ddot{y}_1 - \ddot{y}_2 = 0$, i.e. $\ddot{y}_1 = \ddot{y}_2$. Equation (8) can therefore be written in the form

$$(m_1 + m_2)\,\ddot{y}_1 = P_{1_y} + P_{2_y}. \tag{9}$$

Equations (7), (9), and (4), determine the motion of the points.

Example 3. A vertical plane Π passing through the z axis directed vertically upwards, rotates about z with a constant angular velocity ω. A heavy point A of the mass m is constrained to the plane Π. (Fig. 320). Determine the motion of this point, assuming that there is no friction.

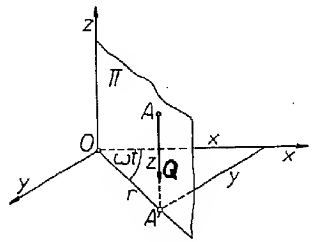

Fig. 320.

Let us assume that the plane Π had the position of the xz-plane at $t = 0$. The equation of the plane Π at the time t will hence be

$$y \cos\omega t - x \sin\omega t = 0. \tag{10}$$

The coordinates x, y, of the point A must therefore satisfy equation (10). Since the force of gravity has the projections, $0, 0, -mg$, on the coordinate axes, from d'Alembert's principle it follows that

$$-m\ddot{x}\,\delta x - m\ddot{y}\,\delta y + (-mg - m\ddot{z})\,\delta z = 0. \tag{11}$$

By (10) the virtual displacement $\delta x, \delta y, \delta z$, satisfies the equation

$$\delta y \cos\omega t - \delta x \sin\omega t = 0. \tag{12}$$

Consequently δz is arbitrary and $\delta x, \delta y$, satisfy equation (12).

Assuming $\delta x = 0$, $\delta y = 0$, in (11), we obtain $(-mg - m\ddot{z})\,\delta z = 0$. Since δz is arbitrary, $-mg - m\ddot{z} = 0$, i.e.

$$\ddot{z} = -g, \tag{13}$$

from which $z = -\frac{1}{2}gt^2 + ct + c'$, where c and c' are certain constants.

From (11) and (13) we obtain $x^{\cdot\cdot}\,\delta x + y^{\cdot\cdot}\,\delta y = 0$; hence

$$x^{\cdot\cdot}\,\delta x \cos\omega t + y^{\cdot\cdot}\,\delta y \cos\omega t = 0, \tag{14}$$

whence by (12) $(x^{\cdot\cdot}\cos\omega t + y^{\cdot\cdot}\sin\omega t)\,\delta x = 0$, and, since δx is arbitrary,

$$x^{\cdot\cdot}\cos\omega t + y^{\cdot\cdot}\sin\omega t = 0. \tag{15}$$

Let us put $r = OA'$, where A' denotes the projections ot A on the xy-plane. Consequently

$$x = r\cos\omega t, \quad y = r\sin\omega t, \tag{16}$$

from which

$$x^{\cdot\cdot} = r^{\cdot\cdot}\cos\omega t - 2r^{\cdot}\omega\sin\omega t - r\omega^2\cos\omega t,$$
$$y^{\cdot\cdot} = r^{\cdot\cdot}\sin\omega t + 2r^{\cdot}\omega\cos\omega t - r\omega^2\sin\omega t$$

and by substituting in (15) $r^{\cdot\cdot} - r\omega^2 = 0$; therefore (cf. example 4, p. 139) $r = c_1e^{\omega t} + c_2e^{-\omega t}$, whence in virtue of (16):

$$x = (c_1e^{\omega t} + c_2e^{-\omega t})\cos\omega t, \quad y = (c_1e^{\omega t} + c_2e^{-\omega t})\sin\omega t.$$

The constants c, c', c_1, c_2, are determined from the initial conditions.

§ 5. Work and kinetic energy in scleronomic systems. Let a holono-mo-scleronomic system composed of n material points of masses $m_1, \ldots, m_n$, and having coordinates $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, be subjected to the action of the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$.

For the moment, let us assume that the constraints are bilateral.

From d'Alembert's principle we have for every virtual displacement $\delta x_i, \delta y_i, \delta z_i$

$$\sum_{i=1}^{n}[(P_{i_x} - m_ix_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_iy_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_iz_i^{\cdot\cdot})\,\delta z_i] = 0. \tag{1}$$

Since the system is scleronomic, the velocities of the points can be considered as virtual displacements (p. 425). Therefore, putting $x_i^{\cdot} = \delta x_i$, $y_i^{\cdot} = \delta y_i$, $z_i^{\cdot} = \delta z_i$, we obtain from (1)

$$\sum_{i=1}^{n}[(P_{i_x} - m_ix_i^{\cdot\cdot})\,x_i^{\cdot} + (P_{i_y} - m_iy_i^{\cdot\cdot})\,y_i^{\cdot} + (P_{i_z} - m_iz_i^{\cdot\cdot})\,z_i^{\cdot}] = 0, \tag{2}$$

i. e.

$$\sum_{i=1}^{n}(P_{i_x}x_i^{\cdot} + P_{i_y}y_i^{\cdot} + P_{i_z}z_i^{\cdot}) - \sum_{i=1}^{n}m_i(x_i^{\cdot\cdot}x_i^{\cdot} + y_i^{\cdot\cdot}y_i^{\cdot} + z_i^{\cdot\cdot}z_i^{\cdot}) = 0. \tag{3}$$

The kinetic energy is expressed by the formula

$$E = \sum_{i=1}^{n}\tfrac{1}{2}m_i(x_i^{\cdot 2} + y_i^{\cdot 2} + z_i^{\cdot 2}),$$

whence $E^{\cdot} = \sum m_i(x_i^{\cdot\cdot}x_i^{\cdot} + y_i^{\cdot\cdot}y_i^{\cdot} + z_i^{\cdot\cdot}z_i^{\cdot})$. Therefore in virtue of (3)

$$E^{\cdot} = \sum_{i=1}^{n}(P_{i_x}x_i^{\cdot} + P_{i_y}y_i^{\cdot} + P_{i_z}z_i^{\cdot}).$$

Integrating both sides of this equation from t_0 to t, we obtain

$$\int_{t_0}^{t}E^{\cdot}\,\mathrm{d}t = \int_{t_0}^{t}\mathrm{d}t\sum_{i=1}^{n}(P_{i_x}x_i^{\cdot} + P_{i_y}y_i^{\cdot} + P_{i_z}z_i^{\cdot}). \tag{4}$$

The left side of formula (4) is equal to $E - E_0$, where E denotes the kinetic energy at the time t, and E_0 the kinetic energy at the time t_0, while the right side expresses the work L_{t_0t} of the forces acting from the time t_0 to t ((II), p. 208). Consequently

$$E - E_0 = L_{t_0t}. \tag{5}$$

Equation (5) expresses the *principle of equivalence of the work of the acting forces and of the kinetic energy.*

Let us now discard the assumption that the constraints are bilateral. Let us assume that in addition to the relations expressed by equalities, the coordinates of the points of the system have to satisfy the inequalities ((15), p. 472):

$$\Phi_r(x_1, \ldots, z_n) \leqq 0 \qquad (r = 1, 2, \ldots, \varrho). \tag{6}$$

D'Alembert's principle in this case has the form

$$\sum_{i=1}^{n}[(P_{i_x} - m_ix_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_iy_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_iz_i^{\cdot\cdot})\,\delta z_i] \leqq 0. \tag{7}$$

Let us assume that the velocity changes in a continuous manner during the motion.

If the position of the system at a certain time t' (where $t_0 \leqq t' \leqq t$) is not a boundary position, i. e. if the $<$ signs hold in the inequalities (6), then the inequalities (6) do not give any conditions on the virtual displacements (p. 471). In this case the virtual displacements are reversible, consequently equation (1) holds and then (2) holds. On the other hand; if the system occupies a boundary position at the time t' (where $t_0 < t' < t$),i. e. if the equality

$$\Phi_r(x_1, \ldots, z_n) = 0$$

holds for a certain r, then according to the assumption that the functions $x_1, \ldots, z_n$, have continuous derivatives with respect to the time t, the function Φ_r will also have a continuous derivative. Moreover, since $\Phi_r \leqq 0$ constantly, the function Φ_r attains a maximum at the time t'. It follows from this that $\Phi_r^{\cdot} = 0$ for $t = t'$; hence

$$\frac{\partial\Phi_r}{\partial x_1}x_1^{\cdot} + \ldots + \frac{\partial\Phi_r}{\partial z_n}z_n^{\cdot} = 0. \tag{8}$$

In view of (8) the virtual displacement $\delta x_1 = x_1^{\cdot}, \ldots, \delta z_n = z_n^{\cdot}$, is a reversible displacement. Hence for this displacement equation (1) holds, and consequently equation (2) holds.

We have therefore proved that equation (2) is satisfied for each instant t' (where $t_0 < t' < t$). From equation (2) — reasoning as before — we obtain formula (4).

Therefore: *the principle of equivalence of the work of the acting forces and of the kinetic energy applies to holonomo-scleronomic systems* (while for unilateral constraints this holds when the velocities of the points vary in a continuous manner).

If the acting forces have a potential V, then $L_{t_0t} = V - V_0$, where V and V_0 denote the corresponding potentials at the instants t and t_0. From (5) we therefore have $E - E_0 = V - V_0$ or $E - V = E_0 - V_0$. Denoting the constant $E_0 - V_0$ by h we obtain

$$E - V = h. \tag{9}$$

We have called $-V$ the potential energy and denoted it by U (p. 216). Consequently

$$E + U = h. \tag{10}$$

We have called the sum $E + U$ the *total energy of the system* (p. 216).

Therefore: *the principle of conservation of total energy applies to holonomo-scleronomic systems* (under the assumption that in the case of unilateral constraints the velocities vary in a continuous manner).

Remark. In general, the principle of equivalence of work and kinetic energy does not hold for rheonomic systems.

For example, if a point is constrained to remain on a moving curve and no forces act on the point, then in spite of this the kinetic energy of the point can change depending on the motion of the curve.

In rheonomic systems the increase in kinetic energy also depends on the work of the forces of reaction, which in general is not zero.

§ 6. Lagrange's equations of the first kind. Let a holonomic system of n material points $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$, be given. Let us denote by $\mathbf{P}_1, \ldots, \mathbf{P}_n$ the forces acting on the points of the system, and by $m_1, \ldots, m_n$, the masses of these points. Let us assume that the constraints are bilateral, defined by the equations ((I), p. 466):

$$F_j(x_1, y_1, z_1, \ldots, x_n, y_n, z_n, t) = 0 \qquad (j = 1, 2, \ldots, m). \tag{1}$$

The virtual displacements of the system satisfy the equations:

$$\sum_{i=1}^{n}\left(\frac{\partial F_j}{\partial x_i}\,\delta x_i + \frac{\partial F_j}{\partial y_i}\,\delta y_i + \frac{\partial F_j}{\partial z_i}\,\delta z_i\right) = 0 \qquad (j = 1, 2, \ldots, m). \tag{2}$$

In virtue of d'Alembert's principle ((II'), p. 475) we have;

$$\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta z_i] = 0. \tag{3}$$

Equation (3) holds for every set of numbers δx_i, δy_i, δz_i, satisfying the system of equations (2). From the considerations on p. 447 it follows that there exist numbers $\lambda_1, \ldots, \lambda_m$, such that equations (I), p. 448 are satisfied at each moment t (where it is necessary to substitute $P_{i_x} - m_i x_i^{\cdot\cdot}$ for P_{i_x}, etc.):

$$\begin{aligned} &P_{i_x} - m_i x_i^{\cdot\cdot} + \sum_{j=1}^{m}\lambda_j \frac{\partial F_j}{\partial x_i} = 0,\\ &P_{i_y} - m_i y_i^{\cdot\cdot} + \sum_{j=1}^{m}\lambda_j \frac{\partial F_j}{\partial y_i} = 0,\\ &P_{i_z} - m_i z_i^{\cdot\cdot} + \sum_{j=1}^{m}\lambda_j \frac{\partial F_j}{\partial z_i} = 0 \qquad (i = 1, 2, \ldots, n). \end{aligned} \tag{4}$$

The numbers $\lambda_1, \ldots, \lambda_m$, depend on t and hence are functions of time; consequently $\lambda_1 = \lambda_1(t), \ldots, \lambda_m = \lambda_m(t)$. From (4) we get:

$$\begin{aligned} m_i x_i^{\cdot\cdot} &= P_{i_x} + \sum_{j=1}^{m}\lambda_j \frac{\partial F_j}{\partial x_i},\\ m_i y_i^{\cdot\cdot} &= P_{i_y} + \sum_{j=1}^{m}\lambda_j \frac{\partial F_j}{\partial y_i},\\ m_i z_i^{\cdot\cdot} &= P_{i_z} + \sum_{j=1}^{m}\lambda_j \frac{\partial F_j}{\partial z_i} \qquad (i = 1, 2, \ldots, n). \end{aligned} \tag{I}$$

Equations (I) are called *Lagrange's equations of the first kind.*

Let the forces $\mathbf{P}_i$ be given as functions of the variables $x_1, \ldots, z_n$, $x_1^{\cdot}, \ldots, z_n^{\cdot}$, t, defining the position of the system at the time t. From equations (I) and (1) we can therefore determine the unknown functions of time $x_1, \ldots, z_n$, defining the motion of the system, as well as the functions $\lambda_1 = \lambda_1(t), \ldots, \lambda_m = \lambda_m(t)$. There are as many unknown functions as there are equations, i. e. $3n + m$.

Let us denote by $\mathbf{R}_1, \ldots, \mathbf{R}_n$, the forces whose projections are defined by the equalities:

$$R_{i_x} = \sum_{j=1}^{m}\lambda_j \frac{\partial F_j}{\partial x_i},\quad R_{i_y} = \sum_{j=1}^{m}\lambda_j \frac{\partial F_j}{\partial y_i},\quad R_{i_z} = \sum_{j=1}^{m}\lambda_j \frac{\partial F_j}{\partial z_i} \qquad (i = 1, 2, \ldots, n). \tag{II}$$

By (I) and (II) we have:

$$m_i x_i^{\cdot\cdot} = P_{i_x} + R_{i_x},\quad m_i y_i^{\cdot\cdot} = P_{i_y} + R_{i_y},\quad m_i z_i^{\cdot\cdot} = P_{i_z} + R_{i_z} \qquad (i = 1, 2, \ldots, n). \tag{5}$$

Denoting by $\mathbf{p}_i$ the acceleration of the point A_i, we can write (5) in the form

$$m_i \mathbf{p}_i = \mathbf{P}_i + \mathbf{R}_i \qquad (i = 1, 2, \ldots, n). \qquad (6)$$

From (6) it follows that the forces $\mathbf{R}_i$ are reactions. For, if we add them to the acting forces, then by (6) we shall be able to regard the system as free. Therefore the reactions are defined by relations (II).

Example 1. Let a point of mass m, subjected to the action of the force $\mathbf{P}$, be constrained to remain on the surface whose equation is

$$F(x, y, z) = 0. \qquad (7)$$

Lagrange's equations (I) assume the form:

$$mx^{\cdot\cdot} = P_x + \lambda \frac{\partial F}{\partial x}, \quad my^{\cdot\cdot} = P_y + \lambda \frac{\partial F}{\partial y}, \quad mz^{\cdot\cdot} = P_z + \lambda \frac{\partial F}{\partial z}. \qquad (8)$$

From equations (7) and (8) we can determine the unknown functions of time x, y, z, and λ.

Equations (8) were obtained in another way (cf. (I), p. 127).

Now let the point be constrained to lie on a moving surface having the equation

$$F(x, y, z, t) = 0. \qquad (9)$$

Equations (I) of Lagrange will then have the form (8), too.

Example 2. Let a material point of mass m be constrained to remain on a sphere whose centre is the origin of a coordinate system, and whose radius r varies together with the time t. Let

$$r = at + r_0, \qquad (10)$$

where the numbers a and r_0 are given. Consequently the coordinates of the point satisfy the equation

$$x^2 + y^2 + z^2 - (at + r_0)^2 = 0. \qquad (11)$$

Let us assume that no forces act on the point. Equations (8) will then assume the form:

$$mx^{\cdot\cdot} = 2\lambda x, \quad my^{\cdot\cdot} = 2\lambda y, \quad mz^{\cdot\cdot} = 2\lambda z. \qquad (12)$$

From equations (12) it follows that the direction of the acceleration passes through the origin of the coordinate system. Hence the motion will be a central motion (p. 85), and it will therefore take place in a plane passing through the origin of the coordinate system (p. 86).

Let us assume that the plane of motion is the xz-plane. Consequently

$$y = 0. \qquad (13)$$

Let us introduce in the xz-plane the polar coordinates r, φ:

$$x = r\cos\varphi, \quad z = r\sin\varphi. \qquad (14)$$

Since the areal velocity is constant, by (I), p. 47,

$$r^2\varphi^{\cdot} = \text{const} = c, \qquad (15)$$

from which by (10) $\varphi^{\cdot} = c / (at + r_0)^2$. Integrating, we get

$$\varphi = -c / (at + r_0)\, a + c_1. \qquad (16)$$

Assuming that $\varphi = 0$ for $t = 0$, we obtain from (16) $c_1 = c / r_0 a$ or

$$\varphi = ct / r_0(at + r_0). \qquad (17)$$

Equations (14), (10), and (17), define the motion of the point. The constant c is obtained from (15) if one knows, for example, the angular velocity $\varphi^{\cdot}$ for $t = 0$.

§ 7. Lagrange's equations of the second kind. We shall now consider equations of motion in which only the generalized coordinates will appear.

Let there be given a holonomic system of n material points whose natural coordinates $x_1, \ldots, z_n$, are defined in terms of the parameters $q_1, \ldots, q_k$, by means of the functions:

$$x_i = f_i(q_1, \ldots, q_k, t), \quad y_i = \varphi_i(q_1, \ldots, q_k, t), \quad z_i = \psi_i(q_1, \ldots, q_k, t) \qquad (1)$$
$$(i = 1, 2, \ldots, n).$$

Let us denote the masses of the points of the system by $m_1, \ldots, m_n$.

Let us assume that the acting forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, depend on the position of the system, on the velocities of the points, and on the time t.

By d'Alembert's principle we have

$$\sum_{i=1}^{n} [(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta z_i] \leqq 0. \qquad (2)$$

Let us note that

$$\sum_{i=1}^{n} [(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta z_i] =$$
$$= \sum_{i=1}^{n} (P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i) - \sum_{i=1}^{n} m_i (x_i^{\cdot\cdot}\,\delta x_i + y_i^{\cdot\cdot}\,\delta y_i + z_i^{\cdot\cdot}\,\delta z_i). \qquad (3)$$

The first sum on the right side of equation (3) represents the virtual work $\delta' L$ of the acting forces. In virtue of (VII), p. 456, we can write it in the form

$$\delta' L = \sum_{i=1}^{n} (P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i) = \sum_{j=1}^{k} Q_j\,\delta q_j, \qquad (4)$$

where Q_j (for $j = 1, 2, \ldots, k$) are the components of the generalized force. By (VI′), p. 456, we therefore have

$$Q_j = \sum_{i=1}^{n}\left(P_{i_x}\frac{\partial x_i}{\partial q_j} + P_{i_y}\frac{\partial y_i}{\partial q_j} + P_{i_z}\frac{\partial z_i}{\partial q_j}\right) \qquad (j = 1, 2, \ldots, k). \tag{5}$$

Taking the derivatives of equations (1) with respect to the time t, we obtain

$$\dot{x}_i = \frac{\partial x_i}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t} \qquad (i = 1, 2, \ldots, n) \tag{6}$$

and similar formulae for $\dot{y}_i$, $\dot{z}_i$.

By hypothesis, the projections P_{i_x}, P_{i_y}, P_{i_z}, are functions of the time t as well as of the variables $x_1, \ldots, z_n, \dot{x}_1, \ldots, \dot{z}_n$, which we can by (1) and (6) express in terms of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k$. Therefore from (5) it follows that Q_j can also be regarded as functions of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$:

$$Q_j = Q_j(q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t) \qquad (j = 1, 2, \ldots, k). \tag{7}$$

By (4) and (7) the first sum on the right side of equation (3) can therefore be expressed in terms of the generalized coordinates.

We shall now consider the second sum on the right side of equation (3). From (1) we obtain ((III), p. 473):

$$\delta x_i = \sum_{j=1}^{k}\frac{\partial x_i}{\partial q_j}\delta q_j,\quad \delta y_i = \sum_{j=1}^{k}\frac{\partial y_i}{\partial q_j}\delta q_j,\quad \delta z_i = \sum_{j=1}^{k}\frac{\partial z_i}{\partial q_j}\delta q_j,\quad (i = 1, 2, \ldots, n). \tag{8}$$

Consequently

$$\begin{aligned}
&\sum_{i=1}^{n} m_i(\ddot{x}_i\,\delta x_i + \ddot{y}_i\,\delta y_i + \ddot{z}_i\,\delta z_i) = \\
&= \sum_{i=1}^{n} m_i\left(\ddot{x}_i\sum_{j=1}^{k}\frac{\partial x_i}{\partial q_j}\delta q_j + \ddot{y}_i\sum_{j=1}^{k}\frac{\partial y_i}{\partial q_j}\delta q_j + \ddot{z}_i\sum_{j=1}^{k}\frac{\partial z_i}{\partial q_j}\delta q_j\right) = \\
&= \sum_{j=1}^{k}\delta q_j\sum_{i=1}^{n} m_i\left(\ddot{x}_i\frac{\partial x_i}{\partial q_j} + \ddot{y}_i\frac{\partial y_i}{\partial q_j} + \ddot{z}_i\frac{\partial z_i}{\partial q_j}\right).
\end{aligned} \tag{9}$$

In virtue of (6) we can regard $\dot{x}_i$ (and similarly $\dot{y}_i$, $\dot{z}_i$) as functions of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k$, as well as of the time t. Assuming that $\dot{x}_i$ denotes the right side of the equality (6) and that $\dot{q}_1, \ldots, \dot{q}_k$, are independent variables, let us calculate the partial derivatives with respect to $\dot{q}_1, \ldots, \dot{q}_k$. We obtain:

$$\partial\dot{x}_i / \partial\dot{q}_1 = \partial x_i / \partial q_1, \ldots, \partial\dot{x}_i / \partial\dot{q}_k = \partial x_i / \partial q_k, \tag{10}$$

i. e.

$$\partial\dot{x}_i / \partial\dot{q}_j = \partial x_i / \partial q_j \qquad (j = 1, 2, \ldots, k). \tag{11}$$

Calculating the partial derivatives of equations (6) with respect to q_j (and at the same time regarding $\dot{q}_1, \ldots, \dot{q}_k$, as independent variables), we obtain

$$\frac{\partial\dot{x}_i}{\partial q_j} = \frac{\partial^2 x_i}{\partial q_1\,\partial q_j}\dot{q}_1 + \ldots + \frac{\partial^2 x_i}{\partial q_k\,\partial q_j}\dot{q}_k + \frac{\partial^2 x_i}{\partial t\,\partial q_j}. \tag{12}$$

On the other hand we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial^2 x_i}{\partial q_j\,\partial q_1}\dot{q}_1 + \ldots + \frac{\partial^2 x_i}{\partial q_j\,\partial q_k}\dot{q}_k + \frac{\partial^2 x_i}{\partial q_j\,\partial t}. \tag{13}$$

Since the order of differentiation does not affect the result, from (12) and (13) we get

$$\frac{\partial\dot{x}_i}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x_i}{\partial q_j}\right). \tag{14}$$

Let us note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}_i\frac{\partial x_i}{\partial q_j}\right) = \ddot{x}_i\frac{\partial x_i}{\partial q_j} + \dot{x}_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x_i}{\partial q_j}\right); \tag{15}$$

hence, in virtue of (11) and (14), we obtain from this the formula

$$\ddot{x}_i\frac{\partial x_i}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}_i\frac{\partial\dot{x}_i}{\partial\dot{q}_j}\right) - \dot{x}_i\frac{\partial\dot{x}_i}{\partial q_j} \tag{16}$$

and similar formulae for the variables y_i, z_i. From these formulae we get for arbitrary j:

$$\begin{aligned}
&\sum_{i=1}^{n} m_i\left(\ddot{x}_i\frac{\partial x_i}{\partial q_j} + \ddot{y}_i\frac{\partial y_i}{\partial q_j} + \ddot{z}_i\frac{\partial z_i}{\partial q_j}\right) = \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} m_i\left(\dot{x}_i\frac{\partial\dot{x}_i}{\partial\dot{q}_j} + \dot{y}_i\frac{\partial\dot{y}_i}{\partial\dot{q}_j} + \dot{z}_i\frac{\partial\dot{z}_i}{\partial\dot{q}_j}\right) - \sum_{i=1}^{n} m_i\left(\dot{x}_i\frac{\partial\dot{x}_i}{\partial q_j} + \dot{y}_i\frac{\partial\dot{y}_i}{\partial q_j} + \dot{z}_i\frac{\partial\dot{z}_i}{\partial q_j}\right).
\end{aligned} \tag{17}$$

The kinetic energy of the system is

$$E = \tfrac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2). \tag{18}$$

In (18) let us substitute for $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$, the right sides of the equalities (6) and of the analogous equalities for $\dot{y}_i$, $\dot{z}_i$. In this way we shall represent E as a function of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$:

$$E = E(q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t). \tag{19}$$

Regarding q_j, $\dot{q}_j$, t, as independent variables, we obtain from (19) and (18)

$$\frac{\partial E}{\partial q_j} = \sum_{i=1}^{n} m_i\left(\dot{x}_i\frac{\partial\dot{x}_i}{\partial q_j} + \dot{y}_i\frac{\partial\dot{y}_i}{\partial q_j} + \dot{z}_i\frac{\partial\dot{z}_i}{\partial q_j}\right), \tag{20}$$

$$\frac{\partial E}{\partial \dot{q}_j} = \sum_{i=1}^{n} m_i \left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} + \dot{y}_i \frac{\partial \dot{y}_i}{\partial \dot{q}_j} + \dot{z}_i \frac{\partial \dot{z}_i}{\partial \dot{q}_j} \right). \tag{21}$$

From (17) we get in virtue of (20) and (21)

$$\sum_{i=1}^{n} m_i \left(\ddot{x}_i \frac{\partial x_i}{\partial q_j} + \ddot{y}_i \frac{\partial y_i}{\partial q_j} + \ddot{z}_i \frac{\partial z_i}{\partial q_j} \right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) - \frac{\partial E}{\partial q_j}, \tag{22}$$

whence by (9)

$$\sum_{i=1}^{n} m_i(\ddot{x}_i\, \delta x_i + \ddot{y}_i\, \delta y_i + \ddot{z}_i\, \delta z_i) = \sum_{j=1}^{k} \delta q_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) - \frac{\partial E}{\partial q_j} \right]. \tag{23}$$

From equations (3), (4), and (23), we obtain

$$\begin{aligned} \sum_{i=1}^{n} [(P_{i_x} - m_i\ddot{x}_i)\, \delta x_i + (P_{i_y} - m_i\ddot{y}_i)\, \delta y_i + (P_{i_z} - m_i\ddot{z}_i)\, \delta z_i] &= \\ = \sum_{j=1}^{k} \delta q_j \left[Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) + \frac{\partial E}{\partial q_j} \right]. \end{aligned} \tag{24}$$

From d'Alembert's principle (2) it therefore follows that

$$\sum_{j=1}^{k} \delta q_j \left[Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) + \frac{\partial E}{\partial q_j} \right] \leq 0. \tag{I}$$

Relations (24) and (I) hold whether the parameters $q_1, \ldots, q_k$, are dependent or not, and whether the constraints are unilateral or bilateral.

In the case of bilateral constraints inequality (I) becomes the equality:

$$\sum_{j=1}^{k} \delta q_j \left[Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) + \frac{\partial E}{\partial q_j} \right] = 0. \tag{I'}$$

Let us assume that the parameters are independent. Consequently $\delta q_1, \ldots, \delta q_k$, are arbitrary numbers. It follows from this that the coefficients of δq_j in (I') are zero, and hence that

$$Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) + \frac{\partial E}{\partial q_j} = 0 \qquad (j = 1, 2, \ldots, k),$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) - \frac{\partial E}{\partial q_j} = Q_j \qquad (j = 1, 2, \ldots, k). \tag{II}$$

Equations (II) are called *Lagrange's equations of the second kind.*

Only the generalized coordinates appear in them.

From equations (II) we can determine $q_1, \ldots, q_k$, as functions of the time t; hence they enable one to determine the motion without passing over to the natural coordinates.

Let us now assume that the parameters are not independent, but must satisfy relations (17), p. 472:

$$\Phi_r(q_1, \ldots, q_k, t) = 0 \qquad (r = 1, 2, \ldots, s). \tag{25}$$

The virtual displacements δq_j consequently satisfy equalities (IV), p. 473,

$$\sum_{j=1}^{k} \frac{\partial \Phi_r}{\partial q_j}\, \delta q_j = 0 \qquad (r = 1, 2, \ldots, s). \tag{26}$$

Equalities (I') hold for every system of numbers δq_j satisfying (26). From considerations analogous to those on p. 447 it follows that for each moment t it is possible to choose numbers $\lambda_1, \ldots, \lambda_s$, satisfying the equations:

$$Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) + \frac{\partial E}{\partial q_j} + \sum_{r=1}^{s} \lambda_r \frac{\partial \Phi_r}{\partial q_j} = 0 \qquad (j = 1, 2, \ldots, k),$$

i. e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_j}\right) - \frac{\partial E}{\partial q_j} = Q_j + \sum_{r=1}^{s} \lambda_r \frac{\partial \Phi_r}{\partial q_j} \qquad (j = 1, 2, \ldots, k). \tag{II'}$$

The Lagrange's multipliers λ_r depend on the time and are therefore functions of the variable t; consequently $\lambda_r = \lambda_r(t)$.

Equations (II') together with (25) enable one to determine the unknown functions of time $q_1, \ldots, q_k$, and $\lambda_1(t), \ldots, \lambda_s(t)$. The number of these equations is $k + s$, i. e. it is equal to the number of unknown functions.

Remark. In forming equations (II) it is first necessary to represent E and Q_j as functions of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$.

In order to obtain Q_j we substitute in the formula for virtual work $\delta' L = \Sigma(P_{i_x}\, \delta x_i + P_{i_y}\, \delta y_i + P_{i_z}\, \delta z_i)$ the expression obtained from (1) for $\delta x_i, \delta y_i, \delta z_i, x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i$, and then we arrange the terms according to $\delta q_1, \ldots, \delta q_k$. The coefficients of $\delta q_1, \ldots, \delta q_k$, will be the components Q_j of the generalized force.

Substituting next for $\dot{x}_i, \dot{y}_i, \dot{z}_i$, in the formula for kinetic energy $E = \frac{1}{2}\Sigma m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$ the derivatives obtained by differentiating (1) with respect to t, we get E as a function of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$.

Having determined E and Q_j as functions of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$, we form the derivatives: $\partial E / \partial q_j$ as well as $\partial E / \partial \dot{q}_j$ and finally

$$\frac{\mathrm{d}}{\mathrm{d}t}(\partial E / \partial \dot{q}_j).$$

Substituting in (II), we obtain Lagrange's equations.

Lagrange's equations in a potential field. Let us assume that the acting forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, have a potential V at each moment t. The potential V is therefore a function of the variables $x_1, \ldots, z_n, t$; in addition:

$$P_{i_x} = \partial V / \partial x_i,\ P_{i_y} = \partial V / \partial y_i,\ P_{i_z} = \partial V / \partial z_i,\ (i = 1, 2, \ldots, n), \quad (27)$$

whence by (5)

$$Q_j = \sum_{i=1}^{n} \left(\frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i} \frac{\partial z_i}{\partial q_j} \right), \qquad (j = 1, 2, \ldots, k). \qquad (28)$$

Expressing the coordinates $x_1, \ldots, z_n$, in V in terms of $q_1, \ldots, q_k$, by means of (1), we can regard V as a function of the variables $q_1, \ldots, q_k$, and of the time t. From (28) we therefore get

$$Q_j = \partial V / \partial q_j \qquad (j = 1, 2, \ldots, k). \qquad (29)$$

From (29) and (27) we see that the components Q_j of the generalized force are expressed in the same way as the coordinates of the forces $\mathbf{P}_i$. Hence we can regard V as the *generalized potential* of the forces $\mathbf{Q}_j$.

From (II) and (29) we get

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_j}\right) - \frac{\partial E}{\partial q_j} = \frac{\partial V}{\partial q_j},$$

i. e.

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_j}\right) - \frac{\partial (E + V)}{\partial q_j} = 0 \quad (j = 1, 2, \ldots, k). \qquad (30)$$

Since V does not depend on the derivatives $\dot{q}_1, \ldots, \dot{q}_k$, it follows that $\partial V / \partial \dot{q}_j = 0$. Consequently

$$\partial E / \partial \dot{q}_j = \partial (E + V) / \partial \dot{q}_j \qquad (j = 1, 2, \ldots, k). \qquad (31)$$

From (30) and (31) we get

$$\frac{d}{dt}\left(\frac{\partial (E + V)}{\partial \dot{q}_j}\right) - \frac{\partial (E + V)}{\partial q_j} = 0 \qquad (j = 1, 2, \ldots, k). \qquad (32)$$

The sum of the kinetic and potential energies, i. e. $E + V$ is called the *kinetic potential*.

Putting

$$W = E + V, \qquad (33)$$

we obtain by (32)

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{q}_j}\right) - \frac{\partial W}{\partial q_j} = 0 \qquad (j = 1, 2, \ldots, k). \qquad \text{(III)}$$

Lagrange's equations of the second kind therefore assume form (III) when the forces have a potential (or — which amounts to the same thing — a kinetic potential) at every moment.

Cyclic coordinates. The coordinate q_j (where j is a certain number) is called *cyclic* if the kinetic potential W does not depend on q_j, i. e. if

$$\partial W / \partial q_j = 0. \qquad (34)$$

If q_j is a cyclic coordinate, then from equations (III) and (34) we obtain

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{q}_j}\right) = 0,$$

whence

$$\partial W / \partial \dot{q}_j = \text{const.} = c. \qquad (35)$$

Equation (35) is a differential equation of the first order. Therefore, if some coordinate q_j is cyclic, then its corresponding equation in Lagrange's equations (III) can be replaced by a differential equation (35) of the first order.

Example 1. Two pulleys of radii R and r are fastened to a common axis. Two heavy material points A_1 and A_2 of masses m_1 and m_2 hang on inextensible strings passing over the pulleys (Fig. 321). Determine the motion of the system, assuming that there is no friction.

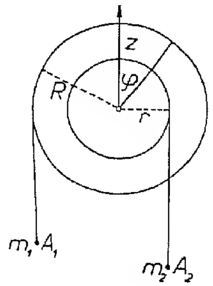

Fig. 321.

Let us assume that the motion takes place in a vertical plane. Let us give the z-axis a direction vertically upwards. Denote by z_1 and z_2 the coordinates of the points m_1 and m_2 at the time t, and by z_1^0 and z_2^0 those at $t = 0$. Let φ denote the angle of rotation of the pulleys, reckoning from the initial position. Assuming the angle of rotation as positive when clockwise, we obtain:

$$z_1 = z_1^0 + R\varphi, \quad z_2 = z_2^0 - r\varphi. \qquad (36)$$

The angle φ therefore determines the position of the system; hence we can take φ as the parameter.

Let I_1 and I_2 be the moments of inertia of the pulleys with respect to the common axis. The kinetic energy is

$$E = \tfrac{1}{2} m_1 \dot{z}_1^2 + \tfrac{1}{2} m_2 \dot{z}_2^2 + \tfrac{1}{2} I_1 \omega^2 + \tfrac{1}{2} I_2 \omega^2, \qquad (37)$$

where ω denotes the angular velocity of the pulleys. By (36) we have $\dot{z}_1 = R\dot{\varphi}$ and $\dot{z}_2 = -r\dot{\varphi}$, and in addition $\omega = \dot{\varphi}$. Putting $I = I_1 + I_2$, we obtain from (37)

$$E = \tfrac{1}{2}(m_1 R^2 + m_2 r^2 + I)\, \dot{\varphi}^2.$$

The potential of the force of gravity is

$$V = -m_1 g z_1 - m_2 g z_2 = -m_1 g(z_1^0 + R\varphi) - m_2 g(z_2^0 - r\varphi),$$

and therefore the kinetic potential $W = E + V$:

$$W = \tfrac{1}{2}(m_1 R^2 + m_2 r^2 + I)\,\varphi^{\cdot 2} - m_1 g(z_1^0 + R\varphi) - m_2 g(z_2^0 - r\varphi),$$

from which:

$$\partial W \,/\, \partial\varphi = -(m_1 R - m_2 r)\, g, \quad \partial W \,/\, \partial\varphi^{\cdot} = (m_1 R^2 + m_2 r^2 + I)\,\varphi^{\cdot}. \quad (38)$$

Lagrange's equation (III), p. 488, in our case has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \varphi^{\cdot}}\right) - \frac{\partial W}{\partial \varphi} = 0;$$

hence in view of (38) $(m_1 R^2 + m_2 r^2 + I)\,\varphi^{\cdot\cdot} + (m_1 R - m_2 r)\, g = 0$, whence

$$\varphi^{\cdot\cdot} = (m_2 r - m_1 R)\, g \,/\, (m_1 R^2 + m_2 r^2 + I). \quad (39)$$

Therefore the angular acceleration is constant.

From (36) we have $z_1^{\cdot\cdot} = R\varphi^{\cdot\cdot}$, and $z_2^{\cdot\cdot} = -r\varphi^{\cdot\cdot}$, consequently the material points will move with a uniformly accelerated motion.

In particular, when $R = r$, we have *Atwood's machine* (p. 193 and 375).

Example 2. A system composed of three rigid rods OA, AB, BC, of equal length l and equal mass m moves under the influence of the force of gravity in a vertical plane Π (Fig. 322). The rods are pinned at A and B, and fixed at O and C, where O and C lie on the horizontal line $OC = l$.

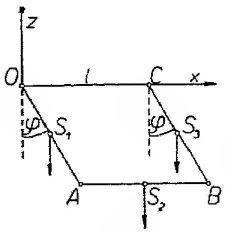

Fig. 322.

Let us choose axes x and z in the plane Π taking O as the origin of the system and giving the x-axis the horizontal direction OC and the z-axis an upward sense. Let us denote by φ the angle which the rods OA and CB make with the vertical, and by S_1, S_2, S_3, the centres of gravity of the rods (assuming that they lie at the geometric centres of these rods).

The angle φ defines the position of the system of rods; consequently φ is a parameter.

The instantaneous motion of the rods OA and BC is an instantaneous rotation about O and C with an angular velocity $\varphi^{\cdot}$. The rod AB moves with an advancing motion (cf. example 1, p. 321) with a velocity $\mathbf{v}$ of the point A, where $|\mathbf{v}| = l|\varphi^{\cdot}|$. The kinetic energy of the system is therefore

$$E = \tfrac{1}{2}I\varphi^{\cdot 2} + \tfrac{1}{2}ml^2\varphi^{\cdot 2} + \tfrac{1}{2}I\varphi^{\cdot 2} = (I + \tfrac{1}{2}ml^2)\,\varphi^{\cdot 2}, \quad (40)$$

where I denotes the moment of inertia of the rod with respect to an end. The coordinates z_1, z_2, z_3, of the centres of gravity S_1, S_2, S_3, are:

$$z_1 = -\tfrac{1}{2}l\cos\varphi, \quad z_2 = -l\cos\varphi, \quad z_3 = -\tfrac{1}{2}l\cos\varphi,$$

consequently the potential of the force of gravity

$$V = -mg(z_1 + z_2 + z_3) = 2mgl\cos\varphi. \quad (41)$$

In virtue of (40) and (41) the kinetic potential $W = E + V$ will therefore be

$$W = (I + \tfrac{1}{2}ml^2)\,\varphi^{\cdot 2} + 2mgl\cos\varphi, \quad (42)$$

whence

$$\partial W \,/\, \partial\varphi = -2mgl\sin\varphi, \quad \partial W \,/\, \partial\varphi^{\cdot} = 2(I + \tfrac{1}{2}ml^2)\,\varphi^{\cdot}. \quad (43)$$

Lagrange's equations (III), p. 488, will assume the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \varphi^{\cdot}}\right) - \frac{\partial W}{\partial \varphi} = 0;$$

hence in virtue of (43) $2(I + \tfrac{1}{2}ml^2)\,\varphi^{\cdot\cdot} + 2mgl\sin\varphi = 0$, whence

$$\varphi^{\cdot\cdot} = -\frac{mgl}{I + \tfrac{1}{2}ml^2}\sin\varphi. \quad (44)$$

Comparing equation (44) with the equation of the simple pendulum ((I), p. 130) we see that the given system of rods will oscillate like a simple pendulum of length $(I + \tfrac{1}{2}ml^2) \,/\, ml$.

Example 3. A line l lies in the vertical plane xz and rotates about the centre O of the coordinate system with a constant angular velocity ω. A heavy point A of mass m is constrained to the line l (Fig. 323). Determine the motion of the point A.

Let us denote by O' the projection of the point O on the line l, by φ the angle between OO' and the x-axis, and let us put $p = OO' = \text{const.}$ Let us assume that the line l has the direction of the z-axis at $t = 0$. Consequently

$$\varphi = \omega t. \quad (45)$$

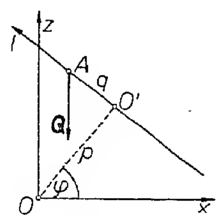

Fig. 323.

Let us give the line l an arbitrary sense and denote by q the coordinate of the point A on the line l, taking the point O' as the origin of this axis. Therefore the coordinates x and z of the point A are:

$$x = p\cos\omega t - q\sin\omega t, \quad z = p\sin\omega t + q\cos\omega t. \quad (46)$$

The system is consequently rheonomic and q is the parameter.

The virtual work is expressed by the formula $\delta' L = -mg\delta z$ (the z-axis has a sense vertically upwards). Since $\delta z = \delta q \cos\omega t$ by (46), $\delta' L = -mg\,\delta q \cos\omega t$. Therefore the generalized force is

$$Q = -mg\cos\omega t. \tag{47}$$

Let us now calculate the kinetic energy E. Differentiating (46), we obtain:

$$\dot{x} = -(p\omega + \dot{q})\sin\omega t - q\omega\cos\omega t,\quad \dot{z} = (p\omega + \dot{q})\cos\omega t - q\omega\sin\omega t;$$

consequently

$$E = \tfrac{1}{2}m(\dot{x}^2 + \dot{z}^2) = \tfrac{1}{2}m[(p\omega + \dot{q})^2 + q^2\omega^2], \tag{48}$$

whence

$$\partial E / \partial q = mq\omega^2,\quad \partial E / \partial \dot{q} = m(p\omega + \dot{q}). \tag{49}$$

By (II), p. 486, Lagrange's equation has in our case the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}}\right) - \frac{\partial E}{\partial q} = Q,$$

from which by (49) and (47) we obtain $m\ddot{q} - mq\omega^2 = -mg\cos\omega t$, i. e.

$$\ddot{q} - q\omega^2 = -g\cos\omega t. \tag{50}$$

The homogeneous equation $\ddot{q} - q\omega^2 = 0$ has the general solution $q = c_1 e^{\omega t} + c_2 e^{-\omega t}$, where c_1 and c_2 are arbitrary constants. A particular solution of equation (50) is, as is easily verified, $q = g\cos\omega t / 2\omega^2$. Therefore the general solution of equation (50) is

$$q = c_1 e^{\omega t} + c_2 e^{-\omega t} + \frac{g}{2\omega^2}\cos\omega t. \tag{51}$$

The constans c_1 and c_2 are determined from initial conditions.

Equations (46) and (51) determine the motion of the point.

Remark. Weight has the potential $V = -mgz$; hence, by (46), $V = -mg(p\sin\omega t + q\cos\omega t)$. In virtue of (48), therefore, the kinetic potential $W = E + V$ is equal to

$$W = \tfrac{1}{2}m[(p\omega + \dot{q})^2 + q^2\omega^2] - mg(p\sin\omega t + q\cos\omega t). \tag{52}$$

By (III), p. 488, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot{q}}\right) - \frac{\partial W}{\partial q} = 0,$$

whence by (52) we obtain equation (50).

Example 4. Two heavy material points A and B of masses $m + \mu$ and m hang at the ends of a weightless and inextensible string passing over a

pulley (Atwood's machine, p. 193 and 375) (Fig. 324). An insect C of mass μ crawls along the string on the side of the weight B. Denoting by h the projection of the vector $\overline{BC}$ on the z-axis having its origin at the centre of the pulley and a direction vertically downwards, we have

$$h = f(t), \tag{53}$$

where $f(t)$ is a given function. Determine the motion of the system of points A, B, C.

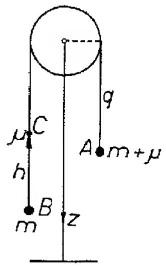

Fig. 324.

Let z_1, z_2, z_3, be the coordinates of the points A, B, C; l the length of the string, r the radius of the pulley, and I the moment of inertia of the pulley with respect to the centre. Taking the coordinate z_1 as the parameter q, we have

$$z_1 = q,\quad z_2 = l - q - r\pi,\quad z_3 = l - q - r\pi + f(t), \tag{54}$$

whence $\delta z_1 = \delta q$, $\delta z_2 = -\delta q$, $\delta z_3 = -\delta q$.

The virtual work of the weights is equal to

$$\delta' L = (m + \mu)\,g\,\delta z_1 + mg\,\delta z_2 + \mu g\,\delta z_3 = (m + \mu)\,g\,\delta q - mg\,\delta q - \mu g\,\delta q = 0;$$

hence the generalized force is

$$Q = 0. \tag{55}$$

The kinetic energy E is

$$E = \tfrac{1}{2}(m + \mu)\,\dot{z}_1^2 + \tfrac{1}{2}m\dot{z}_2^2 + \tfrac{1}{2}\mu\dot{z}_3^2 + \tfrac{1}{2}I\omega^2, \tag{56}$$

where ω denotes the angular velocity of the pulley. From (54) we have:

$$\dot{z}_1 = \dot{q},\quad \dot{z}_2 = -\dot{q},\quad \dot{z}_3 = -\dot{q} + \dot{f}. \tag{57}$$

Since $r|\omega| = |\dot{z}_1| = |\dot{q}|$ it follows that $\omega^2 = \dot{q}^2 / r^2$, whence by (56) and (57)

$$E = \tfrac{1}{2}(m + \mu)\,\dot{q}^2 + \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}\mu(\dot{q} - \dot{f})^2 + \tfrac{1}{2}I\dot{q}^2 / r^2.$$

From this

$$\frac{\partial E}{\partial q} = 0,\quad \frac{\partial E}{\partial \dot{q}} = (2m + 2\mu + I / r^2)\,\dot{q} - \mu\dot{f}. \tag{58}$$

Lagrange's equations (II), p. 486, will assume the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}}\right) - \frac{\partial E}{\partial q} = Q.$$

By (55) and (58) we obtain from this

$$(2m + 2\mu + I / r^2)\, q^{\cdot\cdot} - \mu f^{\cdot\cdot} = 0 \tag{59}$$

and after integration

$$(2m + 2\mu + I / r^2)\, q - \mu f(t) = c_1 t + c_2. \tag{60}$$

The constants c_1 and c_2 are determined from the initial conolitions.

Let us assume that at $t = 0$:

$$f(0) = 0, \quad f^{\cdot}(0) = 0, \quad z_1 = q = q_0, \quad z_1^{\cdot} = q^{\cdot} = 0.$$

From equation (60) and its derivative we obtain:

$$c_2 = (2m + 2\mu + I / r^2)\, q_0, \quad c_1 = 0. \tag{61}$$

Putting $k = 2m + 2\mu + I / r^2$, we get from formulae (60) and (61) $q = \frac{\mu}{k} \cdot f(t) + q_0$, and consequently by (54):

$$z_1 = \frac{\mu}{k} f(t) + q_0, \quad z_3 = l - r\pi - q_0 + \frac{k-\mu}{k} f(t). \tag{62}$$

Since $k - \mu > 0$, it follows from (62) that if the insect C crawls up the string, then the weight A will also go up. At the moment the insect reaches the pulley, i. e. the height $z_3 = 0$, we shall have, as follows from (62),

$$z_1 = q_0 - \frac{\mu}{k - \mu}(l - r\pi - q_0).$$

Example 5. A material point A of mass m moves along the xy-plane under the action of a central force $\mathbf{P}$ whose projection P on the radius vector $\overline{OA}$ (where O denotes the origin of the coordinate system) is a function of the distance $r = OA$ and

$$P = f(r). \tag{63}$$

Let us introduce the polar coordinates r, φ. The coordinates x, y, of the point A will therefore be expressed by the formulae:

$$x = r \cos\varphi, \quad y = r \sin\varphi. \tag{64}$$

The polar coordinates r, φ, are consequently independent parameters.

From (64) we obtain after differentiating:

$$x^{\cdot} = r^{\cdot} \cos\varphi - r\varphi^{\cdot} \sin\varphi, \quad y^{\cdot} = r^{\cdot} \sin\varphi + r\varphi^{\cdot} \cos\varphi.$$

Therefore the kinetic energy is

$$E = \tfrac{1}{2}m(x^{\cdot 2} + y^{\cdot 2}) = \tfrac{1}{2}m(r^{\cdot 2} + r^2\varphi^{\cdot 2}). \tag{65}$$

Since the field is a central field and the force depends on the distance,

the field is a potential field (p. 101). Consequently by (63) and (3), p. 101, the potential is

$$V = \int P \,\mathrm{d}r = \int f(r)\,\mathrm{d}r, \tag{66}$$

and the kinetic potential $W = E + V$ is by (65) and (66),

$$W = \tfrac{1}{2}m(r^{\cdot 2} + r^2\varphi^{\cdot 2}) + \int P\,\mathrm{d}r. \tag{67}$$

Since the kinetic potential W does not depend on φ, it follows that φ is a cyclic coordinate, whence (p. 489) $\partial W / \partial \varphi^{\cdot} = \text{const.}$, i. e.

$$mr^2\varphi^{\cdot} = \text{const.} \tag{68}$$

Lagrange's equation for the coordinate r has the form ((III), p. 488, with r instead of q_j)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial r^{\cdot}}\right) - \frac{\partial W}{\partial r} = 0. \tag{69}$$

From (67) we get

$$mr^{\cdot\cdot} - mr\varphi^{\cdot 2} - P = 0. \tag{70}$$

From (68) we have $\varphi^{\cdot} = \text{const} / mr^2 = c / r^2$, where c is a certain constant. Substituting this value of $\varphi^{\cdot}$ in (70), we get $m(r^{\cdot\cdot} - c^2 / r^3) = P$, i. e.

$$r^{\cdot\cdot} - c^2 / r^3 = f(r) / m.$$

From this equation we can determine r as a function of the time t.

Example 6. Motion of a point on a surface of revolution. A curve lying in the xz-plane and having the equation

$$z = f(x), \tag{71}$$

generates a surface of revolution S by rotating about the z-axis. The equation of the surface S is therefore the equation

$$z = f(\sqrt{x^2 + y^2}). \tag{72}$$

Let us introduce polar coordinates r, φ, in the xy-plane. Then

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = f(r). \tag{73}$$

The variables r, φ, are therefore independent parameters.

A material point of mass m, constrained to remain on the surface S, is subjected to the action of a force $\mathbf{P}$. Determine Lagrange's equations of the second kind.

We shall first determine the generalized forces. From equations (73) we have:

$$\delta x = \delta r \cos\varphi - r\,\delta\varphi \sin\varphi, \quad \delta y = \delta r \sin\varphi + r\,\delta\varphi\cos\varphi,$$
$$\delta z = f'(r)\,\delta r. \tag{74}$$

The virtual work is

$$\delta' L = P_x\,\delta x + P_y\,\delta y + P_z\,\delta z. \tag{75}$$

Substituting (74) in (75), we obtain

$$\begin{aligned}\delta' L = (P_x \cos\varphi + P_y \sin\varphi + P_z f'(r))\,\delta r +\\ + (-P_x \sin\varphi + P_y \cos\varphi)\, r\,\delta\varphi.\end{aligned} \tag{76}$$

The coefficients of δr and $\delta\varphi$ are the generalized forces. Let us denote them by Q_r and Q_φ. Consequently:

$$Q_r = P_x \cos\varphi + P_y \sin\varphi + P_z f'(r),\quad Q_\varphi = (-P_x \sin\varphi + P_y \cos\varphi)\, r. \tag{77}$$

Let us now determine the kinetic energy. Differentiating (73), we get:

$$\dot x = \dot r \cos\varphi - r\dot\varphi \sin\varphi,\quad \dot y = \dot r \sin\varphi + r\dot\varphi \cos\varphi,\quad \dot z = f'(r)\,\dot r. \tag{78}$$

The kinetic energy $E = \frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2)$, whence by (78)

$$E = \tfrac{1}{2}m[(1 + f'^2(r))\,\dot r^2 + r^2\dot\varphi^2], \tag{79}$$

and from this

$$\partial E / \partial r = m[f'(r)\, f''(r)\, \dot r^2 + r\dot\varphi^2],\qquad \partial E / \partial\varphi = 0, \tag{80}$$

$$\partial E / \partial\dot r = m[1 + f'^2(r)]\,\dot r,\qquad \partial E / \partial\dot\varphi = mr^2\dot\varphi. \tag{81}$$

From equations (II), p. 486, putting $q_1 = r$, $q_2 = \varphi$, $Q_1 = Q_r$, $Q_2 = Q_\varphi$, we obtain by (80) and (81):

$$m\frac{\mathrm{d}}{\mathrm{d}t}[(1 + f'^2(r))\,\dot r] - m[f'(r)\, f''(r)\, \dot r^2 + r\dot\varphi^2] = Q_r, \tag{82}$$

$$m\mathrm{d}(r^2\dot\varphi) / \mathrm{d}t = Q_\varphi. \tag{83}$$

The generalized forces Q_r and Q_φ are given by formulae (77).

Let us assume that the motion takes place in a potential field, e. g. in a gravitational field. The potential will then be $V = -mgz$ (when the z-axis has a sense vertically upwards). Hence by (73) we have

$$V = -mg\, f(r), \tag{84}$$

whence for the kinetic potential $W = E + V$:

$$\frac{\partial W}{\partial r} = \frac{\partial E}{\partial r} - mg\, f'(r),\quad \frac{\partial W}{\partial \dot r} = \frac{\partial E}{\partial \dot r},\quad \frac{\partial W}{\partial \varphi} = 0,\quad \frac{\partial W}{\partial \dot\varphi} = \frac{\partial E}{\partial \dot\varphi}.$$

It follows from this that the coordinate φ is cyclic. For the coordinate r equations (III), p. 488, assume the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[(1 + f'^2(r))\,\dot r] - [f'(r)\, f''(r)\, \dot r^2 + r\dot\varphi^2 - g\, f'(r)] = 0. \tag{85}$$

Since φ is a cyclic coordinate, $\partial W / \partial\dot\varphi = \text{const}$, i. e.

$$r^2\dot\varphi = \text{const} = c. \tag{86}$$

From the theorem on the conservation of total energy (p. 105) it follows that $E - V = \text{const}$; hence by (79) and (84)

$$[1 + f'^2(r)]\,\dot r^2 + r^2\dot\varphi^2 + g\, f(r) = \text{const} = c_1. \tag{87}$$

From the equations of the first order (86) and (87) we can determine the motion of the point.

Example 7. Spherical coordinates. We shall investigate the motion of a free material point $A(x, y, z)$ moving under the influence of a force **P** in a spherical coordinate system r, ϑ, φ, where $r = OA$, O denotes the origin of the coordinate system (x, y, z), ϑ is the angle between OA and the z-axis, and φ the angle between the x-axis and the projection OA' of the segment OA on the xy-plane (Fig. 325).

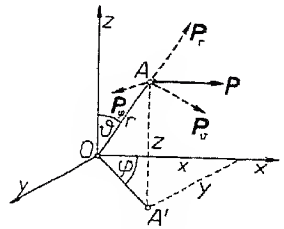

Fig. 325.

We have:

$$x = r\sin\vartheta\cos\varphi,\quad y = r\sin\vartheta\sin\varphi,\quad z = r\cos\vartheta. \tag{88}$$

Since the material point is free, the parameters r, ϑ, φ, are independent. From (88) we get:

$$\begin{aligned}\delta x &= \delta r\sin\vartheta\sin\varphi + r\,\delta\vartheta\cos\vartheta\cos\varphi - r\,\delta\varphi\sin\vartheta\sin\varphi,\\ \delta y &= \delta r\sin\vartheta\sin\varphi + r\,\delta\vartheta\cos\vartheta\sin\varphi + r\,\delta\varphi\sin\vartheta\cos\varphi,\\ \delta z &= \delta r\cos\vartheta - r\,\delta\vartheta\sin\vartheta.\end{aligned} \tag{89}$$

The virtual work is equal to $\delta' L = P_x\,\delta x + P_y\,\delta y + P_z\,\delta z$, whence by (89)

$$\begin{aligned}\delta' L = (P_x\sin\vartheta\cos\varphi + P_y\sin\vartheta\sin\varphi + P_z\cos\vartheta)\,\delta r +\\ + r(P_x\cos\vartheta\cos\varphi + P_y\cos\vartheta\sin\varphi - P_z\sin\vartheta)\,\delta\vartheta +\\ + r\,.\,\sin\vartheta(-P_x\sin\varphi + P_y\cos\varphi)\,\delta\varphi.\end{aligned} \tag{90}$$

The coefficients of δr, $\delta\vartheta$, and $\delta\varphi$, are the components of the generalized force. Let us denote them by Q_r, Q_ϑ, and Q_φ. Consequently:

$$\begin{aligned}Q_r &= P_x\sin\vartheta\cos\varphi + P_y\sin\vartheta\sin\varphi + P_z\cos\vartheta,\\ Q_\vartheta &= r(P_x\cos\vartheta\cos\varphi + P_y\cos\vartheta\sin\varphi - P_z\sin\vartheta),\\ Q_\varphi &= r\sin\vartheta(-P_x\sin\varphi + P_y\cos\varphi).\end{aligned} \tag{91}$$

Let Π be a plane passing through OA and the z-axis. From the point A let us draw the axes Θ, Φ, perpendicular to OA: the axis Θ in the plane Π, and the axis Φ perpendicular to Π. Let us give the axes senses in the direction of the increase of the angles ϑ, φ, and let us denote by P_r, P_ϑ, P_φ, the components of the force **P** in the directions $\overline{OA}$, Θ, Φ, respectively. It is easy to shew that in virtue of (91):

$$Q_r = P_r,\quad Q_\vartheta = rP_\vartheta,\quad Q_\varphi = r\sin\vartheta P_\varphi. \tag{92}$$

The kinetic energy is $E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. The derivatives $\dot{x}, \dot{y}, \dot{z}$, are obtained from (89) by writing $\dot{r}, \dot{\vartheta}, \dot{\varphi}$, instead of $\delta r, \delta\vartheta, \delta\varphi$. Substituting the values obtained, we get

$$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2\sin^2\vartheta + r^2\dot{\vartheta}^2), \tag{93}$$

whence:

$$\partial E / \partial r = mr(\dot{\varphi}^2\sin^2\vartheta + \dot{\vartheta}^2), \quad \partial E / \partial\vartheta = mr^2\dot{\varphi}^2\sin\vartheta\cos\vartheta, \quad \partial E / \partial\varphi = 0, \tag{94}$$

$$\partial E / \partial\dot{r} = m\dot{r}, \quad \partial E / \partial\dot{\vartheta} = mr^2\dot{\vartheta}, \quad \partial E / \partial\dot{\varphi} = mr^2\dot{\varphi}\sin^2\vartheta. \tag{95}$$

Putting in Lagrange's equations (II), p. 486:

$$q_1 = r, \quad q_2 = \varphi, \quad q_3 = \vartheta,$$

we get by (94) and (95):

$$\begin{gathered} m\ddot{r} - mr(\dot{\varphi}^2\sin^2\vartheta + \dot{\vartheta}^2) = Q_r, \\ m\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\varphi}\sin^2\vartheta) = Q_\varphi, \\ m\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\vartheta}) - mr^2\dot{\varphi}^2\sin\vartheta\cos\vartheta = Q_\vartheta . \end{gathered} \tag{96}$$

Equations (96) are the equations of motion in spherical coordinates.

§ 8. Hamilton's canonical equations. Let $q_1, \ldots, q_k$, be independent parameters. The kinetic energy E is in the general case a function of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k$, and of the time t. Regarding these variables as independent let us put

$$\frac{\partial E}{\partial \dot{q}_j} = p_j \qquad (j = 1, 2, \ldots, k). \tag{I}$$

The expressions (I) are called *generalized impulses*.

In virtue of (I), p_j are functions of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$; we can therefore write

$$p_j = \Phi_j(q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t) \qquad (j = 1, 2, \ldots, k). \tag{1}$$

It can be proved under rather general assumptions that equations (1) can be solved for the variables $\dot{q}_1, \ldots, \dot{q}_k$. Consequently

$$\dot{q}_j = \boldsymbol{\Phi}_j(q_1, \ldots, q_k, p_1, \ldots, p_k, t) \qquad (j = 1, 2, \ldots, k). \tag{2}$$

The kinetic potential $W = E + V$ is a function of the variables $q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t$:

$$W = F(q_1, \ldots, q_k, \dot{q}_1, \ldots, \dot{q}_k, t). \tag{3}$$

Substituting (2) in (3), we get

$$W = \mathbf{F}(q_1, \ldots, q_k, p_1, \ldots, p_k, t). \tag{4}$$

The function $\mathbf{F}$ is therefore a function compounded of the function F by means of the functions $\boldsymbol{\Phi}_j$. From the theorem on the derivative of a compound function we obtain:

$$\frac{\partial \mathbf{F}}{\partial q_i} = \frac{\partial F}{\partial q_i} + \sum_{j=1}^{k} \frac{\partial F}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial q_i}, \quad \frac{\partial \mathbf{F}}{\partial p_i} = \sum_{j=1}^{k} \frac{\partial F}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial p_i} \qquad (i = 1, 2, \ldots, k). \tag{5}$$

Since V does not depend on the derivatives $\dot{q}_1, \ldots, \dot{q}_k$, it follows that $\partial V / \partial \dot{q}_j = 0$; from $W = V + E$ we have

$$\partial W / \partial \dot{q}_j = \partial E / \partial \dot{q}_j. \tag{6}$$

Hence by (I) and (3) we have $\partial F / \partial \dot{q}_j = p_j$. From (5) we consequently get:

$$\frac{\partial \mathbf{F}}{\partial q_i} = \frac{\partial F}{\partial q_i} + \sum_{j=1}^{k} p_j \frac{\partial \dot{q}_j}{\partial q_i}, \quad \frac{\partial \mathbf{F}}{\partial p_i} = \sum_{j=1}^{k} p_j \frac{\partial \dot{q}_j}{\partial p_i} \qquad (i = 1, 2, \ldots, k). \tag{7}$$

Let us put

$$H = \sum_{j=1}^{k} p_j \dot{q}_j - W \tag{II}$$

and assume that $\dot{q}_j$ and W are functions of the variables $q_1, \ldots, q_k, p_1, \ldots, p_k, t$, i. e. that they denote the functions (2) and (4). Then

$$H = \sum_{j=1}^{k} p_j \dot{q}_j - \mathbf{F}. \tag{8}$$

Forming partial derivatives, we obtain from (8):

$$\frac{\partial H}{\partial q_i} = \sum_{j=1}^{k} p_j \frac{\partial \dot{q}_j}{\partial q_i} - \frac{\partial \mathbf{F}}{\partial q_i}, \quad \frac{\partial H}{\partial p_i} = \dot{q}_i + \sum_{j=1}^{k} p_j \frac{\partial \dot{q}_j}{\partial p_i} - \frac{\partial \mathbf{F}}{\partial p_i}; \tag{9}$$

hence in virtue of (7):

$$\partial H / \partial q_i = -\partial F / \partial q_i, \quad \partial H / \partial p_i = \dot{q}_i. \tag{10}$$

Lagrange's equations (III), p. 488, have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot{q}_j}\right) - \frac{\partial W}{\partial q_j} = 0 \qquad (j = 1, 2, \ldots, k). \tag{11}$$

By (I) and (6) $\partial W / \partial \dot{q}_j = p_j$. From equation (11) we get

$$\dot{p}_j = \partial W / \partial q_j \qquad (j = 1, 2, \ldots, k), \tag{12}$$

whence by (3) $\partial F / \partial q_j = \dot{p}_j$, and hence by (10):

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \qquad (i = 1, 2, \ldots, k). \tag{III}$$

The function H is called *Hamilton's function*, and equations (III) are called *Hamilton's canonical equations*.

The variables p_i or the generalized impulses are therefore defined by equations (I), and the function H by equations (II). In equations (III) the function H is a function of the variables q_i, p_i, t. Equations (III) consequently form a system of differential equations of the first order, where the unknown functions are q_i and p_i as functions of the time t.

The investigation of motions of systems having a potential is therefore reduced to the examination of differential equations of the form (III). Hence the name *canonical equations*.

Scleronomic systems. Let us assume that a system is scleronomic. By (I), p. 498, we therefore have

$$\sum_{j=1}^{k} p_j q_j^{\cdot} = \sum_{j=1}^{k} \frac{\partial E}{\partial q_j^{\cdot}} q_j^{\cdot}. \tag{13}$$

Let the natural coordinates be expressed by the functions:

$$x_i = f_i(q_1, \ldots, q_k), \quad y_i = \varphi_i(q_1, \ldots, q_k), \quad z_i = \psi_i(q_1, \ldots, q_k) \tag{14}$$
$$(i = 1, 2, \ldots, n).$$

Consequently

$$x_i^{\cdot} = \frac{\partial x_i}{\partial q_1} q_1^{\cdot} + \ldots + \frac{\partial x_i}{\partial q_k} q_k^{\cdot}. \tag{15}$$

Forming partial derivatives with respect to $q_j^{\cdot}$, we get:

$$\delta x_i^{\cdot} / \delta q_j^{\cdot} = \delta x_i / \delta q_j,$$
$$\delta y_i^{\cdot} / \delta q_j^{\cdot} = \delta y_i / \delta q_j, \quad \delta z_i^{\cdot} / \delta q_j^{\cdot} = \delta z_i / \delta q_j. \tag{16}$$

We have

$$E = \tfrac{1}{2}\sum_{i=1}^{n} m_i(x_i^{\cdot 2} + y_i^{\cdot 2} + z_i^{\cdot 2}); \tag{17}$$

hence

$$\frac{\partial E}{\partial q_j^{\cdot}} = \sum_{i=1}^{n} m_i \left(x_i^{\cdot} \frac{\partial x_i^{\cdot}}{\partial q_j^{\cdot}} + y_i^{\cdot} \frac{\partial y_i^{\cdot}}{\partial q_j^{\cdot}} + z_i^{\cdot} \frac{\partial z_i^{\cdot}}{\partial q_j^{\cdot}} \right),$$

whence by (16)

$$\frac{\partial E}{\partial q_j^{\cdot}} q_j^{\cdot} = \sum_{i=1}^{n} m_i \left(x_i^{\cdot} \frac{\partial x_i}{\partial q_j} q_j^{\cdot} + y_i^{\cdot} \frac{\partial y_i}{\partial q_j} q_j^{\cdot} + z_i^{\cdot} \frac{\partial z_i}{\partial q_j} q_j^{\cdot} \right),$$

and from this

$$\sum_{j=1}^{k} \frac{\partial E}{\partial q_j^{\cdot}} q_j^{\cdot} = \sum_{i=1}^{n} m_i \left[x_i^{\cdot} \left(\frac{\partial x_i}{\partial q_1} q_1^{\cdot} + \ldots + \frac{\partial x_i}{\partial q_k} q_k^{\cdot} \right) + \right.$$
$$\left. + y_i^{\cdot} \left(\frac{\partial y_i}{\partial q_1} q_1^{\cdot} + \ldots + \frac{\partial y_i}{\partial q_k} q_k^{\cdot} \right) + z_i^{\cdot} \left(\frac{\partial z_i}{\partial q_1} q_1^{\cdot} + \ldots + \frac{\partial z_i}{\partial q_k} q_k^{\cdot} \right) \right].$$

Therefore in virtue of (15)

$$\sum_{j=1}^{k} \frac{\partial E}{\partial q_j^{\cdot}} q_j^{\cdot} = \sum_{i=1}^{n} m_i(x_i^{\cdot 2} + y_i^{\cdot 2} + z_i^{\cdot 2}) = 2E,$$

whence by (13)

$$\sum_{j=1}^{k} p_j q_j^{\cdot} = 2E. \tag{18}$$

From (II), p. 499, and (18) we get

$$H = 2E - W. \tag{19}$$

According to the definition ((33), p. 488), we have $W = E + V$, where V is the potential. From (19) it therefore follows that

$$H = E - V. \tag{20}$$

Now, $E - V$ is the total energy of the system.

Therefore: *in scleronomic systems Hamilton's function H denotes the total energy of the system.*

Let us assume that the potential V does not depend on the time. H is then a function of the variables $q_1, \ldots, q_k, p_1, \ldots, p_k$ only. Consequently

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_{i=1}^{k} \left(\frac{\partial H}{\partial q_i} q_i^{\cdot} + \frac{\partial H}{\partial p_i} p_i^{\cdot} \right). \tag{21}$$

From equations (III) we obtain

$$\frac{\partial H}{\partial q_i} q_i^{\cdot} + \frac{\partial H}{\partial p_i} p_i^{\cdot} = - p_i^{\cdot} q_i^{\cdot} + q_i^{\cdot} p_i^{\cdot} = 0,$$

whence by (21) $\mathrm{d}H / \mathrm{d}t = 0$, i. e. $H = \text{const.}$

We have therefore proved that if a scleronomic system moves in a potential field, then it is subject to the principle of the conservation of total energy.

Example 1. A free material point of mass m moves in a potential field having a potential V.

Let us take the natural coordinates x, y, z, as parameters. The generalized impulses will be defined by relations (I), p. 498, if we substitute x, y, z, for q_1, q_2, q_3. Consequently:

$$p_1 = \partial E / \partial x^{\cdot}, \quad p_2 = \partial E / \partial y^{\cdot}, \quad p_3 = \partial E / \partial z^{\cdot}. \tag{22}$$

Since $E = \frac{1}{2}m(x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2})$,

$$p_1 = mx^{\cdot}, \quad p_2 = my^{\cdot}, \quad p_3 = mz^{\cdot}. \tag{23}$$

We see from this that p_1, p_2, p_3, are the projections of the momentum on the coordinate axes.

Determining from (23) $x^{\cdot}, y^{\cdot}, z^{\cdot}$, we obtain

$$E = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2). \tag{24}$$

Since the system is scleronomic, Hamilton's function H denotes its total energy. Consequently $H = E - V$, whence by (24)

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) - V.$$

From this $\partial H / \partial p_1 = p_1 / m$, etc., $\partial H / \partial x = - \partial V / \partial x$, etc. Hamilton's equations (III) therefore assume the form:

$$p_1^{\cdot} = \partial V / \partial x, \quad p_2^{\cdot} = \partial V / \partial y, \quad p_3^{\cdot} = \partial V / \partial z, \tag{25}$$

$$x^{\cdot} = p_1 / m, \quad y^{\cdot} = p_2 / m, \quad z^{\cdot} = p_3 / m. \tag{26}$$

Determining p_1, p_2, p_3 from (26) and substituting in (25), we obtain Newton's equations:

$$mx^{\cdot\cdot} = \partial V / \partial x, \quad my^{\cdot\cdot} = \partial V / \partial y, \quad mz^{\cdot\cdot} = \partial V / \partial z.$$

Example 2. A material point of mass m is constrained to remain on the surface of a cylinder of revolution $x^2 + y^2 = r^2$. The point is acted on by an elastic force $\mathbf{P}$ whose projections are:

$$P_x = - k^2 mx, \quad P_y = - k^2 my, \quad P_z = - k^2 mz, \tag{27}$$

where k is a certain constant.

The elastic force — as is easily verified — has the potential

$$V = - \tfrac{1}{2}k^2 m(x^2 + y^2 + z^2). \tag{28}$$

Let us introduce the polar coordinates r, φ, in the xy-plane. Therefore $x = r \cos\varphi$, and $y = r \sin\varphi$, whence (because $r = \text{const}$) we have $x^{\cdot 2} + y^{\cdot 2} = r^2\varphi^{\cdot 2}$.

The kinetic energy is consequently equal to

$$E = \tfrac{1}{2}m(x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}) = \tfrac{1}{2}m(r^2\varphi^{\cdot 2} + z^{\cdot 2}). \tag{29}$$

The variables φ, z, can be taken as independent parameters. Denoting by p_1 and p_2 the corresponding generalized impulses and writing φ, z, instead of q_1, q_2, we obtain from (I) $\partial E / \partial \varphi^{\cdot} = p_1$ and $\partial E / \partial z^{\cdot} = p_2$, whence by (29):

$$p_1 = mr^2\varphi^{\cdot}, \quad p_2 = mz^{\cdot}. \tag{30}$$

Determining $\varphi^{\cdot}$ and $z^{\cdot}$ from (30) and substituting in (29) we obtain

$$E = \frac{1}{2m}[p_1^2 / r^2 + p_2^2].$$

By (28) $V = - \frac{1}{2}k^2 m(r^2 + z^2)$; hence Hamilton's function ((20), p. 501) assumes the form

$$H = E - V = \frac{1}{2m}[p_1^2 / r^2 + p_2^2] + \tfrac{1}{2}k^2 m(r^2 + z^2). \tag{31}$$

Consequently Hamilton's equations (III) are:

$$p_1^{\cdot} = - \partial H / \partial \varphi, \quad p_2^{\cdot} = - \partial H / \partial z, \quad \varphi^{\cdot} = \partial H / \partial p_1, \quad z^{\cdot} = \partial H / \partial p_2,$$

and hence by (31):

$$p_1^{\cdot} = 0, \quad p_2^{\cdot} = - k^2 mz, \quad \varphi^{\cdot} = p_1 / mr^2, \quad z^{\cdot} = p_2 : m. \tag{32}$$

The last two of the equations (32) are equivalent to equations (30).

The first of the equations (32) gives $p_1 = \text{const}$; hence by (30) $mr^2\varphi^{\cdot} = \text{const}$, i. e. $\varphi^{\cdot} = \text{const}$. The projections of the point on the horizontal plane will therefore go around the base of the cylinder with a uniform motion.

In virtue of (30) the second of the equations (32) gives $mz^{\cdot\cdot} = - k^2 mz$, i. e. $z^{\cdot\cdot} + k^2 z = 0$. Comparing it with equation (2), p. 110, we see that the projection of the point on the z-axis will execute a simple harmonic motion.

CHAPTER XI

VARIATIONAL PRINCIPLES OF MECHANICS

§ 1. Variation without the variation of time. In this paragraph we shall give certain information from the calculus of variations necessary for the understanding of what follows.

Variation of a function. Let us take under consideration the motion of a point along the x axis, defined by the function

$$x = x(t) \qquad (t_0 \leqq t \leqq t_1). \tag{1}$$

Let there be given a function

$$T = F(x, x^{\cdot}, t) \tag{2}$$

continuous and having continuous partial derivatives of the first and second orders in a certain region D of the variables $x, x^{\cdot}, t$.

Let us next take under consideration an arbitrary motion along the x-axis, defined by the function

$$\mathbf{x} = \mathbf{x}(t) \qquad (t_0 \leqq t \leqq t_1). \tag{3}$$

Let us assume that it is possible to choose a number $\varepsilon > 0$ such that if

$$|\mathbf{x}(t) - x(t)| < \varepsilon, \quad |\mathbf{x}^{\cdot}(t) - x^{\cdot}(t)| < \varepsilon \qquad (t_0 \leqq t \leqq t_1), \tag{4}$$

then the function T and its partial derivatives will be continuous functions in the interval $\langle t_0, t_1\rangle$ when $\mathbf{x}(t)$ and $\mathbf{x}^{\cdot}(t)$ are substituted in (2) for x and $x^{\cdot}$, respectively.

Let us put:

$$\delta x = \mathbf{x} - x, \quad \delta x^{\cdot} = \mathbf{x}^{\cdot} - x^{\cdot}, \tag{5}$$

where x and $\mathbf{x}$ denote the functions $x(t)$ and $\mathbf{x}(t)$. Consequently δx and $\delta x^{\cdot}$ are functions of the time t, defined in the interval $\langle t_0, t_1\rangle$.

One should note the difference in the meaning of the symbol δx in chapters IX, X and now. Before, the symbol δx denoted a number, and now it denotes a function of the time t.

By (5) we have:

$$\delta x^{\cdot} = \frac{\mathrm{d}(\delta x)}{\mathrm{d}t}, \tag{I}$$

$$\mathbf{x} = x + \delta x, \quad \mathbf{x}^{\cdot} = x^{\cdot} + \delta x^{\cdot}. \tag{6}$$

Let

$$\mathbf{T} = F(\mathbf{x}, \mathbf{x}^{\cdot}, t) = F(x + \delta x, x^{\cdot} + \delta x^{\cdot}, t),$$

where x denotes the function (1), and δx is defined by (5). From Taylor's formula we get

$$\begin{aligned}\mathbf{T} - T &= F(x + \delta x, x^{\cdot} + \delta x^{\cdot}, t) - F(x, x^{\cdot}, t) = \\ &= \frac{\partial T}{\partial x}\,\delta x + \frac{\partial T}{\partial x^{\cdot}}\,\delta x^{\cdot} + R.\end{aligned} \tag{7}$$

The remainder R can be written in the form

$$R = (|\delta x| + |\delta x^{\cdot}|)\,\eta, \tag{8}$$

where η is a function of the time t and depends on x, δx, $\delta x^{\cdot}$, and where η tends to zero uniformly when the functions δx and $\delta x^{\cdot}$ tend to zero uniformly. Therefore, if $|\delta x|$ and $|\delta x^{\cdot}|$ are small, then $|\eta|$ is small, and consequently $|R|$ is of a still higher order of smallness. We express this briefly by saying that R is *"infinitesimal"* as compared with $|\delta x| + |\delta x^{\cdot}|$.

Let us put

$$\delta T = \frac{\partial T}{\partial x}\,\delta x + \frac{\partial T}{\partial x^{\cdot}}\,\delta x^{\cdot}. \tag{II}$$

Hence by (7)

$$\mathbf{T} - T = \delta T + R. \tag{9}$$

The expression δT is called the *variation of the function* $T = F(x, x^{\cdot}, t)$ *at the place* $x = x(t)$ or *for the function* $x = x(t)$.

In formula (II) the function δx is an arbitrary function of time having continuous first and second derivatives in the interval $\langle t_0, t_1\rangle$. This follows from (5), where $\mathbf{x}$ is an arbitrary function having continuous first and second derivatives. In virtue of (I) the symbol $\delta x^{\cdot}$ denotes the derivative of the function δx with respect to the time t.

The variation δT therefore depends on the functions x and δx.

For purposes of differentiation we shall call δx *the variation of the independent variable* (*or of the function*) x, and δT *the variation of the dependent variable* (*or of the function*) T.

The motion $\mathbf{x} = \mathbf{x}(t) = x + \delta x$, will be called a *comparative motion*.

The variation δT therefore denotes approximately the increment of the function T when we pass from a point in the given motion at the moment t

to a point in the comparative motion at the same moment t. In virtue of (8) the difference R between the variation δT and the true increment $\mathbf{T} - T$ is "infinitesimal" as compared with the sum $|\delta x| + |\delta x^{\cdot}|$.

Since we are investigating the increment of the function T in the given motion and in the comparative motion at the same instant t, the variation δT is also called the *variation without the variation of time* in order to differentiate it from another kind of variation with which we shall meet later.

The variation δT is obtained by forming formally the differential of the function $F(x, x^{\cdot}, t)$ under the assumption that $t = \text{const}$ (i. e. $\mathrm{d}t = 0$) and then writing δ instead of d. We often write $\delta F(x, x^{\cdot}, t)$ instead of δT or briefly δF.

Example 1. Let

$$T = \alpha x^2 + \beta x^{\cdot 2} t + \gamma t^2,$$

where α, β, γ, are constants. We have:

$$\frac{\partial T}{\partial x} = 2\alpha x, \qquad \frac{\partial T}{\partial x^{\cdot}} = 2\beta x^{\cdot} t;$$

consequently by (II)

$$\delta T = 2\alpha x\, \delta x + 2\beta x^{\cdot} t\, \delta x^{\cdot},$$

where δx is an arbitrary function.

Variation of an integral. Let us consider the integral

$$I = \int_{t_0}^{t_1} F(x, x^{\cdot}, t)\, \mathrm{d}t \tag{10}$$

and let

$$\mathbf{I} = \int_{t_0}^{t_1} F(x + \delta x, x^{\cdot} + \delta x^{\cdot}, t)\, \mathrm{d}t.$$

We have

$$\mathbf{I} - I = \int_{t_0}^{t_1} [F(x + \delta x, x^{\cdot} + \delta x^{\cdot}, t) - F(x, x^{\cdot}, t)]\, \mathrm{d}t,$$

and hence by (7) and (II)

$$\mathbf{I} - I = \int_{t_0}^{t_1} \delta F\, \mathrm{d}t + \int_{t_0}^{t_1} R\, \mathrm{d}t. \tag{11}$$

The expression

$$\int_{t_0}^{t_1} \delta F\, \mathrm{d}t$$

is called the *variation of the integral* (10) and we denote it by δI.

Therefore according to the definition

$$\delta I = \delta \int_{t_0}^{t_1} F\, \mathrm{d}t = \int_{t_0}^{t_1} \delta F\, \mathrm{d}t. \tag{III}$$

As before, the variation of the integral (10) represents approximately the increment of the integral when we pass from a given motion to a comparative motion. The difference between the true increment and the variation δI is "infinitesimal" in comparison with $|\delta x| + |\delta x^{\cdot}|$.

Variation of a derivative. Let the function

$$x = \Phi(q, t) \tag{12}$$

be given.

Let us assume that q is a certain function of the time t; hence

$$q = q(t) \qquad (t_0 \leqq t \leqq t_1). \tag{13}$$

Regarding q in formula (12) as an independent variable function, we obtain for the variation of the dependent variable x

$$\delta x = \frac{\partial x}{\partial q}\, \delta q, \tag{14}$$

where δq denotes an arbitrary function continuous together with its first and second derivatives in the interval $\langle t_0, t_1 \rangle$.

Let us form the derivative of (12). We get

$$x^{\cdot} = \frac{\partial x}{\partial q} q^{\cdot} + \frac{\partial x}{\partial t};$$

the variation $\delta x^{\cdot}$ of this function is therefore

$$\delta x^{\cdot} = \frac{\partial x^{\cdot}}{\partial q}\, \delta q + \frac{\partial x^{\cdot}}{\partial q^{\cdot}}\, \delta q^{\cdot}.$$

From (12) we see that the derivatives $\delta x / \delta q$ and $\delta x / \delta t$ do not depend on $q^{\cdot}$, because x does not depend on $q^{\cdot}$. Hence we obtain

$$\delta x^{\cdot} = \left(\frac{\partial^2 x}{\partial q^2} q^{\cdot} + \frac{\partial^2 x}{\partial q\, \partial t} \right) \delta q + \frac{\partial x}{\partial q}\, \delta q^{\cdot}. \tag{15}$$

Forming the derivative of (14) with respect to t, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta x) = \left(\frac{\partial^2 x}{\partial q^2} q^{\cdot} + \frac{\partial^2 x}{\partial q\, \partial t} \right) \delta q + \frac{\partial x}{\partial q}\, \delta q^{\cdot},$$

whence by (15)

$$\delta x^{\cdot} = \frac{\mathrm{d}(\delta x)}{\mathrm{d}t}. \tag{16}$$

Comparing (16) with formula (I) we see that both formulae have the same form. The difference lies in the fact that in formula (I) x is an independent variable and in (16) it is a dependent variable.

Formula (I) holds, therefore, regardless of whether x is a dependent or independent variable. It follows from this that *the variation is inter-*

changeable with the derivative, i. e. we obtain the same result by first forming the variation and then taking the derivative or conversely.

Variation of a compound function. Let the functions:

$$T = F(x, x^{\cdot}, t), \tag{17}$$

$$x = \Phi(q, t) \tag{18}$$

be given.

Let us assume that q is a function of the variable t:

$$q = q(t) \qquad (t_0 \leqq t \leqq t_1). \tag{19}$$

Forming the derivative of (18), we obtain

$$x^{\cdot} = \frac{\partial x}{\partial q} q^{\cdot} + \frac{\partial x}{\partial t}. \tag{20}$$

Substituting in (17) for x and $x^{\cdot}$ their values from (18) and (20), we obtain T as a function of the variables $q, q^{\cdot}, t$, i. e.

$$T = \Psi(q, q^{\cdot}, t), \tag{21}$$

and hence as its variation

$$\delta T = \frac{\partial T}{\partial q}\delta q + \frac{\partial T}{\partial q^{\cdot}}\delta q^{\cdot}. \tag{22}$$

From the theorem on the derivative of a compound function we obtain:

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial T}{\partial x^{\cdot}}\frac{\partial x^{\cdot}}{\partial q}, \qquad \frac{\partial T}{\partial q^{\cdot}} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial q^{\cdot}} + \frac{\partial T}{\partial x^{\cdot}}\frac{\partial x^{\cdot}}{\partial q^{\cdot}}, \tag{23}$$

and by (18):

$$\frac{\partial x}{\partial q^{\cdot}} = 0. \tag{24}$$

Substituting (24) in (23) and then in (22), we obtain

$$\partial T = \left(\frac{\partial T}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial T}{\partial x^{\cdot}}\frac{\partial x^{\cdot}}{\partial q}\right)\delta q + \frac{\partial T}{\partial x^{\cdot}}\frac{\partial x^{\cdot}}{\partial q^{\cdot}}\delta q^{\cdot} =$$
$$= \frac{\partial T}{\partial x}\left(\frac{\partial x}{\partial q}\delta q\right) + \frac{\partial T}{\partial x^{\cdot}}\left(\frac{\partial x^{\cdot}}{\partial q}\delta q + \frac{\partial x^{\cdot}}{\partial q^{\cdot}}\delta q^{\cdot}\right).$$

It is easy to verify that the expressions in the last two parentheses are variations of the functions (18) and (20), and therefore equal to δx and $\delta x^{\cdot}$. Consequently

$$\partial T = \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial x^{\cdot}}\delta x^{\cdot}. \tag{25}$$

Formula (25) represents the *variation of the compound function* (21), where δx and $\delta x^{\cdot}$ denote the variations of the functions (18) and (20). Let us note that (25) also represents the variation of function (17). We see from this that formula (25), i. e. formula (II), p. 505, holds regardless of whether x is the dependent or independent variable function.

Similarly, when t = const, the formula for the differential

$$\mathrm{d}T = \frac{\partial T}{\partial x}\mathrm{d}x + \frac{\partial T}{\partial x^{\cdot}}\mathrm{d}x^{\cdot}$$

holds regardless of whether x and $x^{\cdot}$ are dependent variables or not.

Let us note that $\delta x^{\cdot}$ in formula (25) is by (16) the derivative of δx.

Systems of points. Let us now define the variation in the case of a system of points.

Let there be given a system of n material points

$$A_1(x_1, y_1, z_1), \quad \ldots, \quad A_n(x_n, y_n, z_n).$$

Let us consider an arbitrary motion of the system (compatible with constraints or not) defined by the functions:

$$x_i = x_i(t),\ y_i = y_i(t),\ z_i = z_i(t),\ (t_0 \leqq t \leqq t_1,\ i = 1, 2, \ldots, n). \tag{26}$$

Let the function

$$T = F(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n, x_1^{\cdot}, \ldots, x_n^{\cdot}, y_1^{\cdot}, \ldots, y_n^{\cdot}, z_1^{\cdot}, \ldots, z_n^{\cdot}, t) \tag{27}$$

be given.

The *variation of function* (27) for a motion defined by functions (26) is given by the expression

$$\begin{aligned} \partial T = {} & \frac{\partial T}{\partial x_1}\delta x_1 + \ldots + \frac{\partial T}{\partial x_n}\delta x_n + \frac{\partial T}{\partial y_1}\delta y_1 + \ldots + \frac{\partial T}{\partial y_n}\delta y_n + \\ & + \frac{\partial T}{\partial z_1}\delta z_1 + \ldots + \frac{\partial T}{\partial z_n}\delta z_n + \frac{\partial T}{\partial x_1^{\cdot}}\delta x_1^{\cdot} + \ldots + \frac{\partial T}{\partial x_n^{\cdot}}\delta x_n^{\cdot} + \frac{\partial T}{\partial y_1^{\cdot}}\delta y_1^{\cdot} + \ldots + \\ & + \frac{\partial T}{\partial y_n^{\cdot}}\delta y_n^{\cdot} + \frac{\partial T}{\partial z_1^{\cdot}}\delta z_1^{\cdot} + \ldots + \frac{\partial T}{\partial z_n^{\cdot}}\delta z_n^{\cdot}, \end{aligned} \tag{28}$$

which we write more compactly as

$$\partial T = \sum_{i=1}^{n}\left(\frac{\partial T}{\partial x_i}\delta x_i + \frac{\partial T}{\partial y_i}\delta y_i + \frac{\partial T}{\partial z_i}\delta z_i + \frac{\partial T}{\partial x_i^{\cdot}}\delta x_i^{\cdot} + \frac{\partial T}{\partial y_i^{\cdot}}\delta y_i^{\cdot} + \frac{\partial T}{\partial z_i^{\cdot}}\delta z_i^{\cdot}\right), \tag{IV}$$

where $\delta x_i, \delta y_i, \delta z_i$ are arbitrary functions continuous together with their first and second derivatives in the interval $\langle t_0, t_1\rangle$, where

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta x_i) = \delta x_i^{\cdot},\ \frac{\mathrm{d}}{\mathrm{d}t}(\delta y_i) = \delta y_i^{\cdot},\ \frac{\mathrm{d}}{\mathrm{d}t}(\delta z_i) = \delta z_i^{\cdot} \quad (i = 1, 2, \ldots, n). \tag{V}$$

The derivatives $\delta T / \partial x_1, \ldots, \delta T / \partial z_n^{\cdot}$ are partial derivatives of the functions (27), in which for $x_1, \ldots, z_n^{\cdot}$, are substituted the corresponding functions (26) and their derivatives.

As before, the variation δT denotes approximately the increment of the function T when we pass from the position of the system at the moment t in the given motion, to the position of the system at the same moment t in the comparative motion:

$$\mathbf{x}_i = x_i + \delta x_i, \ \mathbf{y}_i = y_i + \delta y_i, \ \mathbf{z}_i = z_i + \delta z_i \qquad (i = 1, 2, \ldots, n).$$

The difference between the true increment and the variation is — as is easily seen — "infinitesimal" in comparison with the sum

$$\sum_{i=1}^{n}(|\delta x_i| + |\delta y_i| + |\delta z_i| + |\delta x_i^{\cdot}| + |\delta y_i^{\cdot}| + |\delta z_i^{\cdot}|).$$

Let us note that the variation δT is obtained by forming the differential of the function T under the assumption that $t = \text{const}$ (i. e. for $dt = 0$) and then writing δ instead of d.

Let there now be given an integral

$$I = \int_{t_0}^{t_1} T \, dt \tag{29}$$

where T denotes the function (27).

The *variation of the integral* (29) for a motion defined by (26) is given by the expression

$$\delta I = \int_{t_0}^{t_1} \delta T \, dt. \tag{30}$$

Therefore

$$\delta \int_{t_0}^{t_1} T \, dt = \int_{t_0}^{t_1} \delta T \, dt. \tag{VI}$$

Example 2. Determine the variation of the kinetic energy

$$E = \tfrac{1}{2}\sum_{i=1}^{n} m_i(x_i^{\cdot 2} + y_i^{\cdot 2} + z_i^{\cdot 2}).$$

We have:

$$\frac{\partial E}{\partial x_i} = 0, \ \frac{\partial E}{\partial y_i} = 0, \ \frac{\partial E}{\partial z_i} = 0, \ \frac{\partial E}{\partial x_i^{\cdot}} = m_i x_i^{\cdot}, \ \frac{\partial E}{\partial y_i^{\cdot}} = m_i y_i^{\cdot}, \ \frac{\partial E}{\partial z_i^{\cdot}} = m_i z_i^{\cdot},$$

consequently

$$\delta E = \Sigma m_i(x_i^{\cdot}\, \delta x_i^{\cdot} + y_i^{\cdot}\, \delta y_i^{\cdot} + z_i^{\cdot}\, \delta z_i^{\cdot}).$$

Example 3. Determine the variation of the function $\sqrt{T}$, where T is a function defined by formula (27).

Forming the differential under the assumption that $t = \text{const}$, we have

$$d\sqrt{T} = \tfrac{1}{2} dT / \sqrt{T}, \quad \text{whence} \quad \delta\sqrt{T} = \tfrac{1}{2}\delta T / \sqrt{T}.$$

Let us assume that the natural coordinates $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$, are defined in terms of the parameters $q_1, \ldots, q_k$, by means of the functions:

$$x_i = f_i(q_1, \ldots, q_k, t), \ y_i = \varphi_i(q_1, \ldots, q_k, t), \ z_i = \psi_i(q_1, \ldots, q_k, t) \quad (i = 1, 2, \ldots, n). \tag{31}$$

We do not assume that the parameters are dependent nor that they are independent. Let

$$q_1 = q_1(t), \quad \ldots, \quad q_k = q_k(t) \qquad (t_0 \leqq t \leqq t_1) \tag{32}$$

be arbitrary functions continuous together with their first and second derivatives in the interval $\langle t_0, t_1 \rangle$. The functions (32) together with (33) define a certain motion of the system which may be compatible with the constraints or not. Differentiating (31), we obtain

$$x_i^{\cdot} = \frac{\partial x_i}{\partial q_1} q_1^{\cdot} + \ldots + \frac{\partial x_i}{\partial q_k} q_k^{\cdot} + \frac{\partial x_i}{\partial t}, \tag{33}$$

and similar formulae for $y_i^{\cdot}$ and $z_i^{\cdot}$. Let us substitute functions (31) for x_i, y_i, z_i, in (27), and functions (33) for $x_i^{\cdot}, y_i^{\cdot}, z_i^{\cdot}$. We obtain T in the form of a function of the variables $q_1, \ldots, q_k, q_1^{\cdot}, \ldots, q_k^{\cdot}, t$:

$$T = \mathbf{F}(q_1, \ldots, q_k, q_1^{\cdot}, \ldots, q_k^{\cdot}, t). \tag{34}$$

Proceeding as in the proof of the theorem on the variation of a compound function (*vide* formula (25), p. 508), it can be shown that the variation of function (34) is also expressed byf ormula (28) or (IV), where $\delta x_i, \delta y_i, \delta z_i$, are the variations of the functions (31), and $\delta x_i^{\cdot}, \delta y_i^{\cdot}, \delta z_i^{\cdot}$, the variations of functions (33) and of analogous ones for $y_i^{\cdot}, z_i^{\cdot}$.

Moreover, as in the proof of the theorem on the variation of a derivative (*vide* formula (16), p. 507) it can be proved that formulae (V), in which x_i, y_i, z_i, denote functions (31), will hold.

Let us form the variations of functions (31). We obtain:

$$\delta x_i = \frac{\partial x_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial x_i}{\partial q_k}\delta q_k, \qquad \delta y_i = \frac{\partial y_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial y_i}{\partial q_k}\delta q_k,$$
$$\delta z_i = \frac{\partial z_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial z_i}{\partial q_1}\delta q_k. \tag{35}$$

Comparing (35) with formulae (III), p. 473, defining the virtual displacements, we see that they have the same formal appearance.

§ 2. Hamilton's principle. Actual motion. Let the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, act on the points $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$, of a system of n material points. Let us assume that the system is holonomic (without friction) and that the constraints are bilateral, defined by the relations:

$$F_j(x_1, y_1, z_1, \ldots, z_n, t) = 0 \qquad (j = 1, 2, \ldots, m). \quad (1)$$

Let us consider an arbitrary system of functions

$$x_i = x_i(t), \quad y_i = y_i(t), \quad z_i = z_i(t) \qquad (i = 1, 2, \ldots, n) \quad (2)$$

continuous together with their first and second derivatives in the interval $\langle t_0, t_1 \rangle$. Functions (2) define a certain motion of the system.

If equations (1) are satisfied at each moment t when functions (2) are substituted for $x_1, \ldots, z_n$, then we say that functions (2) define the *motion* of the system *compatible with the constraints* or a *possible motion*.

The motion of the system which will actually take place under the action of the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, is called the *actual motion*.

There can be a variety of actual motions, because this depends on the initial conditions.

Obviously an actual motion is always possible, because it satisfies relations (1). Conversely, however, not every possible motion is an actual motion.

For example, if a heavy point is constrained to remain constantly on a vertical line l (without friction), then the actual motion is a motion in which the acceleration is directed vertically downwards and equal in magnitude to the gravitational acceleration. On the other hand, a motion compatible with the constraints is every motion in which the point remains on the line l, in particular, a uniform motion as well as a motion in which the acceleration is not constant; these motions are obviously not actual motions.

From d'Alembert's principle it follows that among motions compatible with the constraints, only that motion is actual which satisfies at every moment the equation (II′), p. 475:

$$\sum_{i=1}^{n} [(P_{i_x} - m_i \ddot{x}_i)\, \delta x_i + (P_{i_y} - m_i \ddot{y}_i)\, \delta y_i + (P_{i_z} - m_i \ddot{z}_i)\, \delta z_i] = 0, \quad (3)$$

where δx_i, δy_i, δz_i, are virtual displacements.

D'Alembert's principle therefore expresses a *characteristic property of actual motions*, distinguishing them from all motions compatible with the constraints.

Similarly, the equations of Lagrange (p. 481 and 486) and of Hamilton (p. 499) distinguish the actual motions from the set of all possible motions compatible with the constraints. However, in this chapter we shall meet

with still other characteristic properties of actual motions expressed by means of integrals and variations. They are the so-called *integral variational principles*.

Comparative motion. Let us consider an arbitrary motion of a system compatible with the constraints, defined by functions (2), as well as the comparative motion:

$$x_i + \delta x_i, \ y_i + \delta y_i, \ z_i + \delta z_i \qquad (i = 1, 2, \ldots, n). \quad (4)$$

Let us choose the variations $\delta x_i, \delta y_i, \delta z_i$, so that the variations of the functions (1) for the given motion (2) are zero:

$$\delta F_j = \frac{\partial F_j}{\partial x_1} \delta x_1 + \ldots + \frac{\partial F_j}{\partial z_n} \delta z_n = 0 \qquad (j = 1, 2, \ldots, m). \quad (5)$$

Comparing equations (5) with equations (I), p. 471, we see that δx_i, δy_i, δz_i, are at each moment the virtual displacements of the system.

If $\delta x_i, \delta y_i, \delta z_i$, are very small, then from (5) it follows that approximately

$$F_j(x_1 + \delta x_1, \ldots, z_n + \delta z_n, t) = 0 \qquad (j = 1, 2, \ldots, m), \quad (6)$$

i.e. that the comparative motion is approximately a motion compatible with the constraints. We express this by saying that the comparative motion (4) is compatible with the constraints for "infinitesimal" variations of δx_i, δy_i, δz_i, satisfying equations (5) (cf. p. 428).

Let us assume that the natural coordinates are defined in terms of the parameters $q_1, \ldots, q_k$, by the functions:

$$x_i = f_i(q_1, \ldots, q_k, t), \ y_i = \varphi_i(q_1, \ldots, q_k, t), \ z_i = \psi_i(q_1, \ldots, q_k, t) \quad (7)$$
$$(i = 1, 2, \ldots, n).$$

Let us further assume that the parameters defining the position of the system compatible with the constraints must satisfy the relations:

$$\Phi_r(q_1, \ldots, q_k, t) = 0 \qquad (r = 1, 2, \ldots, s). \quad (8)$$

Let us consider an arbitrary system of functions continuous together with their first and second derivatives in the interval $\langle t_0, t_1 \rangle$:

$$q_1 = q_1(t), \ldots, q_k = q_k(t) \qquad (t_0 \leqq t \leqq t_1). \quad (9)$$

Let us assume, finally, that functions (8) become identically equal to zero when functions (9) are substituted for $q_1, \ldots, q_k$.

Under these assumptions, substituting functions (9) in functions (7), we obtain functions of the time t defining a motion compatible with the constraints.

Let us consider a comparative motion $q_1 + \delta q_1, \ldots, q_k + \delta q_k$ and choose $\delta q_1, \ldots, \delta q_k$, such that for the given motion (9) the variations of the functions (8) are equal to zero:

$$\delta \Phi_r = \frac{\partial \Phi_r}{\partial q_1}\, \delta q_1 + \ldots + \frac{\partial \Phi_r}{\partial q_k}\, \delta q_k = 0 \qquad (r = 1, 2, \ldots, s). \tag{10}$$

Comparing (10) with formulae (IV), p. 473, we see that the variations $\delta q_1, \ldots, \delta q_k$, are virtual displacements at every moment t.

If $\delta q_1, \ldots, \delta q_k$, are very small, then by (10) we obtain approximately

$$\Phi_r(q_1 + \delta q_1, \ldots, q_k + \delta q_k, t) = 0 \qquad (r = 1, 2, \ldots, s). \tag{11}$$

Hence motion (11) will also be approximately compatible with the constraints. Using the same kind of expression as on p. 513, we can therefore say that if $\delta q_1, \ldots, \delta q_k$, are "infinitesimal" and satisfy equations (10), then the comparative motion is compatible with the constraints.

Hamilton's principle for natural coordinates. Let a system of n material points $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$, of masses $m_1, \ldots, m_n$, be acted on by the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, depending on the variables $x_1, \ldots, z_n, \dot{x}_1, \ldots, \dot{z}_n, t$.

Therefore:

$$P_{i_x} = F_i(x_1, \ldots, z_n, \dot{x}_1, \ldots, \dot{z}_n, t), \quad P_{i_y} = \Phi_i, \quad P_{i_z} = \Psi_i. \tag{12}$$

Let us assume that the system is holonomic (without friction) and that the constraints are bilateral. Let us consider the arbitrary functions:

$$x_i = x_i(t), \quad y_i = y_i(t), \quad z_i = z_i(t) \qquad (t_0 \leqq t \leqq t_1), \tag{13}$$

defining the motion of a system compatible with the constraints.

The kinetic energy of motion (13) is

$$E = \tfrac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2). \tag{14}$$

Let us form the variation of the kinetic energy for the motion (13) (cf. example 2, p. 510):

$$\delta E = \sum_{i=1}^{n} m_i(\dot{x}_i\, \delta \dot{x}_i + \dot{y}_i\, \delta \dot{y}_i + \dot{z}_i\, \delta \dot{z}_i). \tag{15}$$

But

$$\dot{x}_i\, \delta \dot{x}_i = \dot{x}_i \frac{\mathrm{d}(\delta x_i)}{\mathrm{d}t} = \frac{\mathrm{d}(\dot{x}_i\, \delta x_i)}{\mathrm{d}t} - \ddot{x}_i\, \delta x_i \tag{16}$$

and similar formulae hold for $\dot{y}_i\, \delta \dot{y}_i$ and for $\dot{z}_i\, \delta \dot{z}_i$. Substituting these values in (15), we therefore get

$$\delta E = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} m_i(\dot{x}_i\, \delta x_i + \dot{y}_i\, \delta y_i + \dot{z}_i\, \delta z_i) - \sum_{i=1}^{n} m_i(\ddot{x}_i\, \delta x_i + \ddot{y}_i\, \delta y_i + \ddot{z}_i\, \delta z_i). \tag{17}$$

Let us take

$$\delta' L = \sum_{i=1}^{n}(P_{i_x}\, \delta x_i + P_{i_y}\, \delta y_i + P_{i_z}\, \delta z_i), \tag{18}$$

where $P_{i_x}, P_{i_y}, P_{i_z}$, denote functions (12), in which for x_i, y_i, z_i, and $\dot{x}_i, \dot{y}_i, \dot{z}_i$, the corresponding functions (13) and their derivatives have been substituted. From formulae (17) and (18) we obtain

$$\delta' L + \delta E = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} m_i(\dot{x}_i\, \delta x_i + \dot{y}_i\, \delta y_i + \dot{z}_i\, \delta z_i) +$$
$$+ \sum_{i=1}^{n}[(P_{i_x} - m_i\ddot{x}_i)\, \delta x_i + (P_{i_y} - m_i\ddot{y}_i)\, \delta y_i + (P_{i_z} - m_i\ddot{z}_i)\, \delta z_i]. \tag{19}$$

Integrating both sides from t_0 to t_1, we obtain

$$\int_{t_0}^{t_1}[\delta' L + \delta E]\, \mathrm{d}t = \sum_{i=1}^{n} m_i(\dot{x}_i\, \delta x_i + \dot{y}_i\, \delta y_i + \dot{z}_i\, \delta z_i)\Big|_{t_0}^{t_1} +$$
$$+ \int_{t_0}^{t_1}\sum_{i=1}^{n}[(P_{i_x} - m_i\ddot{x}_i)\, \delta x_i + (P_{i_y} - m_i\ddot{y}_i)\, \delta y_i + (P_{i_z} - m_i\ddot{z}_i)\, \delta z_i]\, \mathrm{d}t. \tag{20}$$

The symbol $\big|_{t_0}^{t_1}$ here means, as usual, that at first t_1 and then t_0 are to be substituted for t, and the resulting values subtracted from each other.

So far we have not used any principles of mechanics. Formula (20) therefore holds for an arbitrary motion (compatible with the constraints or not) defined by functions (13) if functions (12) are defined for this motion.

Let us now assume that motion (13) is an actual motion and that the variations $\delta x_i, \delta y_i, \delta z_i$, are virtual motions at every moment t.

Then from d'Alembert's principle it follows that at every moment t

$$\sum_{i=1}^{n}[(P_{i_x} - m_i\ddot{x}_i)\, \delta x_i + (P_{i_y} - m_i\ddot{y}_i)\, \delta y_i + (P_{i_z} - m_i\ddot{z}_i)\, \delta z_i] = 0. \tag{21}$$

From formula (20) we obtain

$$\int_{t_0}^{t_1}(\delta' L + \delta E)\, \mathrm{d}t = \sum_{i=1}^{n} m_i(\dot{x}_i\, \delta x_i + \dot{y}_i\, \delta y_i + \dot{z}_i\, \delta z_i)\Big|_{t_0}^{t_1}. \tag{22}$$

Let us assume, in addition, that the system is at the same position at t_0 and t_1 in motion (13) and in the comparative motion; i. e. that $\delta x_i, \delta y_i, \delta z_i$ are zero for $t = t_0$ and $t = t_1$.

Under this assumption the right side of the equality (22) becomes zero and we obtain

$$\int_{t_0}^{t_1}(\delta' L + \delta E)\, \mathrm{d}t = 0. \tag{I}$$

Therefore: *equality* (I) *holds for an actual motion if* δx_i, δy_i, δz_i, *are virtual displacements at every moment and if they are zero for* $t = t_0$ *and* $t = t_1$.

This theorem is known as *Hamilton's principle.*

Hamilton's principle therefore gives a certain property of actual motions. We shall prove that *this property is characteristic,* i. e. that among the motions compatible with the constraints only actual motions satisfy Hamilton's principle. With this in view it is sufficient to prove that a motion compatible with the constraints and satisfying Hamilton's principle also satisfies d'Alembert's principle.

Proof. Let us assume that a motion defined by functions (13) and compatible with the constraints satisfies Hamilton's principle (I). If this did not satisfy d'Alembert's principle at a certain moment t' (where $t_0 < t' < t_1$), then it would be possible to find numbers $\overline{\delta x_i}$, $\overline{\delta y_i}$, $\overline{\delta z_i}$, defining the virtual displacement of the system at the moment t such that

$$\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\overline{\delta x_i} + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\overline{\delta y_i} + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\overline{\delta z_i}] \neq 0 \tag{23}$$

for $t = t'$.

Let us choose the variations $\delta' x_i$, $\delta' y_i$, $\delta' z_i$, such that they define the virtual displacement of the system at each moment t and such that at the moment t'

$$\delta' x_i = \overline{\delta x_i}, \quad \delta' y_i = \overline{\delta y_i}, \quad \delta' z_i = \overline{\delta z_i};$$

from this by (23)

$$\sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta' x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta' y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta' z_i] =$$
$$= A \neq 0 \tag{24}$$

for $t = t'$.

Let us suppose, for instance, that $A > 0$ for $t = t'$. From the continuity of the motion it follows that in a certain small interval $\langle t', t'' \rangle$ A is also greater than zero.

Consequently

$$A > 0, \quad \text{when} \quad t' \leqq t \leqq t''. \tag{25}$$

Let $\alpha(t)$ be an arbitrary function continuous together with its first and second derivatives in the interval $\langle t_0, t_1 \rangle$, positive for $t' < t < t''$ and zero outside of this interval. Let us put:

$$\delta x_i = \alpha(t)\,\delta' x_i, \quad \delta y_i = \alpha(t)\,\delta' y_i, \quad \delta z_i = \alpha(t)\,\delta' z_i.$$

From this by (24) and (25)

$$\int_{t_0}^{t_1} \sum_{i=1}^{n}[(P_{i_x} - m_i x_i^{\cdot\cdot})\,\delta x_i + (P_{i_y} - m_i y_i^{\cdot\cdot})\,\delta y_i + (P_{i_z} - m_i z_i^{\cdot\cdot})\,\delta z_i]\,\mathrm{d}t =$$
$$= \int_{t_0}^{t_1} A\,\alpha(t)\,\mathrm{d}t = \int_{t'}^{t''} A\,\alpha(t)\,\mathrm{d}t > 0. \tag{26}$$

Since the variations δx_i, δy_i, δz_i, represent the virtual displacements at every moment t and by assumption are equal to zero at $t = t_0$ and $t = t_1$, from formula (20) we therefore obtain by (26)

$$\int_{t_0}^{t_1} (\delta' L + \delta E)\,\mathrm{d}t > 0,$$

contrary to the assumption that the given motion satisfies Hamilton's principle.

In this manner we have proved that *the principles of d'Alembert and of Hamilton are equivalent.*

Example. A heavy point of mass m is constrained to remain on the sphere $x^2 + y^2 + z^2 - r^2 = 0$. We have (taking the z-axis directed vertically upwards):

$$\delta' L = -mg\,\delta z, \quad \delta E = m(x^{\cdot}\,\delta x^{\cdot} + y^{\cdot}\,\delta y^{\cdot} + z^{\cdot}\,\delta z^{\cdot});$$

consequently by Hamilton's principle (I), p. 516,

$$\int_{t_0}^{t_1} [-g\,\delta z + x^{\cdot}\,\delta x^{\cdot} + y^{\cdot}\,\delta y^{\cdot} + z^{\cdot}\,\delta z^{\cdot}]\,\mathrm{d}t = 0.$$

This formula holds for an actual motion under the assumption that δx, δy, δz, are virtual displacements at every moment, i. e. that they satisfy the equation

$$x\,\delta x + y\,\delta y + z\,\delta z = 0,$$

and, in addition, become zero at $t = t_0$ and $t = t_1$.

Hamilton's principle for generalized coordinates. Let the natural coordinates be defined as functions of the parameters $q_1, \ldots, q_k$:

$$x_i = f_i(q_1, \ldots, q_k, t), \;\; y_i = \varphi_i(q_1, \ldots, q_k, t), \;\; z_i = \psi_i(q_1, \ldots, q_k, t) \tag{27}$$
$$(i = 1, 2, \ldots, n).$$

Let us assume that the parameters defining the positions of the system compatible with the constraints satisfy the equations

$$\Phi_r(q_1, \ldots, q_k, t) = 0 \qquad (r = 1, 2, \ldots, s) \tag{28}$$

and let us consider an arbitrary actual motion of the system defined by the functions:

$$q_i = q_i(t) \qquad (i = 1, 2, \ldots, k). \tag{29}$$

The variations of the functions (27) for this motion are

$$\delta x_i = \frac{\partial x_i}{\partial q_1}\,\delta q_1 + \ldots + \frac{\partial x_i}{\partial q_k}\,\delta q_k \tag{30}$$

and similarly for δy_i and δz_i.

Let us further assume that δq_i are virtual displacements at every moment t, i. e. that they satisfy equations (IV), p. 473:

$$\frac{\partial \Phi_r}{\partial q_1}\,\delta q_1 + \ldots + \frac{\partial \Phi_r}{\partial q_k}\,\delta q_k = 0 \qquad (r = 1, 2, \ldots, s); \tag{31}$$

consequently δx_i, δy_i, δz_i, are also virtual displacements (p. 473).

Finally, let us assume that δq_i are zero for $t = t_0$ and $t = t_1$; from (30) it follows that δx_i, δy_i, δz_i, will also be zero for $t = t_0$ and $t = t_1$. Since the variation is interchangeable with the derivative (p. 507), the variations of the first derivatives of the functions (27), i. e. $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, are equal to the derivatives of the functions δx_i, δy_i, δz_i.

In virtue of (15), p. 514, and (18), p. 515, we can write Hamilton's principle (I), p. 515, in the form:

$$\int_{t_0}^{t_1}\Big[\sum_{i=1}^{n}(P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i) + \sum_{i=1}^{n} m_i(x_i'\,\delta x_i' + y_i'\,\delta y_i' + z_i'\,\delta z_i')\Big]\,\mathrm{d}t = 0, \tag{32}$$

where x_i, y_i, z_i, are functions defining the actual motion; δx_i, δy_i, δz_i, are the virtual displacements at every moment, assuming the value zero at $t = t_0$ and $t = t_1$; and $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, are derivatives of the functions δx_i, δy_i, δz_i. As follows from the considerations of example 3, p. 510, equation (32) will also be satisfied if we assume that the functions x_i, y_i, z_i, given by the formulae (27) and (29), define the actual motion, while δx_i and $\delta x_i'$ are the variations of the functions (27) and their derivatives, where δq_i are virtual displacements equal to zero for $t = t_0$ and $t = t_1$.

Under these assumptions we have ((4), p. 483)

$$\delta' L = \sum_{i=1}^{n}(P_{i_x}\,\delta x_i + P_{i_y}\,\delta y_i + P_{i_z}\,\delta z_i) = \sum_{i=1}^{n} Q_j\,\delta q_j, \tag{33}$$

where Q_j denote the components of the generalized force. Moreover, from the theorem on the variation of a compound function (p. 508) we have

$$\sum_{i=1}^{n} m_i(x_i'\,\delta x_i' + y_i'\,\delta y_i' + z_i'\,\delta z_i') = \delta\Big(\tfrac{1}{2}\sum_{i=1}^{n} m_i(x_i'^2 + y_i'^2 + z_i'^2)\Big) = \delta E, \tag{34}$$

where the functions x_i, y_i, z_i, are given by formulae (27) and (29) defining the actual motion, $\delta x_i'$, $\delta y_i'$, $\delta z_i'$ are the variations of the derivatives of the functions (27), and E the kinetic energy expressed in terms of the

parameters $q_1, \ldots, q_k$. By (32), (33) and (34) Hamilton's principle can therefore be written in the form (I), p. 515, where $\delta' L$ is defined by formula (33) and the kinetic energy E is expressed in terms of the generalized coordinates $q_1, \ldots, q_k$.

Therefore: *Hamilton's principle also holds for generalized coordinates under the assumption that δq_i are virtual displacements equal to zero for $t = t_0$ and $t = t_1$.*

Hamilton's principle in a potential field. Let us assume that a system of forces has a potential V. Consequently ((1), p. 434)

$$\delta' L = \delta V. \tag{35}$$

From Hamilton's principle we therefore obtain

$$\int_{t_0}^{t_1}[\delta V + \delta E]\,\mathrm{d}t = 0 \quad \text{or} \quad \int_{t_0}^{t_1}\delta(V + E)\,\mathrm{d}t = \delta\int_{t_0}^{t_1}(V + E)\,\mathrm{d}t = 0.$$

The expression $W = E + V$ was called the kinetic potential (p. 488). Hence

$$\delta\int_{t_0}^{t_1} W\,\mathrm{d}t = 0. \tag{II}$$

Therefore: *the variation of the integral of the kinetic potential is equal to zero for an actual motion if the variations δx_i, δy_i, δz_i, represent the virtual displacement of the system at every moment and if they are equal to zero at $t = t_0$ and $t = t_1$.*

Formula (35) holds for generalized coordinates (cf. (39), 463). Since Hamilton's principle also holds for generalized coordinates, (II) is satisfied for an actual motion under the assumption that the kinetic potential W is expressed in terms of the parameters $q_1, \ldots, q_k$, and the variation was formed for an actual motion, where δq_i are virtual displacements equal to zero at $t = t_0$ and $t = t_1$.

Holonomo-scleronomic systems in a potential field. Let a holonomo-scleronomic system be given in which the forces have a potential

$$V = V(x_1, \ldots, z_n). \tag{36}$$

Let us assume that the motion of a system defined by the functions:

$$x_i = x_i(t),\ y_i = y_i(t),\ z_i = z_i(t) \quad (t_0 \leq t \leq t_1;\ i = 1, 2, \ldots, n) \tag{37}$$

is an actual motion for which the kinetic energy in $\langle t_0, t_1\rangle$ does not vanish, i. e.:

$$E \neq 0 \quad \text{for} \quad t_0 \leq t \leq t_1. \tag{38}$$

By Hamilton's principle (I), p. 515, and (35) we have

$$\int_{t_0}^{t_1}\left[\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\,\delta x_i+\frac{\partial V}{\partial y_i}\,\delta y_i+\frac{\partial V}{\partial z_i}\,\delta z_i\right)\right]\mathrm{d}t+$$
$$+\int_{t_0}^{t_1}\left[\sum_{i=1}^{n}m_i(\dot{x}_i\,\delta\dot{x}_i+\dot{y}_i\,\delta\dot{y}_i+\dot{z}_i\,\delta\dot{z}_i)\right]\mathrm{d}t=0, \tag{39}$$

where δx_i, δy_i, δz_i, are the virtual displacements at every moment and are equal to zero at $t = t_0$ and $t = t_1$. Let

$$t=\vartheta(\tau) \qquad (\tau_0\leqq\tau\leqq\tau_1) \tag{40}$$

be an arbitrary function of the variable τ, continuous together with its first and second derivatives in the interval $\langle\tau_0,\tau_1\rangle$ and satisfying the conditions:

$$\vartheta(\tau_0)=t_0,\quad \vartheta(\tau_1)=t_1,\quad \vartheta'(\tau)<0 \qquad (\tau_0\leqq\tau\leqq\tau_1), \tag{41}$$

where ϑ' denotes the derivative with respect to τ. Substituting (40) in (37), we obtain:

$$x_i=x_i(\vartheta(\tau))=\xi_i(\tau),\quad y_i=\eta_i(\tau),\quad z_i=\zeta_i(\tau), \tag{42}$$
$$(\tau_0\leqq\tau\leqq\tau_i;\ i=1,2,\ldots,n).$$

Functions (42) represent the parametric equations of the path of the points of the system in terms of the parameter τ.

Denoting by x_i', y_i', z_i', the derivatives of the functions (42), and by t' the derivative of function (40) with respect to the parameter τ, we obtain:

$$\dot{x}_i=x_i'\,/\,t',\quad \dot{y}_i=y_i'\,/\,t',\quad \dot{z}_i=z_i'\,/\,t', \tag{43}$$

whence for the kinetic energy

$$E=\tfrac{1}{2}\sum_{i=1}^{n}m_i(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2)=\tfrac{1}{2}\sum_{i=1}^{n}\frac{m_i(x_i'^2+y_i'^2+z_i'^2)}{t'^2}.$$

From the principle of conservation of total energy we have

$$E-V=h,\quad \text{where}\quad h=\text{const}, \tag{45}$$

whence by (44)

$$t'=\sqrt{\frac{\tfrac{1}{2}\sum_{i=1}^{n}m_i(x_i'^2+y_i'^2+z_i'^2)}{h+V}}. \tag{46}$$

Formula (46) holds, because by (45) $h + V = E$ and by (38) $E \neq 0$.

Let us assume that the coordinates x_i, y_i, z_i, appearing in V (cf. formula (36)) are expressed in formula (46) by functions (42). In virtue of (40) the variations δx_i, δy_i, δz_i, which are functions of the variable t, can be regarded as functions of the variable τ.

Denoting by $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, the derivatives with respect to τ, we obtain:

$$\delta\dot{x}_i=\delta x_i'\,/\,t',\quad \delta\dot{y}_i=\delta y_i'\,/\,t',\quad \delta\dot{z}_i=\delta z_i'\,/\,t'. \tag{47}$$

Expressing the variable t in (39) in terms of τ by means of function (40), we get by (41), (43), and (47),

$$\int_{\tau_0}^{\tau_1}\left[\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\,\delta x_i+\frac{\partial V}{\partial y_i}\,\delta y_i+\frac{\partial V}{\partial z_i}\,\delta z_i\right)\right]t'\mathrm{d}\tau+$$
$$+\int_{\tau_0}^{\tau_1}\left[\sum_{i=1}^{n}\frac{m_i(x_i'\,\delta x_i'+y_i'\,\delta y_i'+z_i'\,\delta z_i')}{t'^2}\right]t'\,\mathrm{d}\tau=0, \tag{48}$$

where x_i, y_i, z_i, δx_i, δy_i, δz_i, are functions of the variable τ. Formula (48) can be written in the form

$$\int_{\tau_0}^{\tau_1}\left[\delta V\cdot t'+\frac{1}{t'}\,\delta\,\tfrac{1}{2}\sum_{i=1}^{n}m_i(x_i'^2+y_i'^2+z_i'^2)\right]\mathrm{d}\tau=0, \tag{49}$$

where the concept of variation is to be understood as before, but now τ appears instead of t. Substituting (46) in (49), we obtain

$$\int_{\tau_0}^{\tau_1}\left[\frac{\delta V}{\sqrt{h+V}}\sqrt{\tfrac{1}{2}\Sigma m_i(x_i'^2+y_i'^2+z_i'^2)}\right]\mathrm{d}\tau+$$
$$\int_{\tau_0}^{\tau_1}\left[+\sqrt{h+V}\cdot\frac{\delta\Sigma\frac{1}{2}m_i(x_i'^2+y_i'^2+z_i'^2)}{\sqrt{\Sigma\frac{1}{2}m_i(x_i'^2+y_i'^2+z_i'^2)}}\right]\mathrm{d}\tau=0. \tag{50}$$

It is easy to verify that the integrand is equal to the variation of

$$2\delta\left[\sqrt{h+V}\sqrt{\tfrac{1}{2}\sum_{i=1}^{n}m_i(x_i'^2+y_i'^2+z_i'^2)}\right],$$

whence by (50)

$$\delta\int_{\tau_0}^{\tau_1}\left[\sqrt{h+V}\sqrt{\sum_{i=1}^{n}m_i(x_i'^2+y_i'^2+z_i'^2)}\right]\mathrm{d}\tau=0. \tag{I}$$

From the assumptions concerning δx_i, δy_i, δz_i, it follows that formula (I) holds for arbitrary functions δx_i, δy_i, δz_i, of the parameter τ, which for every value τ are the virtual displacements in the position of the system defined by functions (42), and which are zero for $\tau=\tau_0$ and $\tau=\tau_1$. The variation is formed for the functions (42), which define the path of the points of the system in an actual motion.

Since the time t does not appear in formula (I), this formula expresses a certain property of the paths of the actual motion.

In particular, let the system consist of one point x, y, z, of mass m, moving without the action of forces. Consequently $V = 0$. From (I) we obtain

$$\delta\int_{\tau_0}^{\tau_1}\sqrt{x'^2 + y'^2 + z'^2}\,\mathrm{d}\tau = 0.$$

Since the differential of arc is $\mathrm{d}s = \sqrt{x'^2 + y'^2 + z'^2}\,\mathrm{d}\tau$,

$$\delta\int_{\tau_0}^{\tau_1}\mathrm{d}s = 0. \tag{51}$$

Let us assume that the point is constrained to the surface S.

Curves having property (51) are the so-called *geodesics*. In differential geometry it is proved that the shortest curve on a surface joining two sufficiently close points of the geodesic is an arc of this geodesic. From (51) it follows, therefore, that *the motion of a point on a surface without the action of forces always takes place along geodesics.*

Remark. Formula (I) is usually proved more easily from the so-called *principle of Maupertuis* (p. 530). However, it is then necessary to make use of certain theorems from the calculus of variations which we do not assume to be known to the reader.

§ 3. Variation with the variation of time. Variation of a function. Let a motion along the x-axis be defined by the function

$$x = x(t) \qquad (t_0 \leqq t \leqq t_1). \tag{1}$$

Let us consider an arbitrary comparative motion

$$\mathbf{x} = \mathbf{x}(t) \qquad (t_0' \leqq t \leqq t_1'), \tag{2}$$

where the moments t_1' and t_0' can be different from t_0 and t_1. Let us denote by Δt and arbitrary function of the time t, continuous together with its first and second derivatives in the interval $\langle t_0, t_1\rangle$ and satisfying the inequality

$$t_0' \leqq t + \Delta t \leqq t_1' \qquad (t_0 \leqq t \leqq t_1). \tag{3}$$

Finally, let

$$\Delta x = \mathbf{x}(t + \Delta t) - x(t); \tag{I}$$

Δx is therefore a function of the time t and denotes the increment of the coordinate x, when we pass from the point A in the given motion at the moment t to the point A' in thecomparative motion at the moment $t + \Delta t$. Forming the derivative of (I), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Delta x) = [\mathbf{x}^{\cdot}(t + \Delta t)]\left(1 + \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right) - x^{\cdot}(t),$$

whence

$$\mathbf{x}^{\cdot}(t + \Delta t) = \left[\frac{\mathrm{d}(\Delta x)}{\mathrm{d}t} + x^{\cdot}(t)\right]\Big/\left(1 + \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right).$$

Subtracting $x^{\cdot}(t)$ from both sides, we get after some easy transformations

$$\mathbf{x}^{\cdot}(t + \Delta t) - x^{\cdot}(t) = \frac{\mathrm{d}(\Delta x)}{\mathrm{d}t} - x^{\cdot}(t)\,\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t} + \varepsilon, \tag{4}$$

where

$$\varepsilon = \eta\,\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}, \qquad \eta = -\left(\frac{\mathrm{d}(\Delta x)}{\mathrm{d}t} - x^{\cdot}(t)\,\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right)\Big/\left(1 + \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right). \tag{5}$$

Therefore, if Δx, Δt, and their derivatives, tend to zero uniformly, then η tends to zero uniformly. Consequently R is "infinitesimal" in comparison with $|\mathrm{d}(\Delta x)\,/\,\mathrm{d}t| + |\mathrm{d}(\Delta t)\,/\,\mathrm{d}t|$.

Let

$$\Delta x^{\cdot} = \frac{\mathrm{d}(\Delta x)}{\mathrm{d}t} - x^{\cdot}\,\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}. \tag{II}$$

Therefore by (4)

$$\mathbf{x}^{\cdot}(t + \Delta t) - x^{\cdot}(t) = \Delta x^{\cdot} + \varepsilon. \tag{6}$$

The left side of equality (6) denotes the increment of the velocity of the points A and A', i. e. the increment of the velocity when we pass from the point A at the moment t in the given motion to the point A' at the moment $t + \Delta t$ in the comparative motion. Consequently $\Delta x^{\cdot}$ represents this increment approximately, with a difference which is "infinitesimal" as compared with the sum $|\mathrm{d}(\Delta x)\,/\,\mathrm{d}t| + |\mathrm{d}(\Delta t)\,/\,\mathrm{d}t|$.

Let us note that by (6) we have in general

$$\Delta x^{\cdot} \neq \frac{\mathrm{d}(\Delta x)}{\mathrm{d}t}. \tag{II$'$}$$

Let the function

$$T = F(x, x^{\cdot}, t) \tag{7}$$

be given.

Let us denote by T the value of the function (7) in a given motion at the moment t, and by $\mathbf{T}$ the value of this function in a comparative motion at the moment $t + \Delta t$. Consequently the difference of these values is

$$\mathbf{T} - T = F(\mathbf{x}(t + \Delta t), \mathbf{x}^{\cdot}(t + \Delta t), t + \Delta t) - F(x(t), x^{\cdot}(t), t). \tag{8}$$

From Taylor's formula we obtain

$$\mathbf{T}-T=\frac{\partial T}{\partial x}(\mathbf{x}(t+\Delta t)-x(t))$$
$$+\frac{\partial T}{\partial x^{\cdot}}(\mathbf{x}^{\cdot}(t+\Delta t)-x^{\cdot}(t))+\frac{\partial T}{\partial t}\Delta t+R, \tag{9}$$

where the remainder R is an "infinitesimal" of higher order than the increments $\mathbf{x}(t+\Delta t)-x(t)$, $\mathbf{x}^{\cdot}(t+\Delta t)-x^{\cdot}(t)$ and Δt.

In virtue of (I), (6), and (9), we obtain

$$\mathbf{T}-T=\frac{\partial T}{\partial x}\Delta x+\frac{\partial T}{\partial x^{\cdot}}\Delta x^{\cdot}+\frac{\partial T}{\partial t}\Delta t+R', \tag{10}$$

where

$$R'=R+\varepsilon\frac{\partial T}{\partial x^{\cdot}},$$

and hence where R' is "infinitesimal" as compared with

$$|\Delta x|+|\Delta t|+|\mathrm{d}(\Delta x)\,/\,\mathrm{d}t|+|\mathrm{d}(\Delta t)\,/\,\mathrm{d}t|. \tag{11}$$

Let us put

$$\Delta T=\frac{\partial T}{\partial x}\Delta x+\frac{\partial T}{\partial x^{\cdot}}\Delta x^{\cdot}+\frac{\partial T}{\partial t}\Delta t. \tag{III}$$

By (10) we have $\mathbf{T}-T=\Delta T+R'$; consequently ΔT denotes approximately the increment of the function T when we pass from the point A at the moment t in the given motion to the point A' at the moment $t+\Delta t$ in the comparative motion, where the error committed is "infinitesimal" as compared with (11).

The expression ΔT is called the *variation together with the variation of time* of the function T for the motion $x=x(t)$, and the function Δt is called the *variation of time*.

In virtue of (I), p. 505, and (5), p. 504, we shall have $\Delta x=\delta x$ for $\Delta t=0$, and by (II) and (I), p. 505, $\Delta x^{\cdot}=\delta x^{\cdot}$. From (III) we therefore get

$$\Delta T=\frac{\partial T}{\partial x}\delta x+\frac{\partial T}{\partial x^{\cdot}}\delta x^{\cdot},$$

i. e. $\Delta T=\delta T$.

Hence, when $\Delta t=0$, the variation together with the variation of time becomes an ordinary variation.

Example. Form the variation together with the variation of time for the function

$$T=\tfrac{1}{2}mx^{\cdot 2}+\alpha xt,$$

where α and m are constants.

We have

$$\Delta T=\alpha t\,\Delta x+mx^{\cdot}\,\Delta x^{\cdot}+\alpha x\,\Delta t;$$

hence by (II)

$$\Delta T=\alpha t\,\Delta x+mx^{\cdot}\frac{\mathrm{d}(\Delta x)}{\mathrm{d}t}+\alpha x\,\Delta t-mx^{\cdot 2}\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}.$$

Systems of points. Let the motion of a system of n material points be defined by the functions:

$$x_i=x_i(t),\ y_i=y_i(t),\ z_i=z_i(t),\ (t_0\leqq t\leqq t_1;\ i=1,2,\ldots,n) \tag{12}$$

and let the function

$$T=F(x_1,\ldots,z_n,x_1^{\cdot},\ldots,z_n^{\cdot},t) \tag{13}$$

be given.

Let us consider an arbitrary comparative motion

$$\mathbf{x}_i=\mathbf{x}_i(t),\ \mathbf{y}_i=\mathbf{y}_i(t),\ \mathbf{z}_i=\mathbf{z}_i(t),\ (t_0'\leqq t\leqq t_1';\ i=1,2,\ldots,n). \tag{14}$$

Let Δt be an arbitrary function of the time t, continuous with its first and second derivatives in the interval $\langle t_0,t_1\rangle$ and satisfying the condition

$$t_0'\leqq t+\Delta t\leqq t_1'\qquad (t_0\leqq t\leqq t_1). \tag{15}$$

Let us put:

$$\Delta x_i=\mathbf{x}_i(t+\Delta t)-x_i(t),\quad \Delta y_i=\mathbf{y}_i(t+\Delta t)-y_i(t),$$
$$\Delta z_i=\mathbf{z}_i(t+\Delta t)-z_i(t), \tag{IV}$$

$$\Delta x_i^{\cdot}=\frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t}-x_i^{\cdot}\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t},\quad \Delta y_i^{\cdot}=\frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t}-y_i^{\cdot}\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t},$$
$$\Delta z_i^{\cdot}=\frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}-z_i^{\cdot}\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}. \tag{V}$$

The expressions Δx_i, Δy_i, Δz_i, denote the increments of the coordinates, and $\Delta x_i^{\cdot}$, $\Delta y_i^{\cdot}$, $\Delta z_i^{\cdot}$, the approximate increments of the derivatives of these coordinates when we pass from the position of the system at the moment t in the given motion to the position of the system at the moment $t+\Delta t$ in the comparative motion. The error made by this approximation is "infinitesimal" as compared with

$$\left|\frac{\mathrm{d}(\Delta x_1)}{\mathrm{d}t}\right|+\left|\frac{\mathrm{d}(\Delta y_1)}{\mathrm{d}t}\right|+\ldots+\left|\frac{\mathrm{d}(\Delta z_n)}{\mathrm{d}t}\right|+\left|\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right|. \tag{16}$$

Let us denote by T the value of the function (13) in the given motion at the moment t, and by $\mathbf{T}$ its value at the moment $t+\Delta t$ in the comparative motion. Putting

$$\Delta T = \sum_{i=1}^{n}\left(\frac{\partial T}{\partial x_i}\Delta x_i + \frac{\partial T}{\partial y_i}\Delta y_i + \frac{\partial T}{\partial z_i}\Delta z_i\right) + $$
$$+ \sum_{i=1}^{n}\left(\frac{\partial T}{\partial \dot{x}_i}\Delta \dot{x}_i + \frac{\partial T}{\partial \dot{y}_i}\Delta \dot{y}_i + \frac{\partial T}{\partial \dot{z}_i}\Delta \dot{z}_i\right) + \frac{\partial T}{\partial t}\Delta t \qquad \text{(VI)}$$

and proceeding as before, we obtain

$$\mathbf{T} - T = \Delta T + R, \qquad (17)$$

where R is "infinitesimal" as compared with the sum

$$\sum_{i=1}^{n}\left(|\Delta x_i| + |\Delta y_i| + |\Delta z_i| + \left|\frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t}\right| + \left|\frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t}\right| + \left|\frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}\right|\right) +$$
$$+ |\Delta t| + \left|\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right|. \qquad (18)$$

The expression ΔT is called the *variation together with the variation of time* of the function T for the motion (12) of a system of points.

The variation ΔT therefore represents approximately the increment of the function T when we pass from the position of the system at the moment t in a given motion to the position of a system at the moment $t + \Delta t$ in a comparative motion, where the difference between the true increment and ΔT is "infinitesimal" as compared with (18).

In virtue of (IV), (V), (5), p. 504, and (I), p. 505, we get for $\Delta t = 0$:

$$\Delta x_i = \delta x_i,\ \Delta y_i = \delta y_i,\ \Delta z_i = \delta z_i,\ \Delta \dot{x}_i = \delta \dot{x}_i,\ \Delta \dot{y}_i = \delta \dot{y}_i,\ \Delta \dot{z}_i = \delta \dot{z}_i,$$

whence by (VI) $\Delta T = \delta T$.

Hence, when $\Delta t = 0$, the variation together with the variation of timo becomes an ordinary variation.

Variation together with the variation of time of an integral. Let the integral

$$I = \int_{t_0}^{t_1} T\,\mathrm{d}t \qquad (19)$$

be given, where T denotes the function (13). Let us denote by Δt_0 and Δt_1 the values of the function Δt at t_0 and t_1.

Let I be the value of the integral (19) for a given motion, and $\mathbf{I}$ the value of this integral for a comparative motion taken between the limits $t_0 + \Delta t_0$ and $t_1 + \Delta t_1$, i. e.

$$\mathbf{I} = \int_{t_0+\Delta t_0}^{t_1+\Delta t_1} T_1\,\mathrm{d}t, \qquad (20)$$

where T_1 denotes the value of the function T at the moment t in the comparative motion. Substituting $t + \Delta t$ for t in (20), we obtain

$$\mathbf{I} = \int_{t_0}^{t_1} \mathbf{T}\left(1 + \frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right)\mathrm{d}t, \qquad (21)$$

where $\mathbf{T}$ denotes the value of the function T at the moment $t + \Delta t$ in the comparative motion.

By (19) and (21) we obtain

$$\mathbf{I} - I = \int_{t_0}^{t_1}\left[(\mathbf{T} - T) + \mathbf{T}\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t,$$

whence by (17) after some easy transformations

$$\mathbf{I} - I = \int_{t_0}^{t_1}\left[\Delta T + T\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t + R', \qquad (22)$$

where

$$R' = \int_{t_0}^{t_1}\left[(\Delta T + R)\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t} + R\right]\mathrm{d}t. \qquad (23)$$

It is easy to verify that R' is "infinitesimal" in comparison with (18). Let us put

$$\Delta I = \Delta\int_{t_0}^{t_1} T\,\mathrm{d}t = \int_{t_0}^{t_1}\left[\Delta T + T\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t. \qquad \text{(VII)}$$

The expression ΔI is called the *variation together with the variation of time of the integral I*.

By (22) and (VII) we have

$$\mathbf{I} - I = \Delta I + R', \qquad (24)$$

ΔI therefore represents approximately the increment of the integral (19) when we pass from the given motion to the comparative motion; we calculate the integral between the limits $t_0 + \Delta t_0$, $t_1 + \Delta t_1$ in the comparative motion. The difference between ΔI and the true increment is "infinitesimal" in comparison with (18).

In the case when $\Delta t = 0$, the variation together with the variation of time becomes — as it is easily seen — an ordinary variation.

§ 4. Maupertuis' principle (of least action). Hölder's transformation. Let a system of material points $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$, of masses $m_1, \ldots, m_n$, be subjected to the action of the forces $\mathbf{P}_1, \ldots, \mathbf{P}_n$, depending on $x_1, \ldots, z_n, \dot{x}_1, \ldots, \dot{z}_n, t$. Therefore

$$P_{i_x} = F_i(x_1, \ldots, z_n, \dot{x}_1, \ldots, \dot{z}_n, t), \quad P_{i_y} = \Phi_i,\ P_{i_z} = \Psi_i. \qquad (1)$$

Let us assume that the system is holonomic without friction and that the constraints are bilateral.

Let us consider an arbitrary motion of the system compatible with the constraints or not, defined by the functions:

$$x_i = x_i(t),\ y_i = y_i(t),\ z_i = z_i(t)\ \ (t_0 \leqq t \leqq t_1;\ i = 1, 2, \ldots, n). \tag{2}$$

The kinetic energy is

$$E = \tfrac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2). \tag{3}$$

Let us form the variation of the kinetic energy together with the variation of time for the motion (2):

$$\Delta E = \sum_{i=1}^{n} m_i(\dot{x}_i\, \Delta\dot{x}_i + \dot{y}_i\, \Delta\dot{y}_i + \dot{z}_i\, \Delta\dot{z}_i). \tag{4}$$

Expressing $\Delta\dot{x}_i$, $\Delta\dot{y}_i$, $\Delta\dot{z}_i$, by means of formulae (V), p. 525, we get

$$\Delta E = \sum_{i=1}^{n} m_i\left[\dot{x}_i \frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} + \dot{y}_i \frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} + \dot{z}_i \frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}\right] - 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}. \tag{5}$$

Transposing the last term to the left and integrating, we obtain

$$\int_{t_0}^{t_1}\left[\Delta E + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t =$$

$$= \int_{t_0}^{t_1}\sum_{i=1}^{n} m_i\left[\dot{x}_i \frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} + \dot{y}_i \frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} + \dot{z}_i \frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}\right]\mathrm{d}t. \tag{6}$$

Integrating by parts, we obtain

$$\int_{t_0}^{t_1} \dot{x}_i \frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} = \dot{x}_i\, \Delta x_i\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \ddot{x}_i\, \Delta x_i\, \mathrm{d}t$$

and similar formulae are obtained for y_i and z_i. Applying them to the right side of equation (6), we get

$$\int_{t_0}^{t_1}\left[\Delta E + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t =$$

$$= \sum_{i=1}^{n} m_i(\dot{x}_i\, \Delta x_i + \dot{y}_i\, \Delta y_i + \dot{z}_i\, \Delta z_i)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\Big[\sum_{i=1}^{n} m_i(\ddot{x}_i\, \Delta x_i + \ddot{y}_i\, \Delta y_i + \ddot{z}_i\, \Delta z_i)\Big]\,\mathrm{d}t. \tag{7}$$

Let us put

$$\Delta' L = \sum_{i=1}^{n}(P_{i_x}\, \Delta x_i + P_{i_y}\, \Delta y_i + P_{i_z}\, \Delta z_i). \tag{8}$$

Integrating formula (8) and adding to both sides of equation (7), we get

$$\int_{t_0}^{t_1}\left[\Delta' L + \Delta E + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t = \sum_{i=1}^{n} m_i(\dot{x}_i\, \Delta x_i + \dot{y}_i\, \Delta y_i + \dot{z}_i\, \Delta z_i)\Big|_{t_0}^{t_1} +$$

$$+ \int_{t_0}^{t_1}\sum_{i=1}^{n}[(P_{i_x} - m_i\ddot{x}_i)\, \Delta x_i + (P_{i_y} - m_i\ddot{y}_i)\, \Delta y_i + (P_{i_z} - m_i\ddot{z}_i)\, \Delta z_i]\,\mathrm{d}t. \tag{I}$$

Formula (I) is called *Hölder's transformation.*

It holds for every motion, whether compatible with the constraints or not (on contition that the functions (1) are defined for this motion).

If we take the ordinary variation δ instead of the variation Δ together with the variation of time, i. e. if we put $\Delta t = 0$, then — as it is easily seen — we obtain formula (20), p. 515.

More general form of Hamilton's principle. Let us assume that functions (2) define an actual motion. In addition, let us assume that the functions $\Delta x_i, \Delta y_i, \Delta z_i$, are virtual displacements of the system at every moment t.

By d'Alembert's principle the integrand on the right side of (I) is zero. Consequently

$$\int_{t_0}^{t_1}\left[\Delta' L + \Delta E + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t = \sum_{i=1}^{n} m_i(\dot{x}_i\, \Delta x_i + \dot{y}_i\, \Delta y_i + \dot{z}_i\, \Delta z_i)\Big|_{t_0}^{t_1}.$$

Let us assume that Δx_i, Δy_i, Δz_i, are equal to zero at $t = t_0$ and $t = t_1$. The right side of the last equality will therefore be zero. Hence we obtain

$$\int_{t_0}^{t_1}\left[\Delta' L + \Delta E + 2E\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t = 0. \tag{II}$$

Equation (II) *holds for an actual motion under the assumption that Δx_i, Δy_i, Δz_i, are virtual displacements at every moment t and are equal to zero for $t = t_0$ and $t = t_1$, while Δt is arbitrary.*

When $\Delta t = 0$ the variation Δ becomes the variation δ. It is easy to see that (II) then assumes the form of Hamilton's principle (I), p. 515. Form (II) of the variational principle is therefore more general than Hamilton's principle. However, it does not represent a more general property. For by (5) and (8) we can write (II) in the form

$$\int_{t_0}^{t_1}\left[\sum_{i=1}^{n}(P_{i_x}\,\Delta x_i + P_{i_y}\,\Delta y_i + P_{i_z}\,\Delta z_i)\right]\mathrm{d}t\, +$$
$$+\int_{t_0}^{t_1}\left[\sum_{i=1}^{n} m_i\left(x_i\,\frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} + y_i\,\frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} + z_i\,\frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}\right)\right]\mathrm{d}t = 0. \tag{9}$$

Since $\Delta x_i, \Delta y_i, \Delta z_i$, are arbitrary functions representing virtual displacements and vanishing at $t = t_0$ and $t = t_1$, while Δt does not appear at all in formula (9), writing $\delta x_i, \delta y_i, \delta z_i$, for $\Delta x_i, \Delta y_i, \Delta z_i$, we obtain Hamilton's principle from (9).

Equation (II) is therefore equivalent to Hamilton's principle.

Maupertuis' principle. Equation (II) holds for an arbitrary Δt, while $\Delta x_i, \Delta y_i, \Delta z_i$, should only be virtual displacements at every moment t, vanishing at $t = t_0$ and $t = t_1$.

Let us now assume that $\Delta x_i, \Delta y_i, \Delta z_i$, and Δt are so chosen that they satisfy, in addition, the condition

$$\Delta' L = \Delta E. \tag{III}$$

By (5) and (8) condition (III) can be written in the form

$$\sum_{i=1}^{n}(P_{i_x}\,\Delta x_i + P_{i_y}\,\Delta y_i + P_{i_z}\,\Delta z_i) =$$
$$= \sum_{i=1}^{n} m_i\left(x_i\,\frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} + y_i\,\frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} + z_i\,\frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}\right) - 2E\,\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}. \tag{10}$$

From (II) and (III) we obtain

$$\int_{t_0}^{t_1}\left[2\Delta E + 2E\,\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right] = 0, \tag{11}$$

whence by formula (VII), p. 527,

$$\Delta\int_{t_0}^{t_1} E\,\mathrm{d}t = 0. \tag{IV}$$

Therefore: *the variation together with the variation of time of the integral of the kinetic energy is zero for an actual motion if $\Delta x_i, \Delta y_i, \Delta z_i$, are virtual displacements at every moment, equal to zero at $t = t_0$ and $t = t_1$, and if condition* (III) *holds* (i. e. if the virtual work for the displacement $\Delta x_i, \Delta y_i, \Delta z_i$, is equal to the variation together with the variation of time of the kinetic energy).

This theorem is called *Maupertuis' principle* or the *principle of the least action.*

Denoting by $\mathrm{d}s_i$ the differential of arc along which the point m_i moves, and by v_i the velocity of the point, we have $\mathrm{d}s_i = v_i\,\mathrm{d}t$. Consequently

$$\int_{t_0}^{t_1} E\,\mathrm{d}t = \int_{t_0}^{t_1}(\tfrac{1}{2}\sum_{i=1}^{n} m_i v_i^2)\,\mathrm{d}t = \tfrac{1}{2}\int_{t_0}^{t_1}\sum_{i=1}^{n} m_i v_i\,\mathrm{d}s_i.$$

The expression $m_i v_i$ is the *momentum* (*"action"*).

On the basis of (IV) it can be proved that under certain assumptions the integral of the kinetic energy for an actual motion has the smallest value among motions satisfying certain conditions. Hence the name *principle of the least action.*

Let us assume that a certain motion compatible with the constraints satisfies Maupertuis' principle, and, in addition, that the kinetic energy does not vanish in $\langle t_0, t_1\rangle$:

$$E \neq 0 \qquad (t_0 \leqq t \leqq t_1). \tag{12}$$

Let the functions $\Delta x_i, \Delta y_i, \Delta z_i$, at every moment t be virtual displacements, equal to zero at $t = t_0$ and $t = t_1$, and arbitrary in other respects. Let us choose Δt so that equality (III), or — which amounts to the same thing — equation (10), holds. In virtue of (12) and (10) we can assume

$$\Delta t = \int_{t_0}^{t_1}\frac{1}{2E}\left[\sum_{i=1}^{n} m_i\left(x_i\,\frac{\mathrm{d}(\Delta x_i)}{\mathrm{d}t} + y_i\,\frac{\mathrm{d}(\Delta y_i)}{\mathrm{d}t} + z_i\,\frac{\mathrm{d}(\Delta z_i)}{\mathrm{d}t}\right) - \right.$$
$$\left. - \sum_{i=1}^{n}(P_{i_x}\,\Delta x_i + P_{i_y}\,\Delta y_i + P_{i_z}\,\Delta z_i)\right]\mathrm{d}t. \tag{13}$$

Formula (IV) holds for $\Delta x_i, \Delta y_i, \Delta z_i, \Delta t$, chosen in the above way, and consequently (11) also holds. By (III) and (11)

$$\int_{t_0}^{t_1}\left[2\Delta E + 2E\,\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right]\mathrm{d}t = \int_{t_0}^{t_1}\left[\Delta' L + \left(\Delta' L + 2E\,\frac{\mathrm{d}(\Delta t)}{\mathrm{d}t}\right)\right]\mathrm{d}t = 0,$$

whence by (10) we obtain formula (9), in which $\Delta x_i, \Delta y_i, \Delta z_i$, satisfy the same conditions as $\delta x_i, \delta y_i, \delta z_i$, in Hamilton's principle. Since, as we have proved (p. 529), (9) is equivalent to Hamilton's principle, the given motion is an actual motion. Hence we see that among these motions compatible with the constraints for which $E \neq 0$, only actual motions satisfy Maupertuis' principle.

Therefore: *Maupertuis' principle represents a characteristic property of those actual motions for which $E \neq 0$.*

Let us assume that a motion takes place in a potential field having a potential V. Consequently:

$$\partial V / \partial x_i = P_{i_x}, \quad \partial V / \partial y_i = P_{i_y}, \quad \partial V / \partial z_i = P_{i_z},$$

$$\Delta' L = \sum_{i=1}^{n} (P_{i_x} \Delta x_i + P_{i_y} \Delta y_i + P_{i_z} \Delta z_i) = \\ = \sum_{i=1}^{n} \left(\frac{\partial V}{\partial x_i} \Delta x_i + \frac{\partial V}{\partial y_i} \Delta y_i + \frac{\partial V}{\partial z_i} \Delta z_i \right).$$

Since V is a function of the coordinates x_i, y_i, z_i, only, it follows that $\Delta' L = \Delta V$, and hence condition (III) can be written in the form $\Delta V = \Delta E$; consequently

$$\Delta (E - V) = 0. \tag{III'}$$

Remark 1. The assumption $E \neq 0$ is essential; this means that if a motion compatible with the constraints satisfies Maupertuis' principle but not the condition $E \neq 0$, then the motion need not be an actual motion.

For example, let some scleronomic system be given. Let us consider a motion in which the system is at rest from t_0 to t_1 in a certain position compatible with the constraints. Therefore we have $E = 0$ constantly, whence by (4) $\Delta E = 0$ constantly also. It follows from this that formula (11), and consequently formula (IV), will be satisfied for arbitrary Δx_i, Δy_i, Δz_i, Δt, and in particular, therefore, also for all those which satisfy formula (III), or — which amounts to the same thing — formula (10). Thus the given motion satisfies Maupertuis' principle. However, it is obvious that for a suitable choice of forces, rest is impossible, i. e. rest is not an actual motion.

Remark 2. If the variation together with the variation of time were replaced in Maupertuis' principle by an ordinary variation, i. e. if we assumed that $\Delta t = 0$, writing δ for Δ, then formulae (III) and (IV) would assume the forms:

$$\delta' L = \delta E, \tag{14}$$

$$\delta \int_{t_0}^{t_1} E \, dt = 0, \text{ whence } \int_{t_0}^{t_1} \delta E \, dt = 0. \tag{15}$$

Therefore: for an actual motion formula (15) holds under the assumption that $\delta x_i, \delta y_i, \delta z_i$, are virtual displacements at every moment, equal to zero at $t = t_0$ and $t = t_1$, and satisfying condition (14).

The principle expressed in this manner, however, would not represent the characteristic properties of actual motions. For assuming e. g. that no forces act on a system, we should have $\delta' L = 0$. Consequently condition (14) would assume the form $\delta E = 0$ and hence it would imply formula (15) for every motion compatible with the constraints. Therefore in this case every motion compatible with the constraints, and not only an actual motion, would satisfy formula (15), i. e. Maupertuis' principle, in which the variation Δ is replaced by the ordinary variation δ.

We see from this that in Maupertuis' principle it is also essential that we form the variation together with the variation of time.

Remark 3. For a motion given in generalized coordinates it can be proved that *in holonomo-scleronomic systems Maupertuis' principle in the form* (IV) *holds under the assumption that the energy E is also expressed in generalized coordinates, and the variations Δq_j are virtual displacements equal to zero for $t = t_0$ and $t = t_1$ and satisfying condition* (III), *in which* $\Delta' L = \Sigma Q_j \Delta q_j$ (i. e. expressed in terms of the generalized forces Q_j).

Formula (IV) does not hold for rheonomic systems and generalized coordinates, and for them Maupertuis' principle is given in another form.